VJEŽBE 9
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FERMI-DIRAC DISTRIBUCIJA

$f(\mu, \varepsilon, T)$

Za sistem identičnih fermiona u termodinamičkoj ravnoteži, prosječan broj u jednočestičnom stanju i:

$$f_\mu(\varepsilon) = \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1}$$

$$\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1} = f(\varepsilon)$$

tako ćemo mi označavati

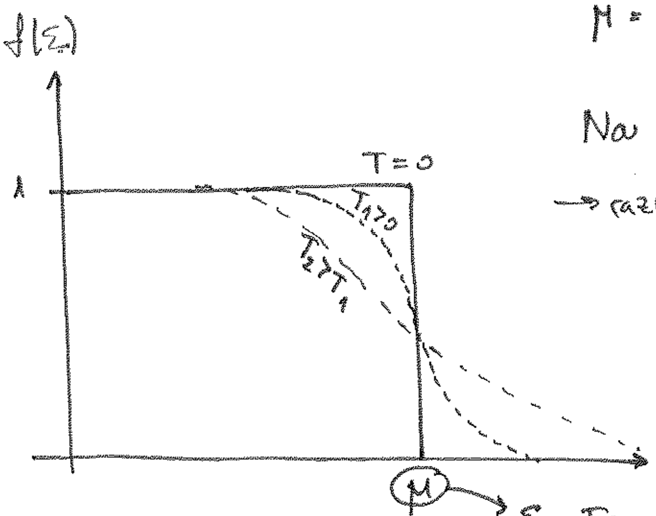
$\mu = \frac{\partial U}{\partial N}$ termodinamička definicija → kad gust

k_B = Boltzmann konst.

T = apsolutna temperatura

ε_i = energija jednočestičnog stanja i

μ = totalni kemijski potencijal



$\mu \rightarrow \varepsilon_F$ Fermijeva energija

Na $T=0$ je F-D raspodjela step funkcija

→ razmaže se s porastom temperature

→ moramo doći na energiju $\sim 1eV$ da bi vidjeli zadoljenje

$(\varepsilon = k_B T)$

$$\frac{1eV}{k_B} \approx 11605\,K$$

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1}$$ → koristit ćemo da je to step-funkcija

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon$$ broj čestica $$N = \sum_\lambda N_i = \sum_i g_i f_i = \int_0^\infty d\varepsilon\, g(\varepsilon) f(\varepsilon)$$

$$U = \int_0^\infty g(\varepsilon)\, \varepsilon\, f(\varepsilon)\, d\varepsilon$$ ukupna energija sustava (unutrašnja energ.)

$$\langle \varepsilon^2 \rangle = \int_0^\infty g(\varepsilon)\, \varepsilon^2 f(\varepsilon)\, d\varepsilon$$

PROSJEČNA VRIJEDNOST NEKE VARIJABLE $\bar{A}$

$$\bar{A} = \frac{\int d\varepsilon\, g(\varepsilon) A f(\varepsilon)}{\int d\varepsilon\, g(\varepsilon) f(\varepsilon)}$$

JAKO DEGENERIRANI FERMIONSKI PLIN

→ fermionski plin na jako niskim temperaturama

$\mu \gg k_B T$

funkcija raspodjele: $f(\varepsilon) = \lim_{T\to 0} \frac{1}{e^{\frac{\varepsilon-\mu}{k_B T}}+1} = \begin{cases} 0 & \varepsilon > \mu \\ 1 & \varepsilon < \mu \end{cases}$

→ Na $T=0$ F-D raspodjela je step funkcija

$$f(\varepsilon) = \underbrace{\Theta(\varepsilon_F - \varepsilon)}_{\text{step funkcija}}$$

$\mu(T=0) = \mu_0 = \varepsilon_F$

↳ kemijski potencijal na apsolutnoj nuli je Fermijeva energija

Na $T=0$ su sva kvantna stanja energije manje od ε_F popunjena, a kvantna stanja energije veće od ε_F su prazna.

Na $T=0$ prosječna energija je različita od nule jer čestice popunjavaju kvantna stanja od stanja minimalne energije pa do Fermijeve en.

