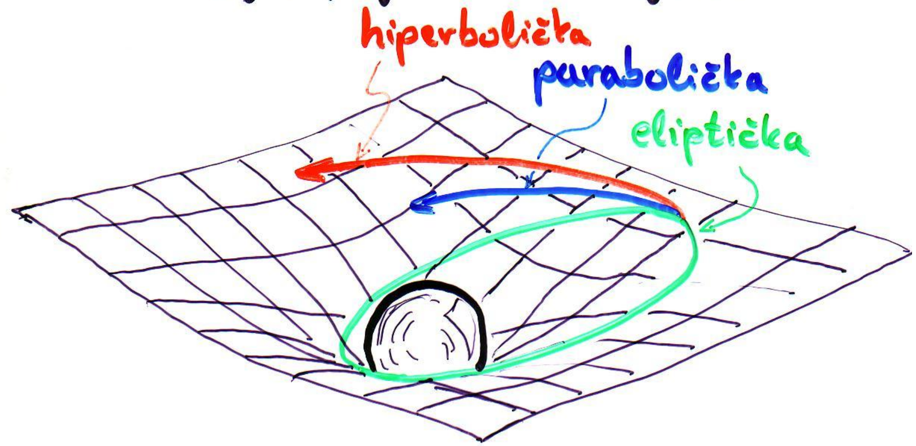
FIZIKALNA

KOZMOLOGIJA

II. VEZA KOZMOLOGIJE I GRAVITACIJE

◊ Veza geometrije i gravitacije

- primjer putanja oko središnjeg tijela, koje stvara zakrivljenost



- gravitacije nema na mjestima iščezavajuće zakrivljenosti
- zakrivljenost prostora-vremena opisana je Riemannovim tenzorom zakrivljenosti $R_{\mu\nu\rho\sigma}$ 1) (antisim. $(\mu\nu)$, $(\rho\sigma)$)

iz relac. u lokalno-inerc. koor. $= \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\nu\rho}(x) + \partial_\nu\partial_\rho g_{\mu\sigma}(x) - \partial_\mu\partial_\rho g_{\nu\sigma} - \partial_\nu\partial_\sigma g_{\mu\rho}\right)$

2) $R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0 \Rightarrow 4^4 = 256$ komp. $\rightarrow$ 20 nezav.

(Narlikar p. 47)

STR (nerazlikovanje inercijalnih sust.)
OTR (nerazlikovanje inercijalnih sust. od sustava u slobodnom padu)

- **STR**: kontrakcija duljina, dilatacija vremena
- **OTR** s "jakim principom ekvivalencije" savijanje zraka svjetlosti, usporavanje vremena u gravitacijskom polju (BeG 3.3)
- **Vježba 2.1** :

Mjera jakosti gravitacije (brzina oslobađanja)

Od Newton-ovog do Einstein-ovog svemira

Newton-ov prostor • $\ell^2 = x^2 + y^2 + z^2$

-ovo vrijeme • t (vrem. interval)

dvije invarijante

Einsteinov prostor

◊ specijalne relativnosti (STR)

- jedinstvena udaljenost u geometriji Minkowskoga

(struktura prostora-vremena takva da su dobro definirani samo intervali VLASTITOG VREMENA)

• $s^2 = c^2(\Delta\tau)^2 = c^2(\Delta t)^2 - (\Delta\vec{x})^2$

$$ds^2 = c^2(d\tau)^2 = c^2(dt)^2 - (d\vec{x})^2 \qquad (1)$$

1. invarijanta (Lorentzovih transf. {potiske + rotac.} koje povezuju INERCIJALNE SUSTAVE)

• c (brzina svjetlosti) je

2. invarijanta (ista za sve opažače)

& kovarijantnost zakona (istih —||—)

◊ Einsteinova opća relativnost (OTR)

proširuje kovarijantnost zakona na širu klasu opažača, koji su u sustavima u slobodnom padu!

