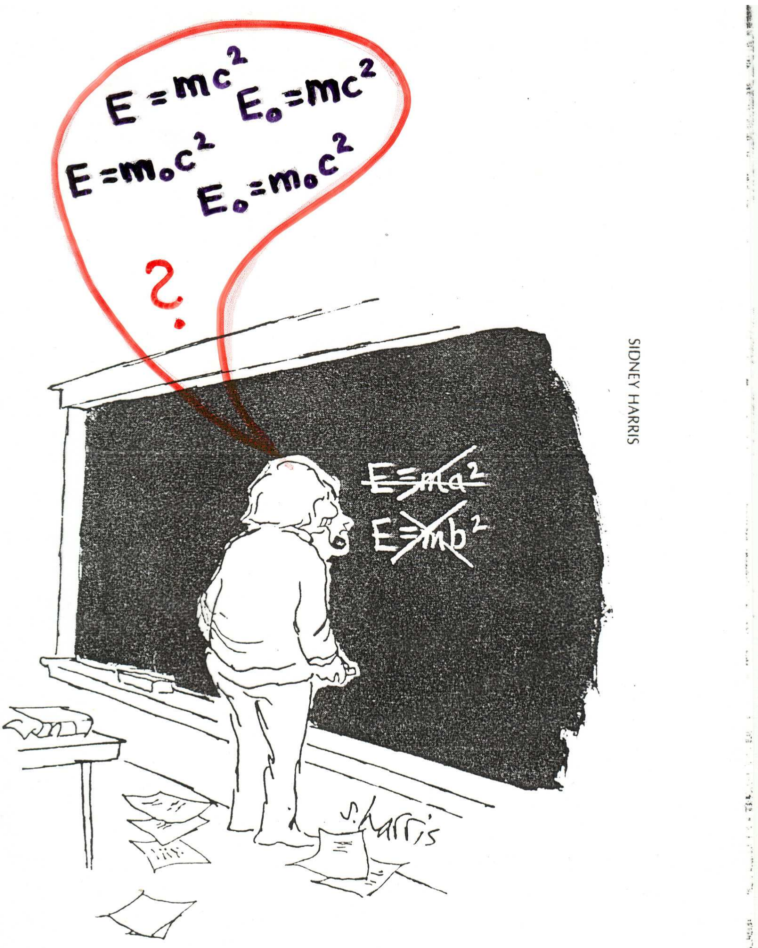
IV. RELATIVISTIČKA (LORENTZOVA) INVARIJANTNOST (FEČ § 2.2, str. 61)



- TENZORSKA POLJA &
- RELATIVISTIČKE JEDNADŽBE (KLEIN-GORDONOVA I DIRACOVA)

PRINCIP SIMETRIJE U RELATIVISTIČKOJ KVANTNOJ

RELATIVISTIČKA

KVANTNA MEHANIKA

- zadire dublje u strukturu prostora i vremena $x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{x})$

kakvo vide elementarne čestice

Hermann Minkowski

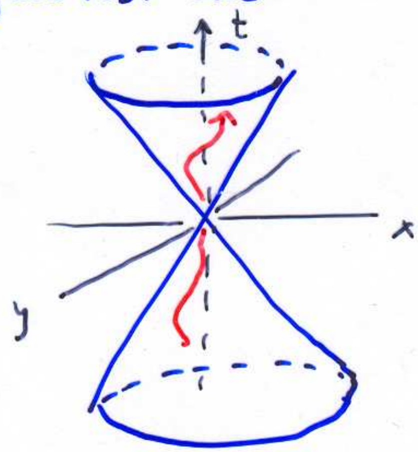
- geometrija s jedinstvenom udaljenosti

$s^2 = c^2t^2 - x^2 - y^2 \; (-z^2)$

> 0 vremenska

$= 0$ svjetlosna

< 0 prostorna



PODSJETNIK NA RELATIVISTIČKE INVARIJANTE

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^2 = x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu = (x^0)^2 - (\vec{x})^2$$

