2. kolokvij iz kvantne fizike

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Zadatak 1

Kako izgleda relacija neodređenosti za opservable L_x i L_y za neko svojstveno stanje čestice u sferno-simetričnom potencijalu?

$$[L_x, L_y] = i\hbar L_z$$

Relacija neodređenosti:

$$\sigma_{L_x}\sigma_{L_y} \geq \frac{1}{2}\left|\langle\psi|[L_x, L_y]|\psi\rangle\right| = \frac{\hbar}{2}\left|\langle\psi|L_z|\psi\rangle\right|$$

Svojstveno stanje čestice u sferno-simetričnom potencijalu podrazumijeva:

$$L_z|\psi\rangle = \hbar m|\psi\rangle$$

$$\Longrightarrow \boxed{\sigma_{L_x}\sigma_{L_y} \geq \frac{\hbar^2}{2}|m|}$$

Zadatak 2

Zadan je 1D sustav dva identična neinteragirajuća fermiona mase m u potencijalu harmoničkog oscilatora frekvencije ω.

(a) Kako izgleda (normirana) valna funkcija osnovnog stanja takvog sustava?

(b) Pretpostavimo da fermioni međudjeluju interakcijom opisanom potencijalom

$$V(x_1, x_2) = \frac{a}{(x_1 - x_2)^2}$$

Odredite popravku energije osnovnog stanja u prvom redu računa smetnje.

Napomena: Svojstvena valna funkcija harmoničkog oscilatora glasi:

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

(a) Valna funkcija mora biti antisimetrična na zamjenu čestica (jer se radi o fermionima), tj. na zamjenu koordinata. Najniže takvo stanje (po energiji) može se konstruirati od od osnovnog i prvog pobuđenog jednočestičnog stanja:

$$\psi(x_1, x_2) = \mathcal{N}\left(\psi_0(x_1)\psi_1(x_2) - \psi_0(x_2)\psi_1(x_1)\right)$$

Za određivanje norme $\mathcal{N}$ koristimo da su $\psi_0(x)$ i $\psi_1(x)$ već normirane na 1 i međusobno ortogonalne.

$$1 = \int_{-\infty}^{\infty} dx_1 dx_2 \, |\Psi(x_1,x_2)|^2$$

$$= |\mathcal{N}|^2 \int_{-\infty}^{\infty} dx_1 dx_2 \Big[|\Psi_0(x_1)|^2 |\Psi_1(x_2)|^2 -$$

$$- \left(\Psi_0^*(x_1)\Psi_1(x_1)\right)\left(\Psi_1^*(x_2)\Psi_0(x_2)\right) -$$

$$- \left(\Psi_1^*(x_1)\Psi_0(x_1)\right)\left(\Psi_0^*(x_2)\Psi_1(x_2)\right) +$$

$$+ |\Psi_0(x_2)|^2 |\Psi_1(x_1)|^2 \Big] =$$

$$= |\mathcal{N}|^2 \Big[\int_{-\infty}^{\infty} dx_1 |\Psi_0(x_1)|^2 \int_{-\infty}^{\infty} dx_2 |\Psi_1(x_2)|^2 -$$

$$- \int_{-\infty}^{\infty} dx_1 \left(\Psi_0^*(x_1)\Psi_1(x_1)\right) \int_{-\infty}^{\infty} dx_2 \left(\Psi_1^*(x_2)\Psi_0(x_2)\right) -$$

$$- \int_{-\infty}^{\infty} dx_1 \left(\Psi_1^*(x_1)\Psi_0(x_1)\right) \int_{-\infty}^{\infty} dx_2 \left(\Psi_0^*(x_2)\Psi_1(x_2)\right) +$$

$$+ \int_{-\infty}^{\infty} dx_1 |\Psi_1(x_1)|^2 \int_{-\infty}^{\infty} dx_2 |\Psi_0(x_2)|^2 \Big] = \underline{2|\mathcal{N}|^2}$$

$$\longrightarrow \mathcal{N} = \frac{1}{\sqrt{2}}$$

$$\Longrightarrow \underline{\Psi(x_1,x_2) = \frac{1}{\sqrt{2}} \left(\Psi_0(x_1)\Psi_1(x_2) - \Psi_1(x_1)\Psi_0(x_2) \right)}$$

$$\underline{\Psi(x_1,x_2) = \frac{m\omega}{\pi\hbar} (x_2 - x_1) \exp\left(-\frac{m\omega}{2\hbar}(x_1^2 + x_2^2)\right)} \, , \quad \underline{E = 2\hbar\omega}$$