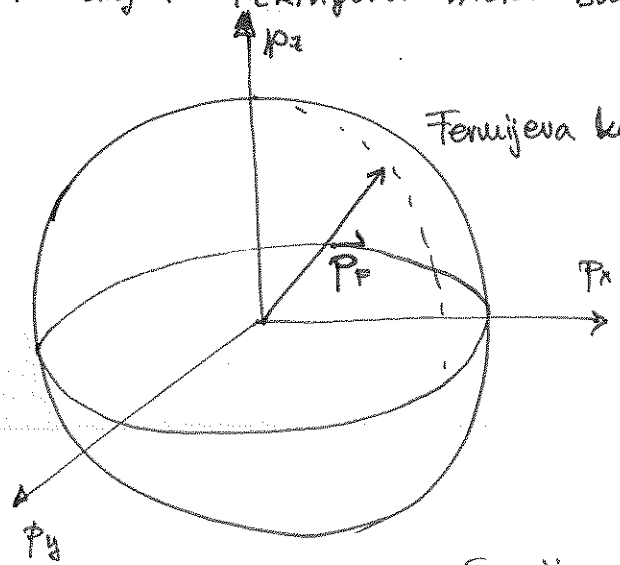
$$\frac{p^2}{2m} \mp \frac{3}{2}k_B T = 0$$

$p_F \equiv$ impuls koji odgovara fermi energiji

$$\varepsilon_F = \frac{p_F^2}{2m}$$

možemo definirati FERMIJEV VALNI BROJ I FERMIJEVU VALNU DULJINU.

$$p_F = \hbar k_F = \hbar \frac{2\pi}{\lambda_f}$$



Sva stanja za koja je impuls:

$$\frac{\vec{p}^2}{2m} = \frac{p_x^2+p_y^2+p_z^2}{2m} < \frac{p_F^2}{2m}$$

su popunjena.

→ površina Fermi kugle razdvaja puna i prazna kvantna stanja

$k_B T_F = \varepsilon_F$ Fermijeva temperatura

$T < T_F$ → kvantna statistika (degenerirani fermionski plin)

$T > T_F$ → klasična statistika

Fermijeva brzina:

$$v_F = \frac{p_F}{m} = \frac{\hbar k_F}{m}$$

Izračunajmo broj čestica ako je poznat Fermijev nivo: (Slobodni elektronski plin, 3D)

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$= \int_0^\infty g(\varepsilon)\, \Theta(\varepsilon_F - \varepsilon)\, d\varepsilon$$

Step funkcija modificira granice integracije (nakon ε_F će integral uvijek biti 0)

$$= \int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon = \frac{V}{\pi \hbar^3} 2^{1/2} m^{3/2} \int_0^{\varepsilon_F} \sqrt{\varepsilon}\, d\varepsilon$$

$$\boxed{N = \frac{V}{\pi^2 \hbar^3} 2^{1/2} m^{3/2} \frac{2}{3} \varepsilon_F^{3/2}}$$

Izrazimo preko Fermijevog valnog vektora k_F

$$\varepsilon_F = \frac{\hbar^2}{2m} k_F^2 \longrightarrow \boxed{N = \frac{V}{\pi^2} \frac{k_F^3}{3}}$$

Izračunajmo sad broj čestica u sustavu, ali koristeći k

$$g(\varepsilon) = \frac{V}{(2\pi)^3} \cdot 2 \int 4\pi k^2 dk$$

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon = \int_0^\infty g(\varepsilon)\, \Theta(k_F - k)\, dk$$

$$= \frac{V}{(2\pi)^3} \cdot 2 \int_0^{k_F} 4\pi k^2 dk = \frac{V}{\pi^2} \int_0^{k_F} k^2 dk$$

3D

$$\rightarrow \boxed{N = \frac{V}{\pi^2} \frac{k_F^3}{3}} \quad \checkmark$$

Sada izračunajmo ukupnu energiju

$$U = \int_0^{\infty} f(\varepsilon)\, g(\varepsilon)\, \varepsilon \, d\varepsilon = \int_0^{\varepsilon_F} g(\varepsilon)\, \varepsilon \, d\varepsilon$$

$$= \frac{V}{\pi^2 \hbar^3} \sqrt{2}\, m^{3/2} \underbrace{\int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon}_{\frac{2}{5}\varepsilon_F^{5/2}}$$

$$= \frac{V}{\pi^2 \hbar^3} \sqrt{2}\, m^{3/2}\, \frac{2}{5} \left(\frac{\hbar^2}{2m} k_F^2 \right)^{5/2}$$

ε_F $\quad N = \frac{1}{3} \frac{V}{\pi^2} k_F^3$

$$= \frac{V}{\pi^2 \hbar^3} \sqrt{2}\, m^{3/2}\, \frac{2}{5}\, \frac{\hbar^5}{2^{5/2}}\, \frac{k_F^5}{m^{5/2}} = \frac{\hbar^2}{2m} \frac{1}{5} \frac{V}{\pi^2} k_F^3 \cdot k_F^2$$

$$\underline{U = \frac{3}{5} N \cdot \varepsilon_F}$$

$$\frac{U}{N} = \frac{3}{5} \varepsilon_F$$

Istu stvar dobivamo računamo li preko valnog vektora:

$$U = \frac{V}{(2\pi)^3} \cdot 2 \int_0^{k_F} 4\pi k^2 \frac{\hbar^2}{2m} k^2 \, dk = \dots = \frac{3}{5} N \varepsilon_F$$

→ Cijeli ovaj postupak treba znati provesti i za 2D i 3D (koristiti pripadne gustoće stanja)

→ pismeni!