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \equiv \partial_\mu &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right), \\ \frac{\partial}{\partial x_\mu} \equiv \partial^\mu &= \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \end{aligned}$$

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \left(\frac{\partial}{\partial \vec{x}}\right)^2$$

$$p^\mu = i\hbar \frac{\partial}{\partial x_\mu} = \left(i\frac{\hbar}{c}\frac{\partial}{\partial t}, -i\hbar\nabla\right)$$

$$p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2 c^2$$

LORENTZOVE TRANSFORMACIJE KAO ROTACIJE U 4-PROSTORU

- **PASIVNA I AKTIVNA TRANSFORMACIJA**

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1) \\ x'^1 &= \gamma(x^1 - \beta x^0) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \,. \end{aligned}$$

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \equiv \Lambda^\mu_\nu x^\nu$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_1 = \begin{pmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

VLASTITE I ORTOKRONE TRANSFORMACIJE

$$x^2 = x'^2 = g_{\mu\nu} x'^{\mu} x'^{\nu} \qquad g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} = g_{\rho\sigma}$$

$$\det \Lambda = \pm 1 \qquad \mathbf{i} \qquad |\Lambda^0_0| \geq +1$$

- **ROTACIJE** $\quad \vec{r} \longrightarrow \vec{r}\,' = R_{(\hat{n},\varphi)} \vec{r}$

$$\psi(\vec{r}) \longrightarrow \psi'(\vec{r}) \quad = \quad U_{(\hat{n}\varphi)} \psi(\vec{r}) = \psi\left(R^{-1}\vec{r}\right)$$

$$U_{(\hat{n}\varphi)} = e^{-i\varphi \hat{n}\cdot\vec{L}}$$

POTISCI

$$\beta_j'' = \frac{\beta_j' + \beta_j}{1 + \beta_j' \beta_j}$$

UVOĐENJEM RAPIDITETA

$$\beta_j = \operatorname{th} \zeta_j \qquad \gamma = \operatorname{ch} \zeta_j$$

$$\zeta_j'' = \operatorname{Arth} \beta_j'' = \operatorname{Arth} \beta_j' + \operatorname{Arth} \beta_j = \zeta_j' + \zeta_j$$

$$\begin{aligned} x_j' &= x_j \operatorname{ch} \zeta_j + t \operatorname{sh} \zeta_j , \\ x_k' &= x_k & \text{za } k \neq j \\ t' &= x_j \operatorname{sh} \zeta_j + t \operatorname{ch} \zeta_j \ . \end{aligned}$$

$$\Lambda_3 = \begin{pmatrix} \operatorname{ch} \zeta & 0 & 0 & \operatorname{sh} \zeta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \operatorname{sh} \zeta & 0 & 0 & \operatorname{ch} \zeta \end{pmatrix}$$

GENERATORI POTISAKA I ROTACIJA

Potisci u koordinatnom prostoru

$$x^\mu \longrightarrow x'^\mu = \left[\Lambda_{(\zeta,\hat{n})}\right]^\mu_{\ \nu} x^\nu$$

praćeni su transformacijom u vektorskom prostoru stanja

$$\psi'(x^\mu) = U_{(\zeta,\hat{\eta})}\psi(x^\mu) = \psi\left\{\left[\Lambda^{-1}_{(\zeta,\hat{n})}\right]^\mu_{\ \nu} x^\nu\right\},$$

gdje je unitarna transformacija U izražena generatorom potiska $\vec{K}$,

$$U_{(\zeta,\hat{n})} = e^{-i\zeta\hat{n}\cdot\vec{K}} \ .$$

$$K_i = \frac{1}{i}\frac{\partial\Lambda_i}{\partial\zeta}\bigg|_{\zeta=0} \ , \qquad J_i = \frac{1}{i}\frac{\partial R_i}{\partial\varphi}\bigg|_{\varphi=0} \qquad K_3 \ = \ -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