(b)

$$\Delta E = \int_{-\infty}^{\infty} dx_1 \, dx_2 \; \Psi^*(x_1, x_2) \, V(x_1, x_2) \, \Psi(x_1, x_2) =$$

$$= \frac{m^2\omega^2}{\pi\hbar^2} \int_{-\infty}^{\infty} dx_1 dx_2 \; \cancel{(x_2 - x_1)^2} \exp\left(-\frac{m\omega}{\hbar}(x_1^2 + x_2^2)\right) \frac{a}{\cancel{(x_1 - x_2)^2}}$$

$$= \frac{m^2\omega^2 a}{\pi\hbar^2} \left(\int_{-\infty}^{\infty} dx_1 \, \exp\left(-\frac{m\omega}{\hbar} x_1^2\right) \right)^2 = \left\{ \begin{array}{l} y = \sqrt{\frac{m\omega}{\hbar}} \, x_1 \\ dx_1 = \sqrt{\frac{\hbar}{m\omega}} \, dy \end{array} \right.$$

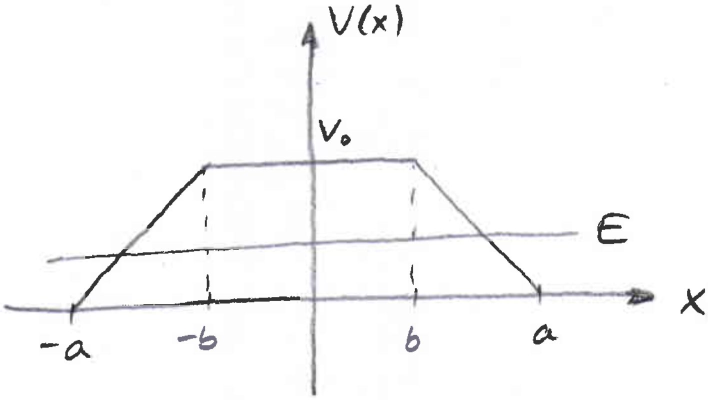
$$= \frac{m^{\cancel{2}}\omega^{\cancel{2}} a}{\pi\hbar^{\cancel{2}}} \frac{\cancel{\hbar}}{\cancel{m\omega}} \left(\int_{-\infty}^{\infty} dy \, e^{-y^2} \right)^2 =$$

$$= \underline{\underline{\frac{m\omega a}{\hbar}}}$$

Zadatak 3

Pomoću WKB metode odredite koeficijent refleksije za česticu energije $E \ll V_0$ koja se raspršuje na 1D potencijalu:

$$V(x) = \begin{cases} 0 & , \quad |x| > a \\ V_0 \dfrac{x+a}{a-b} & , \quad -a < x < -b \\ V_0 & , \quad |x| < b \\ V_0 \dfrac{x-a}{b-a} & , \quad b < x < a \end{cases}$$



$$\underline{\underline{R = 1 - T}}$$

$$T = \exp\left[- \frac{2\sqrt{2m}}{\hbar} \int_{x_1}^{x_2} dx\, K(x) \right] \quad , \quad K(x) = \sqrt{V(x) - E}$$