Invarijantni interval (1) poopćuje se na

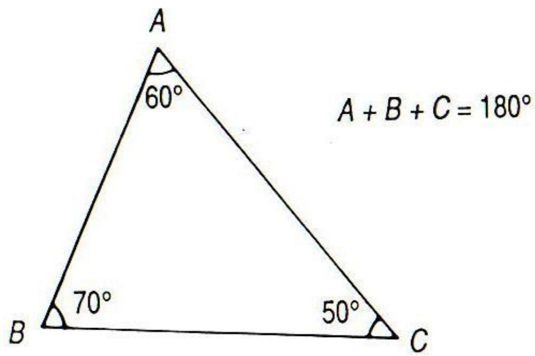
$$ds^2 = c^2(d\tau)^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu \qquad (2)$$

(1) → (2) znači

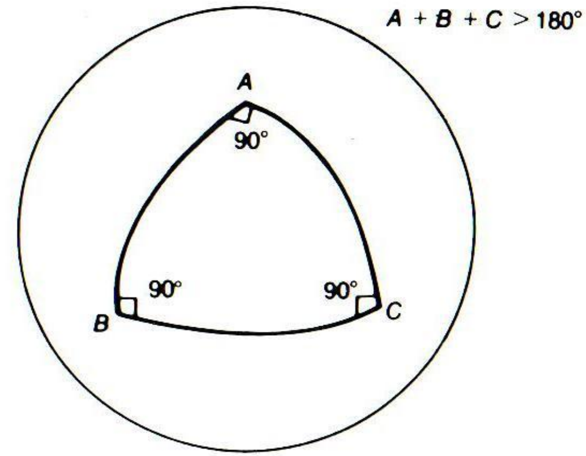
napuštanje prostora

"ravnozemaca" (flatlanders)

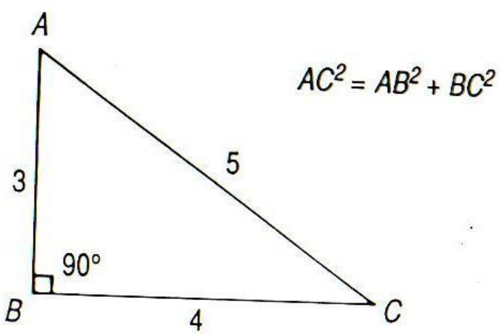
U općoj relativnosti (OTR)



(a)



(a)



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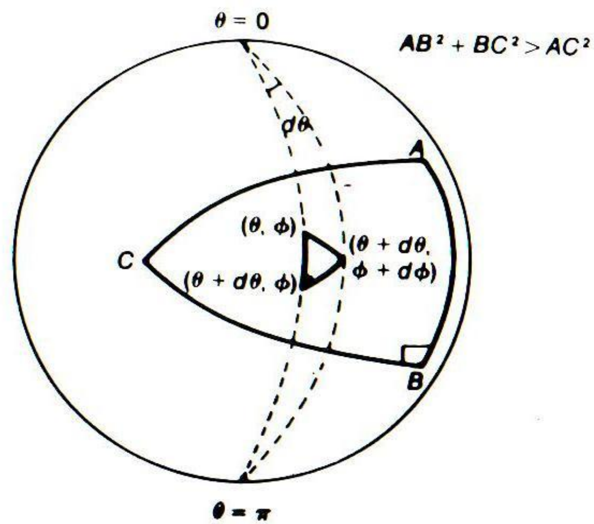


(b)

prijelaz s (1) na (2)
odgovara prijelazu s
ravnih na zakrivljene trokute

Slabi i jaki oblik principa ekvivalencije

- Geodetska jednadžba i usporedba s Lorentzovom silom
- Vježba 2.2: prikaz slobodnog pada u općim koordinatama i Newtonova granica

Princip ekvivalencije i geodetska jednadžba

Svođenje gravitacije na geometriju*) omogućuje da $\boxed{\text{zakrivljene svjetske crte ravnog prostora-vremena}}$ svedemo na $\boxed{\text{ravne svjetske crte zakrivljenog prost.-vrem.}}$

*) pomoću SLABOG PRINCIPA EKVIVALENCIJE:

Gravitacija se može ukloniti lokalno (u malim prost-vrem. intervalima) izborom sustava u slobodnom padu ($\xi^\alpha(x)$)

(poništavanje inercijalne i gravitacijske sile ⇒ Newtonova ekvivalencija inercijalne i gravitacijske mase)

Oko točke P

$g_{\mu\nu} \to \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$; vlast. vrijeme $ds = c\,dt_0$

trajektorija je pravac $\left.\dfrac{d^2\xi^\alpha}{ds^2}\right|_P = 0$; $ds^2 = \eta_{\mu\nu}\, d\xi^\mu d\xi^\nu$