REPREZENTACIJE LORENTZOVIH TRANSFORMACIJA

Lorentzove transf. (HLG) $\qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

rotacije i potisci $\in L_+^{\uparrow}$ $\quad$ ortokrone, $\Lambda^0{}_0 \geq 1$; $\quad$ vlastite, $\det\Lambda = +1$

reprezentirane s $\qquad U(\Lambda) = e^{-\frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}}$

$$M^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

$\psi' = U(\Lambda)\psi$

$\epsilon_{kij} M^{ij} = J_k$

$M^{0i} = K_i$

$[K_i, K_j] = -i\epsilon_{ijk} J_k$

$[J_i, J_j] = i\epsilon_{ijk} J_k$

$[J_i, K_j] = i\epsilon_{ijk} K_k$

The Lorentz group

Rotations J_i Boosts K_i

$$\left.\begin{aligned}[J_i, J_j] &= i\varepsilon_{ijk} J_k \\ [J_i, K_j] &= i\varepsilon_{ijk} K_k \\ [K_i, K_j] &= -i\varepsilon_{ijk} J_k\end{aligned}\right\}$$

Generate the group SO(3,1)

To construct representations a more convenient (non-Hermitian) basis is

$$N_i = \tfrac{1}{2}(J_i + iK_i)$$

$$[N_i, N_j] = i\varepsilon_{ijk} N_k$$

$$[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk} N_k^\dagger$$

$$[N_i, N_j^\dagger] = 0$$

Representations $J_i = N_i + N_i^\dagger$

(n, m) $J = n + m$

$(0,0)$ scalar $J=0$

$(\tfrac{1}{2},0), (0,\tfrac{1}{2})$ LH and RH spinors $J=\tfrac{1}{2}$

$(\tfrac{1}{2},\tfrac{1}{2})$ vector $J=1$, etc

& POINCAREOVIH TRANSF.

Poincaréove transf. (NLG)

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

infinitez.

$$U(\Lambda, a) = I - \frac{i}{2}\omega_{\rho\sigma} M^{\rho\sigma} + i a_\mu P^\mu$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\mu\rho} M^{\sigma\nu} - \quad \ldots \quad)$$