(2) Odredite potenciju n s kojom fermionska gustoća stanja ovisi o energiji ako se na $T=0$ u intervalu $[0, \varepsilon_F/4]$ nalazi 50% čestica.

$$g(\varepsilon) = \text{const} \cdot \varepsilon^n$$

$$x = 0{,}5$$

$$x = \frac{N(0, \varepsilon_F/4)}{N(0, \varepsilon_F)} = \frac{\int_0^{\varepsilon_F/4} g(\varepsilon)\, d\varepsilon}{\int_0^{\varepsilon_F} g(\varepsilon)\, d\varepsilon} = \frac{c\int_0^{\varepsilon_F/4} \varepsilon^n\, d\varepsilon}{c\int_0^{\varepsilon_F} \varepsilon^n\, d\varepsilon}$$

$$x = \frac{\cancel{\frac{1}{n+1}}\, \varepsilon^{n+1}\Big|_0^{\varepsilon_F/4}}{\cancel{\frac{1}{n+1}}\, \varepsilon^{n+1}\Big|_0^{\varepsilon_F}} = \frac{\cancel{\varepsilon_F^{n+1}}}{4^{n+1}} \cdot \frac{1}{\cancel{\varepsilon_F^{n+1}}} = \frac{1}{4^{n+1}}$$

$$0.5 = \frac{1}{4^{n+1}}$$

$$\frac{1}{2} 4^{n+1} = 1 \longrightarrow 4^{n+1} = 2$$

$$2^{2(n+1)} = 2^1$$

$$2(n+1) = 1$$

$$n+1 = 1/2 \longrightarrow \boxed{n = -1/2}$$

(2) Zadan je plin slobodnih fermiona na apsolutnoj nuli s Fermijevom energijom μ_0.
Odredite udio fermiona s energijom manjom od $\mu_0/3$ ako je gustoća stanja proporcionalna kvadratu energije.

$$g(\varepsilon) = \text{const} \cdot \varepsilon^2$$

$$x = \frac{N(0, \mu_0/3)}{N(0, \mu_0)} = \frac{\cancel{\text{const}} \int_0^{\mu_0/3} \varepsilon^2 d\varepsilon}{\cancel{\text{const}} \int_0^{\mu_0} \varepsilon^2 d\varepsilon} = \frac{\frac{1}{3} \varepsilon^2 \Big|_0^{\mu_0/3}}{\frac{1}{3} \varepsilon^2 \Big|_0^{\mu_0}}$$

$$= \frac{\cancel{\frac{1}{3}} \left(\frac{\mu_0}{3}\right)^2}{\cancel{\frac{1}{3}} \mu_0^2} = \frac{\frac{\cancel{\mu_0^2}}{9}}{\cancel{\mu_0^2}} = \frac{1}{9} \qquad \boxed{x = 1/9}$$

(2) Odredite gustoću stanja dvodimenzionalnog fermiona energetske disperzije $\varepsilon = ck^6$ na $T=0$.

$$g(\varepsilon) = \frac{2V}{(2\pi)^2} \int_0^\infty 2\pi k \, \delta(\varepsilon - \varepsilon(k)) dk = \frac{4\pi V}{4\pi^2} \int_0^\infty k \, \delta(\underbrace{\varepsilon - ck^6}_{z(k)}) dk$$

$z(k) = 0$

$\varepsilon = ck^6$

$$\boxed{k_0 = \oplus \sqrt[6]{\frac{\varepsilon}{c}}} = \left(\frac{\varepsilon}{c}\right)^{1/6}$$

$$\frac{\partial z(k)}{\partial k} = -6ck^5$$

$$\left|\frac{\partial z}{\partial k}\right| = 6ck^5$$

$$g(\varepsilon) = \frac{V}{\pi} \frac{k_0}{\left|\frac{\partial z}{\partial k}\right|} = \frac{V}{\pi} \frac{k_0}{6ck_0^5} = \frac{V}{6c\pi} \frac{1}{k_0^4} = \frac{V}{6c\pi} \left(\frac{c^{1/6}}{\varepsilon^{1/6}}\right)^4$$

$$= \frac{V}{6c\pi} \frac{c^{2/3}}{\varepsilon^{2/3}} = \frac{V}{6c^{1/3}} \varepsilon^{-2/3}$$

$$\boxed{g(\varepsilon) = \frac{V}{6\pi c^{1/3}} \varepsilon^{-2/3}}$$