Treći kolokvij

Kvantna fizika

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Zadatak 1

Izrazite operatore S_x i S_y preko operatora podizanja i spuštanja spina. Pokažite da je operator $S^2 = S_x^2 + S_y^2 + S_z^2$ dijagonalan u bazi svojstvenih stanja operatora S_z. Čestica ima spin 1/2.
Rješenje:

- definicija operatora podizanja i spuštanja spina

$$S_+ = S_x + iS_y, \quad S_- = S_x - iS_y \tag{1}$$

- invertiramo gornje relacije

$$S_x = \frac{1}{2}(S_+ + S_-), \quad S_y = \frac{1}{2i}(S_+ - S_-) \tag{2}$$

- računamo operator $\vec{S}^2$, uzimajući u obzir da operatori S_+^2 S_-^2 poništavaju svojstvena stanja operatora S_z

$$\vec{S}^2 = S_x^2 + S_y^2 + S_z^2 = \frac{1}{2}\left(S_+S_- + S_-S_+\right) + S_z^2 \tag{3}$$

- djelujemo operatorom $\vec{S}^2$ na svojstvena stanja operatora S_z

$$\vec{S}^2 \left|\frac{1}{2}\right\rangle = \left(\frac{1}{2}S_+S_- + \frac{1}{2}S_-S_+ + S_z^2\right)\left|\frac{1}{2}\right\rangle = \frac{3}{4}\hbar^2\left|\frac{1}{2}\right\rangle, \tag{4}$$

$$\vec{S}^2 \left|-\frac{1}{2}\right\rangle = \left(\frac{1}{2}S_+S_- + \frac{1}{2}S_-S_+ + S_z^2\right)\left|-\frac{1}{2}\right\rangle = \frac{3}{4}\hbar^2\left|-\frac{1}{2}\right\rangle \tag{5}$$

- svojstvena stanja operatora S_z su ujedno i svojstvena stanja operatora $\vec{S}^2$

- operator $\vec{S}^2$ je dijagonalan u bazi svojstvenih stanja operatora S_z

$$\vec{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

- točno napisani operatori S_x i S_y nose 1 bod, a točno napisan operator $\vec{S}^2$ u dijagonalnoj formi 1 bod

Zadatak 2

Čestica u centralnom potencijalu ima orbitalni angularni moment $l = 2\hbar$ i spin $s = 1\hbar$. Nađite energijske nivoe i degeneracije povezane sa spin-orbit interakcijom oblika

$$H_{so} = A\boldsymbol{L} \cdot \boldsymbol{S}.$$

<u>Rješenje:</u>

- radimo u bazi svojstvenih stanja operatora $\{H, \vec{J}^2, J_z, \vec{L}^2, \vec{S}^2\}$
- iskoristimo definiciju operatora $\vec{J}$

$$\vec{J}^2 = \left(\vec{L} + \vec{S}\right)^2 = \vec{L}^2 + 2\vec{L} \cdot \vec{S} + \vec{S}^2 \Longrightarrow \vec{L} \cdot \vec{S} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right) \quad (7)$$

- Hamiltonijan spin-orbit međudjelovanja možemo napisati u sljedećem obliku

$$H_{so} = \frac{A}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right) \quad (8)$$

- izračunamo matrični element

$$E_{so} = \langle jm| H_{so} |jm\rangle = \frac{A}{2} \langle jm(ls)| \vec{J}^2 - \vec{L}^2 - \vec{S}^2 |(ls)jm\rangle \quad (9)$$

$$E_{so} = \frac{A}{2}\hbar^2 \left[j(j+1) - l(l+1) - s(s+1)\right] \quad (10)$$

- zadan je orbitalni angularni moment $2\hbar$, a spin $1\hbar$

$$E_{so} = \langle jm| H_{so} |jm\rangle = \frac{A}{2}\hbar^2 \left[j(j+1) - 8\right]$$

- ograničenja na zakretni impuls j zadana su pravilom trokuta

$$|l - s| \leq j \leq l + s$$

- $j = 1$: $E_{so} = -3A\hbar^2$, $d = 2j + 1 = 3$
- $j = 2$: $E_{so} = -A\hbar^2$, $d = 2j + 1 = 5$
- $j = 3$: $E_{so} = 2A\hbar^2$, $d = 2j + 1 = 7$
- matrični element (8) nosi 0.5 bodova, dok svaki nivo zajedno s degeneracijom nosi 0.5 bodova

Zadatak 3

Sustav ukupnog angularnog momenta $J = \frac{5}{2}$ i projekcije na z-os $M = -\frac{1}{2}$ se sastoji od dvije čestice angularnih momenata $j_1 = \frac{3}{2}$ i $j_2 = 1$. Polazeći od stanja $|5/2\ -5/2\rangle$, djelujući operatorom J_+ generirajte stanje $|5/2\ -1/2\rangle$. Koje su moguće vrijednosti projekcija m_1 i m_2, te pripadajuće vjerojatnosti?