$x_1, x_2 \sim$ klasične točke obrata:

$$V_0 \frac{x_1 + a}{b-a} = E \longrightarrow x_1 = (a-b)\frac{E}{V_0} - a \equiv -x_0$$

$$V_0 \frac{x_2 - a}{b-a} = E \longrightarrow x_2 = (b-a)\frac{E}{V_0} + a \equiv x_0$$

$$\int_{-x_0}^{x_0} dx\,\sqrt{V(x)-E} = 2\int_0^{x_0} dx\,\sqrt{V(x)-E} =$$

$$= 2\int_0^{b} dx\,\sqrt{V(x)-E} + 2\int_b^{x_0} dx\,\sqrt{V(x)-E} =$$

$$= 2\int_0^{b} dx\,\sqrt{V_0-E} + 2\int_b^{x_0} dx\,\sqrt{V_0\,\frac{x-a}{b-a}-E} =$$

$$= 2b\sqrt{V_0-E} + 2\int_b^{x_0} dx\,\sqrt{\frac{V_0}{b-a}\,x + \left(V_0\,\frac{a}{a-b} - E\right)} =$$

$$= 2b\sqrt{V_0-E} + \frac{2}{3}\,\frac{(b-a)}{V_0}\left[\left(\frac{V_0}{b-a}\,x + \frac{a}{a-b}\,V_0 - E\right)^{3/2}\right]\Bigg|_b^{x_0} =$$

$$= 2b\sqrt{V_0-E} + \frac{4}{3}\,\frac{(a-b)}{V_0}\,(V_0-E)^{3/2}$$

$$\Longrightarrow \quad R = 1 - \exp\left[-\frac{4b}{\hbar}\sqrt{2m(V_0-E)} - \frac{8}{3}\,\frac{(a-b)}{\hbar V_0}\sqrt{2m}\,(V_0-E)^{3/2}\right]$$

Zadatak 4

Neka se čestica naboja q nalazi u sferno-simetričnom potencijalu. Ako na sustav nametnemo homogeno električno polje $\vec{E} = E\hat{e}$, gdje jedinični vektor $\hat{e}$ određuje smjer električnog polja, odredite postoji li smjer polja za koji je opservabla $\vec{L}^2$ konstanta gibanja. Može li se polje usmjeriti na način da je L_z konstanta gibanja? Objasnite rezultate.

$$H_0 = \frac{p^2}{2m} + V(r)$$

Bez električnog polja su $\vec{L}^2$ i L_z konstante gibanja za sferno-simetrični potencijal $V(r)$, jer:

$$[H_0, \vec{L}^2] = 0 \quad i \quad [H_0, L_z] = 0$$

Kada uključimo električno polje imamo:

$$H = H_0 + q\phi(r)$$

Za električno polje uzmemo općenito: $\vec{E} = E\hat{e}$

Za elektrostatski potencijal imamo:

$\vec{E} = -\vec{\nabla}\phi$, pa možemo uzeti $\underline{\phi(\vec{r}) = -E\hat{e}\cdot\vec{r}}$

$$\phi(\vec{r}) = -E\left(e_x r\sin\vartheta\cos\varphi + e_y r\sin\vartheta\sin\varphi + e_z r\cos\vartheta\right)$$

Treba provjeriti možemo li izabrati $\hat{e}$ da bude:

$[H, L_z] = 0$ i $[H, \vec{L}^2] = 0$, tj. $[\phi, L_z] = 0$ i $[\phi, \vec{L}^2] = 0$

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}$$

$$[\Phi(\vec{r}), L_z] = -i\hbar E \frac{\partial}{\partial \varphi}\left(e_x r \sin\vartheta \cos\varphi + e_y r \sin\vartheta \sin\varphi + e_z r \cos\vartheta\right)$$

$$= -i\hbar E r \sin\vartheta \left(-e_x \sin\varphi + e_y \cos\varphi\right) = 0$$

$\Rightarrow$ jedini izbor je $\underline{e_x = e_y = 0}$.