U opć. koordinatama x^μ

$$\frac{d\xi^\alpha}{ds} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{ds} \Rightarrow$$

$$\Rightarrow 0 = \frac{d^2x^\lambda}{ds^2} + \underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}}_{\equiv \Gamma^\lambda_{\mu\nu}} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \Rightarrow g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

$$\Rightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)$$

Einsteinova kozmološka jednadžba

Od OTR do KOZMOLOGIJE

Narlikar '93, Ch 3

◇ Einsteinov svemir (1917)

- zamišljen kao posvuda ispunjen materijom

uz dodatne pretpostavke SIMETRIJA:

- HOMOGENOST & IZOTROPNOST

uz ondašnje vjerovanje astronoma da nema širenja ili sažimanja, pretpostavlja

- STATIČNOST

↳ omogućava izoliranje vremenske koordinate:

$$ds^2 = c^2 dt^2 - \alpha_{ij} dx^i dx^j \quad (i,j = 1,2,3)$$

konst. (zbog homogenosti)

$\nexists \; dt\,dx^i$ (zbog izotropnosti)

Nadalje, Einsteinov izbor 3-D prostora,

- ZATVORENOG

vodi na odabir 3-D površine S_3 (površine 4-D hipersfere radijusa S) dane j-bom (u Kartezijevim koor.)

$$(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = S^2$$

[Narlikar §3.3 $k = +1$

ibid. §3.4 za negativnu $(k = -1)$ ili iščezavajuću zakrivlj.]

Einsteinova 3D hipersfera

& inzistiranje na statičkom rješenju ("Einsteinova najveća zabluda")

Prijelazom na koordinate unutar površine

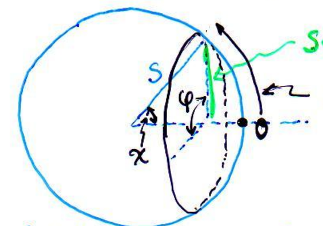
$x_4 = S\cos\chi \qquad 0 \le \chi \le \pi$

$x_1 = S\sin\chi\cos\vartheta \qquad 0 \le \vartheta \le \pi$

$x_2 = S\sin\chi\sin\vartheta\cos\varphi \qquad 0 \le \varphi \le 2\pi$

$x_3 = S\sin\chi\sin\vartheta\sin\varphi$

Zbog zora zamislimo 2-D plohu $\vartheta = \pi/2$
$\longrightarrow$ 3D prostor dobijemo kao volumen dobiven revolucijom te površine



$S\sin\chi \equiv Sr$

$\rho(r) = S\int_0^r \frac{dr}{(1-r^2)^{1/2}} = S\arcsin r = S\chi$

Opseg ekvatora $(\vartheta = \pi/2)$ $= S\int_0^{2\pi} r\,d\vartheta = 2\pi Sr$

$d\sigma^2 = S^2(d\chi^2 + \sin^2\chi\, d\varphi^2)$

- Interval duljine na plohi S_3

$$d\sigma^2 = S^2\left[d\chi^2 + \sin^2\chi\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]$$

uz $r = \sin\chi \in [0, 1]$

$$= S^2\left[\frac{dr^2}{1-r^2} + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]$$

površina $S_3 = 4\pi S^2\sin^2\chi$

volumen $S_3 = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \int_0^{\pi} d\chi\, S^3\sin^2\chi\sin\vartheta = 2\pi^2 S^3$

$$ds^2 = c^2dt^2 - d\sigma^2$$

$$= c^2dt^2 - \underbrace{S^2(t)}\left[\frac{dr^2}{1-kr^2} + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right]$$

poopće $S^2(t)$; $k = \pm 1, 0$

Svođenje gravitacije na zakrivljenost i Einsteinova gravitacijska jednadžba

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\varkappa T_{\mu\nu}$$

$$\varkappa = \frac{8\pi G_N}{c^4}$$

- **10 jedn. polja**
  - **- 3 pogleda na Einst. jedn.**
  - **- Newtonova granica**

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

- **4 jedn. gibanja (BeG App. A)**
  - **Usporedba geodetske jednadžbe i Lorentzove sile**

Tvar bez gibanja ...