Rješenje:

- koristimo relaciju

$$J_+ |JM\rangle = \hbar\sqrt{J(J+1) - M(M+1)}\,|JM+1\rangle \tag{11}$$

- polazimo od stanja s najnižom projekcijom zakretnog impulsa i djelujemo operatorom J_+

$$J_+ \left|\frac{5}{2} - \frac{5}{2}\right\rangle = \left(J_{1+}\left|\frac{3}{2} - \frac{3}{2}\right\rangle\right)|1 - 1\rangle + \left|\frac{3}{2} - \frac{3}{2}\right\rangle (J_{2+}|1 - 1\rangle) \tag{12}$$

- dolazimo do sljedeće relacije

$$\left|\frac{5}{2} - \frac{3}{2}\right\rangle = \sqrt{\frac{3}{5}}\left|\frac{3}{2} - \frac{1}{2}\right\rangle |1 - 1\rangle + \sqrt{\frac{2}{5}}\left|\frac{3}{2} - \frac{3}{2}\right\rangle |10\rangle \tag{13}$$

- još jednom djelujemo operatorom J_+

$$J_+ \left|\frac{5}{2} - \frac{3}{2}\right\rangle = \sqrt{\frac{3}{5}}\left(J_{1+}\left|\frac{3}{2} - \frac{1}{2}\right\rangle\right)|1 - 1\rangle + \sqrt{\frac{3}{5}}\left|\frac{3}{2} - \frac{1}{2}\right\rangle (J_{2+}|1 - 1\rangle) \tag{14}$$

$$+\sqrt{\frac{2}{5}}\left(J_{1+}\left|\frac{3}{2} - \frac{3}{2}\right\rangle\right)|10\rangle + \sqrt{\frac{2}{5}}\left|\frac{3}{2} - \frac{3}{2}\right\rangle (J_{2+}|10\rangle) \tag{15}$$

- izračnamo pojedine članove na desnoj strani

$$J_+ \left|\frac{5}{2} - \frac{3}{2}\right\rangle = \hbar\sqrt{8}\left|\frac{5}{2} - \frac{1}{2}\right\rangle \tag{16}$$

$$J_+ \left|\frac{3}{2} - \frac{3}{2}\right\rangle = \sqrt{3}\hbar\left|\frac{3}{2} - \frac{1}{2}\right\rangle \tag{17}$$

$$J_+ \left|\frac{3}{2} - \frac{1}{2}\right\rangle = 2\hbar\left|\frac{3}{2}\frac{1}{2}\right\rangle \tag{18}$$

$$J_+ |10\rangle = \hbar\sqrt{2}\,|11\rangle \tag{19}$$

$$\tag{20}$$

- dolazimo do sljedeće relacije

$$\sqrt{8}\left|\frac{5}{2}-\frac{1}{2}\right\rangle = 2\sqrt{\frac{3}{5}}\left|\frac{3}{2}\frac{1}{2}\right\rangle|1-1\rangle + \sqrt{\frac{6}{5}}\left|\frac{3}{2}-\frac{1}{2}\right\rangle|10\rangle \tag{21}$$

$$+\sqrt{\frac{6}{5}}\left|\frac{3}{2}-\frac{1}{2}\right\rangle|10\rangle + \frac{2}{\sqrt{5}}\left|\frac{3}{2}-\frac{3}{2}\right\rangle|11\rangle \tag{22}$$

- podijelimo prethodni izraz s $\sqrt{8}$

$$\left|\frac{5}{2}-\frac{1}{2}\right\rangle = \sqrt{\frac{3}{10}}\left|\frac{3}{2}\frac{1}{2}\right\rangle|1-1\rangle + \sqrt{\frac{3}{5}}\left|\frac{3}{2}-\frac{1}{2}\right\rangle|10\rangle + \frac{1}{\sqrt{10}}\left|\frac{3}{2}-\frac{3}{2}\right\rangle|11\rangle \tag{23}$$

- sustav se može nalaziti u sljedećim stanjima
  - $m_1 = 1/2$ i $m_1 = -1$, vjerojatnost $3/10$
  - $m_1 = -1/2$ i $m_1 = 0$, vjerojatnost $6/10$
  - $m_1 = -3/2$ i $m_1 = 1$, vjerojatnost $1/10$
- suma vjerojatnosti iznosi 1
- točno izračunat $J_+^2|5/2-5/2\rangle$ nosi 0.5 bodova, a svaka točna projekcija sa vjerojatnosti još 0.5 bodova

Zadatak 4

U Bornovoj aproksimaciji pronađite diferencijalni udarni presjek za česticu mase m koja se raspršuje na potencijalu

$$V(r) = Ae^{-\frac{r^2}{a^2}}$$

Rješenje:

- amplitude raspršenja u Bornovoj aproksimaciji

$$f_B(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r \sin qr V(r) dr, \quad q = 2k\sin\frac{\theta}{2} \tag{24}$$