$\Rightarrow$ L_z je konstanta gibanja ako je električno polje u z-smjeru.

$$\vec{L}^2 = -\frac{\hbar^2}{\sin\vartheta}\frac{\partial}{\partial\theta}\left(\sin\vartheta \frac{\partial}{\partial\theta}\right) - \frac{\hbar^2}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} =$$

$$= -\hbar^2 \cot\theta \frac{\partial}{\partial\theta} - \hbar^2 \frac{\partial^2}{\partial\theta^2} - \frac{\hbar^2}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

$$[\Phi(\vec{r}), \vec{L}^2] = -\hbar^2 E r$$

$$= -\frac{\hbar^2 E r}{\sin^2\theta}\left(\sin\vartheta\cos\theta\frac{\partial}{\partial\theta} - \sin^2\vartheta\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial\varphi^2}\right)\Big(\sin\vartheta\left(e_x\cos\varphi + e_y\sin\varphi\right) +$$

$$+ e_z\cos\vartheta\Big) =$$

$$= -\frac{\hbar^2 E r}{\sin^2\vartheta}\Big(e_x \sin\vartheta\cos^2\vartheta\cos\varphi + e_y\sin\vartheta\cos^2\vartheta\sin\varphi - e_z\sin^2\vartheta\cos\vartheta -$$

$$- e_x\sin^3\theta\cos\varphi - e_y\sin^3\vartheta\sin\varphi - e_z\sin^2\vartheta\cos\theta -$$

$$- e_x\sin\vartheta\cos\varphi - e_y\sin\vartheta\sin\varphi\Big) =$$

$$= -\frac{\hbar^2 E r}{\sin\theta}\Big[e_x(\cos^2\theta - \sin^2\theta - 1)\cos\varphi +$$

$$+ e_y(\cos^2\theta - \sin^2\theta - 1)\sin\varphi -$$

$$- 2e_z \sin\theta\cos\theta \Big] = 0$$

$\Rightarrow$ komutator je nula jedino za $e_x = e_y = e_z = 0$

$\Rightarrow$ Nema smjera električnog polja za kojeg je $\vec{L}^2$ konstanta gibanja.

$\vec{E} = E\hat{z}$ ne narušava cilindričnu simetriju uz koju je vezan L_z, ali $\vec{E} = E\hat{e}$ narušava sfernu simetriju uz koju je vezan $\vec{L}^2$.

Zadatak 5

Čestica se nalazi u 3D sustavu opisanom Hamiltonijanom H_0. Dva svojstvena stanja sustava imaju jednaku svojstvenu energiju E_0. Valna funkcija jednog od stanja je simetrična u prostoru, a druga je asimetrična. Ako u sustav dodamo malu smetnju $V(\vec{r})$ koja je antisimetrična u prostoru, odredite popravku energije tih stanja u prvom redu računa smetnje te valne funkcije koje odgovaraju tim stanjima. Rezultate izrazite preko neiščezavajućih matričnih elemenata.

$$\psi_1(\vec{r}) = \psi_1(-\vec{r}) \quad , \quad H_0\psi_1 = E_0\psi_1$$

$$\psi_2(\vec{r}) = -\psi_2(-\vec{r}) \quad , \quad H_0\psi_2 = E_0\psi_2$$

$$V(\vec{r}) = -V(-\vec{r})$$

$$\langle\psi_1|H_0|\psi_1\rangle = \langle\psi_2|H_0|\psi_2\rangle = E_0$$

$$\langle\psi_1|H_0|\psi_2\rangle = \langle\psi_2|H_0|\psi_1\rangle = 0 \quad \longrightarrow \quad H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$$

$$|\psi_1\rangle \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |\psi_2\rangle \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Matrični elementi matrice smetnje:

$$\langle\psi_1|V|\psi_1\rangle = \int d^3r\, |\psi_1(\vec{r})|^2 V(\vec{r}) = \int d^3r\, |\psi_1(-\vec{r})|^2 V(-\vec{r}) =$$

$$= -\int d^3r\, |\psi(\vec{r})|^2 V(\vec{r}) = \underline{0}$$

$$\langle \psi_2 | V | \psi_2 \rangle = \int d^3r \, |\psi_2(\vec{r})|^2 V(\vec{r}) = \int d^3r \, |\psi_2(-\vec{r})|^2 V(-\vec{r}) =$$