- Za svemir ispunjen prašinom u mirovanju RHS Einst. j-be $T^0_0 = \rho_0 c^2$; $T^1_1 = T^2_2 = T^3_3 = 0$

$\Rightarrow$ dvije nezavisne j-be

$$\lambda - \frac{3}{S^2} = -\frac{8\pi G}{c^2}\rho_0 \quad ; \quad \lambda - \frac{1}{S^2} = 0$$

bez smislenog (statičkog) rješenja !

Einstein uvodi **KOZMOLOŠKI ČLAN**

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$$

$$\boxed{S = \left(\frac{1}{\Lambda}\right)^{1/2} = \frac{c}{2\sqrt{\pi G \rho_0}}} \quad \left\{ \begin{array}{l} S \doteq 10^{29}\ \mathrm{cm} \\ \Lambda \doteq 10^{-58}\ \mathrm{cm}^{-2} \end{array} \right.$$

(uz $\rho_0 \doteq 10^{-31}\ \mathrm{g/cm^3}$)

Gibanje bez tvari

de Sitter-ov svemir (1917)

Suprotno Einsteinovom očekivanju, jednadžbe OTR daju rješenje i za prostor-vrijeme bez materije ($T_{\mu\nu}=0$)

→ s intervalom

$$ds^2 = c^2\left(1-\frac{H^2R^2}{c^2}\right)dt^2 - \frac{dR^2}{1-\left(\frac{H^2R^2}{c^2}\right)} - R^2\left(d\vartheta^2+\sin^2\vartheta\, d\varphi^2\right)$$

veza:

$$\Lambda = \frac{3H^2}{c^2}$$

prividno statični svemir

→ u novim koordinatama $(t, r, \vartheta, \varphi)$

$$ds^2 = c^2dt^2 - e^{2Ht}\left[dr^2 + r^2\left(d\vartheta^2+\sin^2\vartheta\, d\varphi^2\right)\right]$$

otkriva prostor tipa Minkowskog, čiji se prostorni dio napuhuje: za razliku od "Einsteinove tvari bez gibanja", ovdje je riječ o "gibanju bez tvari" (Eddington)

Svođenje gravitacije na geometriju

General Relativity : the simplest relativistic theory of gravitaty consistent with data. Gravity described as a geometric property of spacetime.

Metric: allows to compute distances between two points

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

$g_{\mu\nu}$: metric tensor

ds^2 : line element, invariant

- in special relativity:

$$ds^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$= c^2 dt^2 - \left(dr^2 + r^2 \left(d\theta^2 + \sin\theta^2 d\varphi^2\right)\right)$$

$\eta_{\mu\nu}$: Minkowski metric

- in a particle's rest frame:

$$ds = c\ d\tau$$

$d\tau$: particle proper time

Metrika idealnog širećeg svemira

- Friedmann-Lemaître-Robertson-Walker (FLRW) metric : metric for a spatially homogeneous and isotropic expanding universe, with scale/expansion factor a(t) and curvature k

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right)$$

expansion factored out

$H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ expansion rate or Hubble parameter

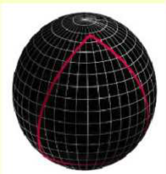
$H_0 \equiv H(t_0) \quad t_0 = today$

r,θ,φ spherical comoving coordinates:

r : dimensionless & stationary wrt expansion

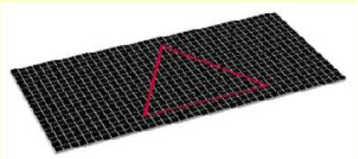
k, gaussian curvature:

closed universe



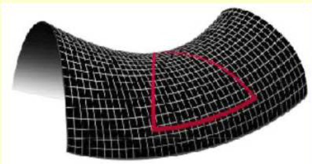
k=+1

flat universe



k=0

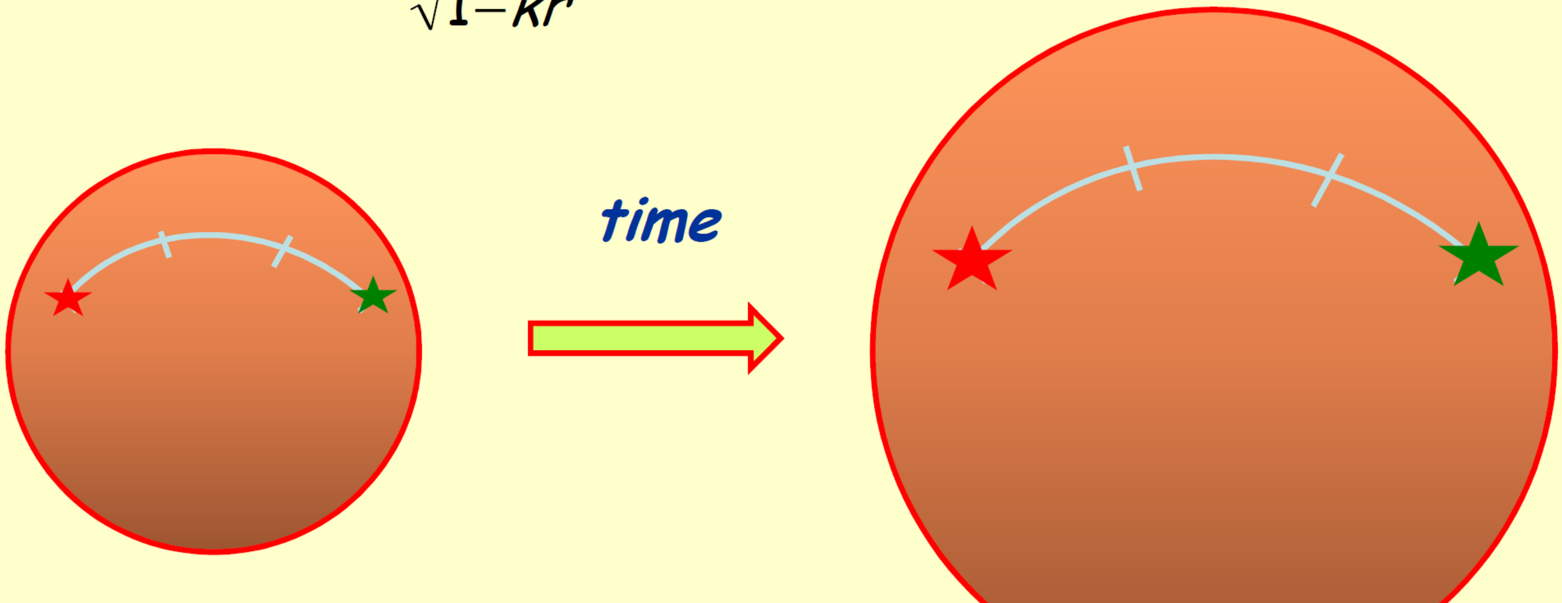
open universe



k=-1

Sugibajuća i fizikalna udaljenost

- Introducing $d\chi \equiv \frac{dr}{\sqrt{1-kr^2}}$ to account for curvature :



$\Delta\chi = 3 \quad D(t_1) = a(t_1)\Delta\chi$

$\Delta\chi = 3 \quad D(t_2) = a(t_2)\Delta\chi \geq D(t_1)$

a(t): expansion factor

χ: comoving coordinate, stationary wrt expansion

D(t) : proper/physical distance $\quad\Delta\chi$: comoving distance

Vježba 2.3: Hubbleov zakon u FLRW metrici

- FLRW metric:

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right)$$

$$ds^2 = c^2 dt^2 - a(t)^2 \left(d\chi^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right)$$

$$r = f(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ sh\chi & k=-1 \end{cases}$$

- Hubble's law:

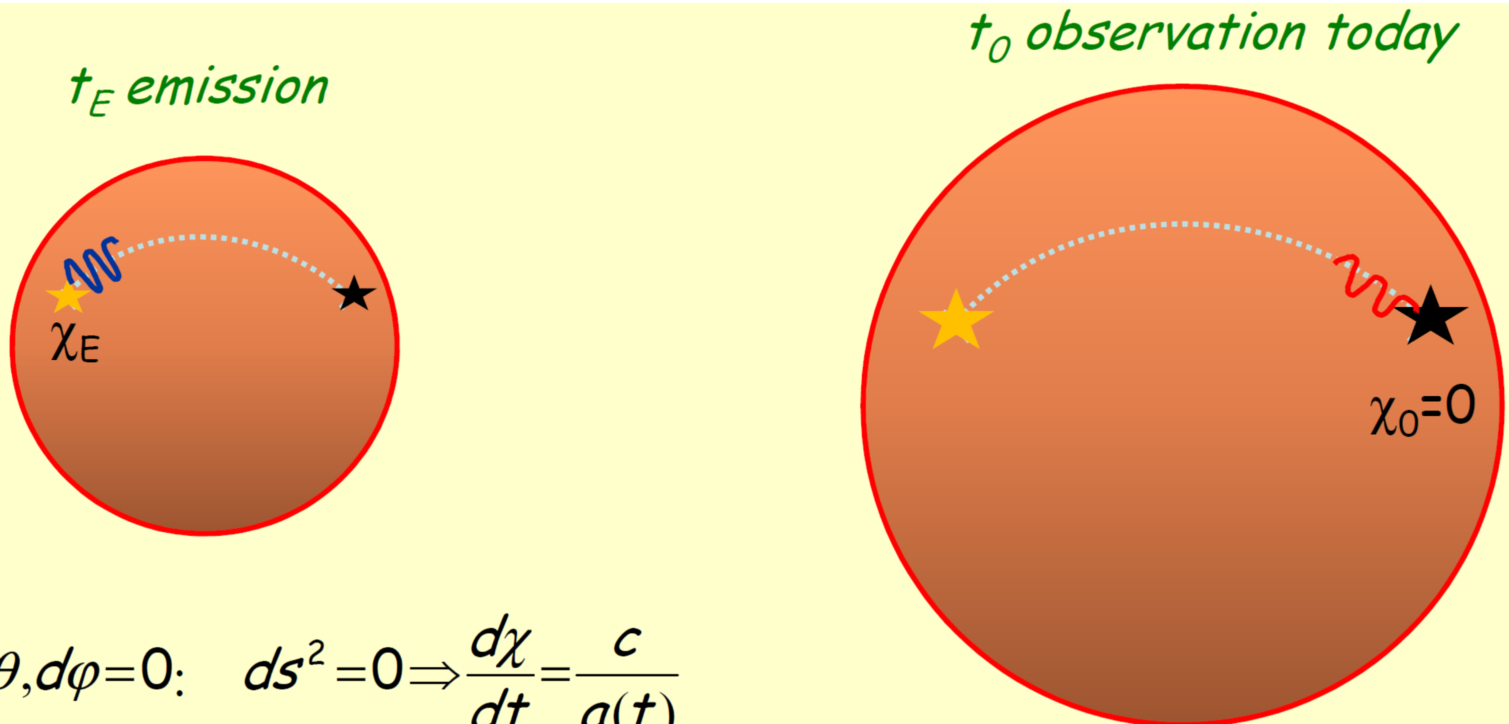
$$D(t) = a(t)\Delta\chi \Rightarrow \frac{dD}{dt} = \dot{a}(t)\Delta\chi = \frac{\dot{a}(t)}{a(t)} D(t) \Rightarrow \frac{dD}{dt} = H(t)D(t)$$

- Particle trajectories :

$ds^2 = 0$ geodesic equation (shortest path in three-space and maximum proper time)

Vježba 2.4:

Crveni pomak pri širenju fotona



$$d\theta, d\varphi = 0: \quad ds^2 = 0 \Rightarrow \frac{d\chi}{dt} = \frac{c}{a(t)}$$

$$\chi_E = \int_{t_E}^{t_0} \frac{cdt}{a(t)} = \int_{t_E+\lambda_E/c}^{t_0+\lambda_0/c} \frac{cdt}{a(t)} = \int_{t_E+\lambda_E/c}^{t_E} \frac{cdt}{a(t)} + \int_{t_E}^{t_0} \frac{cdt}{a(t)} + \int_{t_0}^{t_0+\lambda_0/c} \frac{cdt}{a(t)} = \int_{t_E}^{t_0} \frac{cdt}{a(t)} + \frac{\lambda_0}{a_0} - \frac{\lambda_E}{a_E}$$

$$\Rightarrow \quad \boxed{\frac{\lambda_0}{\lambda_{emitted}} \equiv 1 + z = \frac{a_0}{a(t_{emission})}}$$

light from distant sources is redshifted