- uvrstimo potencijal

$$f_B(\theta) = -\frac{2mA}{\hbar^2 q}\int_0^\infty re^{-\frac{r^2}{a^2}} \sin qr dr \tag{25}$$

- parcijalno integriramo

$$du = re^{-\frac{r^2}{a^2}} \Longrightarrow u = \int re^{-\frac{r^2}{a^2}} dr = \frac{1}{2}a^2 \int e^{-t} dt = -\frac{1}{2}a^2 e^{-\frac{r^2}{a^2}} \tag{26}$$

$$v = \sin qr \Longrightarrow dv = q\cos qr dr \tag{27}$$

- vratimo se početnom integralu

$$\int_0^\infty re^{-\frac{r^2}{a^2}} \sin qr dr = -\frac{1}{2}a^2 e^{-\frac{r^2}{a^2}} \sin qr \Big|_0^\infty + \frac{q}{2}a^2 \int_0^\infty e^{-\frac{r^2}{a^2}} \cos qr dr \tag{28}$$

- radi se o tabličnom integralu (Bronštejn)

$$\int_0^\infty re^{-\frac{r^2}{a^2}} \sin qr dr = \frac{q}{2}a^2 \frac{\sqrt{\pi}}{2} a e^{-\frac{q^2 a^2}{4}} \tag{29}$$

- vratimo se amplitudi raspršenja

$$f_B(\theta) = -\frac{mAa^3\sqrt{\pi}}{2\hbar^2} e^{-\frac{1}{4}q^2 a^2} \tag{30}$$

- diferencijalni udarni presjek

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2 = \frac{m^2 A^2 a^6 \pi}{4\hbar^4} e^{-\frac{1}{2}q^2 a^2}, \quad q = 2k\sin\frac{\theta}{2} \tag{31}$$

- amplituda raspršenja nosi 1.5 bodova, a diferencijalni udarni presjek 0.5 bodova

Zadatak 5

Koristeći operatore podizanja i spuštanja, izračunajte očekivane vrijednosti $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$ i $\langle p^2 \rangle$ u n-tom pobuđenom stanju harmoničkog oscilatora. Pokažite da je u osnovnom stanju ispunjena relacija $\Delta x \Delta p = \frac{\hbar}{2}$.

Rješenje:

- operatori podizanja i spuštanja

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right), \tag{32}$$

- izrazimo $\hat{x}$ i $\hat{p}$ pomoću operatora $\hat{a}$ i $\hat{a}^\dagger$

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger \right), \quad p = i\sqrt{\frac{m\omega\hbar}{2}} \left(a^\dagger - a \right) \tag{33}$$

- očekivane vrijednosti operatora x i p

$$\langle n | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle n | a + a^\dagger | n \rangle = 0, \tag{34}$$

$$\langle n | p | n \rangle = i\sqrt{\frac{m\omega\hbar}{2}} \langle n | -a + a^\dagger | n \rangle = 0 \tag{35}$$

- kvadriramo operatore x i p

$$x^2 = \frac{\hbar}{2m\omega} \left(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger \right) \tag{36}$$

$$p^2 = -\frac{m\omega\hbar}{2} \left(aa - aa^\dagger - a^\dagger a + a^\dagger a^\dagger \right) \tag{37}$$

- očekivane vrijednosti operatora x^2 i p^2

$$\langle n | x^2 | n \rangle = \frac{\hbar}{2m\omega} \langle n | a^\dagger a + aa^\dagger | n \rangle = 0, \tag{38}$$

$$\langle n | p^2 | n \rangle = \frac{m\omega\hbar}{2} \langle n | aa^\dagger + a^\dagger a | n \rangle = 0 \tag{39}$$

- koristimo relacije

$$a | n \rangle = \sqrt{n} | n-1 \rangle, \quad a^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle \tag{40}$$

- slijedi

$$\langle n | a^\dagger a | n \rangle = \sqrt{n} \langle n | a^\dagger | n-1 \rangle = n \tag{41}$$

$$\langle n | aa^\dagger | n \rangle = \sqrt{n+1} \langle n | a | n+1 \rangle = n+1 \tag{42}$$

- tražene očekivane vrijednosti

$$\langle n|\, x^2 \,|n\rangle = \frac{\hbar}{2m\omega}(2n+1) \tag{43}$$

$$\langle n|\, p^2 \,|n\rangle = \frac{m\omega\hbar}{2}(2n+1) \tag{44}$$

- računamo odstupanja u osnovnom stanju

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \tag{45}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{m\omega\hbar}{2}} \tag{46}$$

- neodređenost u osnovnom stanju

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{47}$$

- očekivane vrijednosti $\langle x\rangle$ i $\langle p\rangle$ nose 0.5 bodova, očekivane vrijednosti $\langle x^2\rangle$ i $\langle p^2\rangle$ 1 bod, a neodređenost u osnovnom stanju 0.5 bodova.

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