$$= -\int d^3r \, |\psi_2(\vec{r})|^2 V(\vec{r}) = \underline{0}$$

$$\langle \psi_1 | V | \psi_2 \rangle = \int d^3r \, \psi_1^*(\vec{r}) \psi_2(\vec{r}) V(\vec{r}) =$$

$$= \int d^3r \, \psi_1^*(-\vec{r}) \psi_2(-\vec{r}) V(-\vec{r}) =$$

$$= \int d^3r \, \psi_1^*(\vec{r}) \left(-\psi_2(\vec{r})\right)\left(-V(\vec{r})\right) =$$

$$= \int d^3r \, \psi_1^*(\vec{r}) \psi_2(\vec{r}) V(\vec{r}) = V$$

$$\langle \psi_2 | V | \psi_1 \rangle = \int d^3r \, \psi_2^*(\vec{r}) \psi_1(\vec{r}) V(\vec{r}) =$$

$$= \int d^3r \, \psi_2^*(-\vec{r}) \psi_1(-\vec{r}) V(-\vec{r}) =$$

$$= \int d^3r \left(-\psi_2^*(\vec{r})\right) \psi_1(\vec{r}) \left(-V(\vec{r})\right) =$$

$$= \int d^3r \, \psi_2^*(\vec{r}) \psi_1(\vec{r}) V(\vec{r}) =$$

$$= \left(\int d^3r \, \psi_1^*(\vec{r}) \psi_2(\vec{r}) V(\vec{r})\right)^* = V^*$$

$$\Longrightarrow \quad \underline{H' = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}}, \quad H = H_0 + H'$$

Prvi red degeneriranog računa smetnje:

$$|\phi\rangle = C_1|\psi_1\rangle + C_2|\psi_2\rangle$$

$$\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Delta E \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \det(H' - \Delta E) = 0 = \det\begin{pmatrix} -\Delta E & V \\ V^* & \Delta E \end{pmatrix} = (\Delta E)^2 - |V|^2$$

$$\Rightarrow \underline{\underline{\Delta E_{\pm} = \pm|V|}}$$

$\underline{\Delta E_+}$: $\quad |\phi_+\rangle = c_1^{(+)}|\psi_1\rangle + c_2^{(+)}|\psi_2\rangle$

$$\begin{pmatrix} -|V| & V \\ V^* & |V| \end{pmatrix}\begin{pmatrix} c_1^{(+)} \\ c_2^{(+)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_1^{(+)} = \frac{V}{|V|} c_2^{(+)}$$

$$|\phi_+\rangle = c_2^{(+)}\left(\frac{V}{|V|}|\psi_1\rangle + |\psi_2\rangle\right)$$

$$\langle\phi_+|\phi_+\rangle = 1 = 2|c_2^{(+)}|^2 \longrightarrow c_2^{(+)} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \underline{|\phi_+\rangle = \frac{1}{\sqrt{2}}\left(\frac{V}{|V|}|\psi_1\rangle + |\psi_2\rangle\right)}$$

$\underline{\Delta E_-:}$ $\quad |\Phi_-\rangle = c_1^{(-)}|\psi_1\rangle + c_2^{(-)}|\psi_2\rangle$

$$\begin{pmatrix} |V| & V \\ V^* & |V| \end{pmatrix} \begin{pmatrix} c_1^{(-)} \\ c_2^{(-)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow c_1^{(-)} = -\frac{V}{|V|} c_2^{(-)}$$

$$\longrightarrow |\Phi_-\rangle = c_2^{(-)} \left(-\frac{V}{|V|} |\psi_1\rangle + |\psi_2\rangle \right)$$

$$\langle \Phi_- | \Phi_- \rangle = 1 = 2 |c_2^{(-)}|^2 \longrightarrow c_2^{(-)} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow |\Phi_-\rangle = \underline{\frac{1}{\sqrt{2}} \left(-\frac{V}{|V|} |\psi_1\rangle + |\psi_2\rangle \right)}$$