$$[M^{\mu\nu}, P^\rho] = -i g^{\mu\rho} P^\nu + i g^{\nu\rho} P^\mu$$

$$[P^\mu, P^\rho] = 0$$

Poincaré-ova algebra

1. Casimirova invarijanta : $P^2 = P^\mu P_\mu$

$$[P^2, M^{\mu\nu}] = 0$$

2. Casimirova invarijanta : $W^2 = W^\mu W_\mu$

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$$

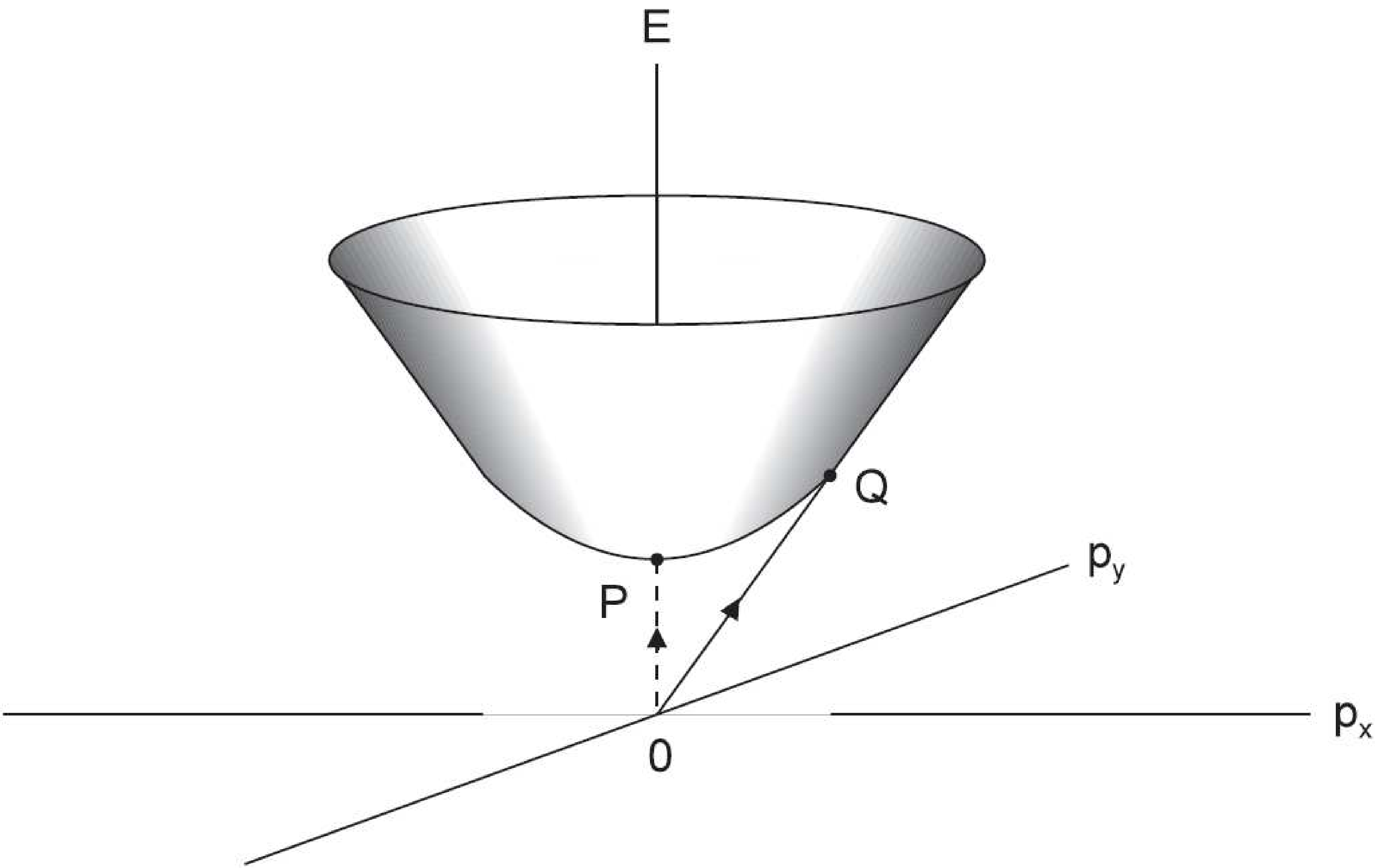
- za masivne č. $W^2 = -\vec{W}^2 = -m^2 s(s+1)$
- za bezmasene č. $W_\mu |p\rangle = \lambda P_\mu |p\rangle$; $\lambda = \dfrac{\vec{P}\cdot\vec{J}}{p_0}$ = helicitet

CASIMIROVE INVARIJANTE

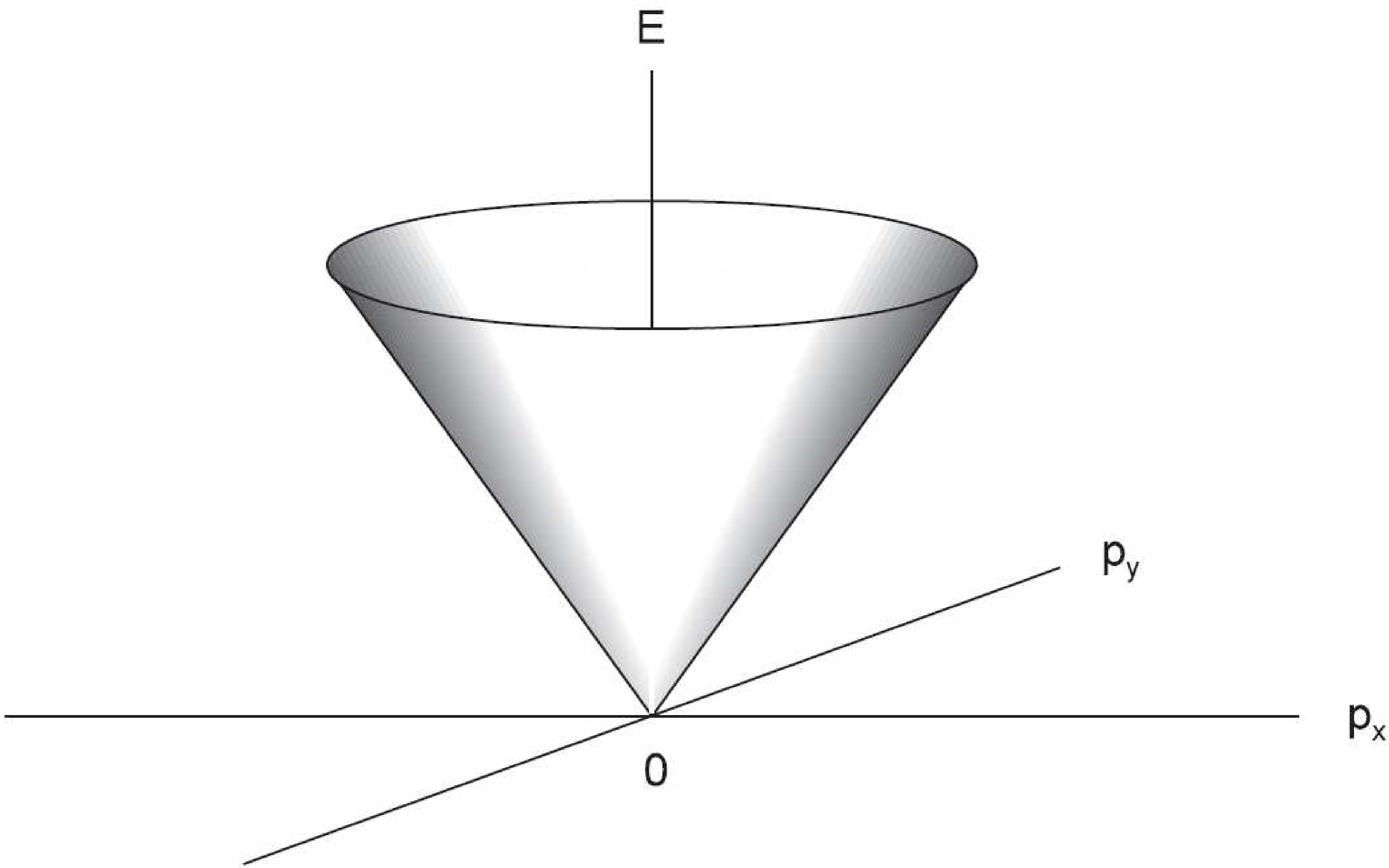
Katalogizacija stanja prema

$P^2 = m^2$	P^0 (syn. inv. na Lor. transf.)	
>0	>0	$W^2 = -m^2 s(s+1)$
>0	<0	
$=0$	>0	$W^2 = 0$ $W^2 = -\rho^2$ (kontin)
$=0$	<0	
<0	virtualne č. prostornog imp.	
$P^\mu \equiv 0$	vakuum	

PLOHA ENERGIJE MASIVNE ČESTICE



PLOHA ENERGIJE BEZMASENE ČESTICE



PRIKAZ ČESTICA LOKALNIM POLJIMA

ČESTICU OPISUJEMO FUNKCIJOM KOJA SE NE MIJENJA PRI TRANSLACIJAMA U PROSTORU I VREMENU (LOKALNIM POLJEM):

$$x^\mu \to x'^\mu = x^\mu + \underbrace{\delta x^\mu}_{=\varepsilon^\mu} \Rightarrow \quad \phi(x) \to \phi'(x) = \phi(x) + \delta_0 \phi(x)$$

za transl. $\delta_0 \phi = \phi'(x) - \phi(x) = -\varepsilon^\mu \partial_\mu \phi = -i \varepsilon^\mu P_\mu \phi$

$$\delta \phi = \phi'(x') - \phi(x)$$

$$= \underbrace{\phi'(x') - \phi(x')}_{\delta_0 \phi} + \phi(x') - \phi(x)$$

$$\phi(x') = \phi(x) + \delta x^\mu \partial_\mu \phi$$

$$= \delta_0 \phi + \delta x^\mu \partial_\mu \phi$$

ε^μ za translacije

$$\delta f = 0 = \delta_0 f + \varepsilon^\mu \partial_\mu f \Rightarrow \delta_0$$

TENZORSKA POLJA

Lokalna polja

invarijantna na translacije ($\in$ NLG)

tenzorska polja (HLG)

$(2S+1)$ - komponentna za spin "S"

SKALARNA $\phi(x) \rightarrow \phi'(x') = \phi(x)$ (tj. $\phi'(x) = \phi(\Lambda^{-1}x)$)

SPINORNA $\psi(x) \rightarrow \psi'(x') = S(\Lambda)\psi(x)$

treba naći !

VEKTORSKA $A^{\mu}(x) \rightarrow A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} A^{\nu}(x)$

rep. Lor. gr. od generatora grupe!

TENZORSKA $U_{\mu\nu}(x) \rightarrow U'_{\mu\nu}(x') = \Lambda_{\mu\rho}\Lambda_{\nu\sigma} U^{\rho\sigma}(x)$

$\equiv T_{\mu\nu,\rho\sigma}$

reducibilna

$U_{\mu\nu} = U^{S}_{\mu\nu} + U^{A}_{\mu\nu}$ — komp. S & A-tenzora transf. se na Λ neovisno; bez miješanja

[10-d] [6-d]

RELATIVISTIČKE INVARIJANTE

- KOMBINACIJE (PRODUKTI) KOJI SE NE MIJENJAJU PRI LORENTZOVIM TRANSFORMACIJAMA

$\phi(x)\,\phi(x)$ skalarnih polja

$A_\mu(x)\,A^\mu(x)$ vektorskih -"-

$\bar{\Psi}(x)\,\gamma_\mu\,\partial^\mu\,\psi(x)$ spinornih -"-

$\bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,A^\mu(x)$ miješanih

$\bar{\psi}(x)\,\psi(x)\,\phi(x)$

RELATIVISTIČKE JEDNADŽBE

$\hbar = c = 1$

Klein-Gordon

$$\boxed{E^2 = \vec{p}^{\,2} + m^2} \Rightarrow E_0 = m$$

$$-\frac{\partial^2}{\partial t^2}\phi = (-\nabla^2 + m^2)\phi \quad \Leftrightarrow \quad (\Box + m^2)\phi = 0$$

$$\frac{\partial}{\partial t}\left[i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\right] +$$

$$+ \nabla\cdot\left[-i\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right)\right] = 0$$

$$\frac{\partial\rho}{\partial t} + \nabla\vec{j} = 0 \qquad \partial_\mu j^\mu = 0$$

$$E < 0 \leftrightarrow \rho < 0$$

KAO LINEARIZIRANA KLEIN-GORDONOVA, DAJE POZITIVNU GUSTOĆU VJEROJATNOSTI

Dirac

$$i\frac{\partial}{\partial t}\Psi(\vec{x},t) = (-i\vec{\alpha}\cdot\nabla + \beta m)\Psi(\vec{x},t) \Longleftrightarrow (i\gamma^\mu\partial_\mu - m)\Psi = 0$$

$$\frac{\partial}{\partial t}\underbrace{[\Psi^\dagger\Psi]}_{\rho > 0} + \nabla[\Psi^\dagger\vec{\alpha}\Psi] = 0 \qquad \partial_\mu j^\mu = 0$$

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi$$