Ukupan broj fotona u crnom tijelu volumena V i temperature T?

$$N = \int_0^\infty g(\varepsilon)\, n(\varepsilon)\, d\varepsilon$$

fotoni = bozoni

$$g(\varepsilon) = \frac{\lambda V}{(2\pi)^3} \int_0^\infty \delta(\varepsilon - \varepsilon_k)\, 4\pi k^2\, dk$$

$\varepsilon = \hbar\omega = \hbar c k = h\nu$

$\lambda = 2$ polarizacija za fotone

$$= \frac{2V}{(2\pi)^3} 4\pi \int_0^\infty \delta(\varepsilon - \varepsilon_k)\, k^2\, dk$$

$$= \frac{\cancel{2} V \cancel{4\pi}}{\cancel{8}\pi^{\cancel{3}\,2}} \int_0^\infty \delta(\varepsilon - \hbar c k)\, k^2\, dk$$

$z = \varepsilon - \hbar c k$

$\rightarrow k_0 = \frac{\varepsilon}{\hbar c}$

$\frac{\partial z}{\partial k} = -\hbar c$

$$= \frac{V}{\pi^2} \frac{k_0^2}{|-\hbar c|} = \frac{V}{\pi^2} \frac{k_0^2}{\hbar c}$$

$$= \frac{V}{\pi^2} \frac{(\varepsilon/\hbar c)^2}{\hbar c} \rightarrow \boxed{g(\varepsilon) = \frac{V}{\pi^2} \frac{\varepsilon^2}{(\hbar c)^3}}$$

$$N = \int_0^\infty \frac{V \varepsilon^2}{\pi^2 (\hbar c)^3} \frac{1}{e^{\beta\varepsilon} - 1} d\varepsilon = \frac{V}{\pi^2 (\hbar c)^3} \int_0^\infty \frac{\varepsilon^2}{e^{\beta\varepsilon} - 1}$$

$n(\varepsilon)$ BOZONI

Supstitucija:

$$\left[\begin{array}{ll} x = \beta\varepsilon & \varepsilon^2 = \frac{x^2}{\beta^2} \\ dx = \beta\, d\varepsilon & \end{array}\right]$$

$$= \frac{V}{\pi^2 (\hbar c)^3} \frac{1}{\beta^3} \underbrace{\int_0^\infty \frac{x^2}{e^x - 1} dx}_{\text{ovo ne ovisi o temperaturi (tj. } \beta)\ \equiv K}$$

$$N = \underbrace{\frac{V}{\pi^2 (\hbar c)^3} K k_B^3}_{\equiv b} T^3$$

$$\underline{N = bVT^3}$$

Pogledajmo još jednom gustoću stanja koju smo izveli

$$g(\varepsilon)d\varepsilon = \frac{V}{\pi^2}\frac{\varepsilon^2}{(\hbar c)^3}d\varepsilon \quad , \quad \text{a za fotone imamo } \varepsilon = h\nu$$

$= \hbar\omega$

Možemo pisati:

$$g(\varepsilon)d\varepsilon = \frac{V}{\pi^2}\frac{h^2\nu^2}{\left(\frac{h}{2\pi}\right)^3 c^3}h\,d\nu \qquad d\varepsilon = h\,d\nu$$

$$= \frac{V}{\cancel{\pi^2}}\frac{\cancel{h^3}\nu^2}{\frac{\cancel{h^3}}{8\pi^{\cancel{3}}}c^3}d\nu$$

$$G(\nu)d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu$$

$$= \frac{8\pi V}{c^3}\nu^2 d\nu$$

→ Broj fotona u rasponu frekvencija $d\nu$ kod frekvencije ν:

$$n(\nu)G(\nu)\,d\nu = \frac{8\pi V}{c^3}\frac{\nu^2}{e^{h\nu/kT}-1}d\nu$$

Svaki foton nosi energiju $h\nu$

→ gustoća energije na frekvenciji ν:

$/: V$

$/\cdot h\nu$

$$u(\nu,T) = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1}$$

PLANCKOVA RASPODJELA

→ Ukupna izračena energija po jedinici volumena:

$$\frac{U}{V} = \int_0^\infty u(\nu,T)\,d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}d\nu =$$

Supstitucija

$$\frac{h\nu}{kT} = s$$

$$\nu^3 = s^3\frac{k^3T^3}{h^3}$$

$$= \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \underbrace{\int_0^\infty \frac{s^3\,ds}{e^s-1}}_{=\frac{\pi^4}{15}}$$

$$d\nu = \frac{kT}{h}ds$$

Stefan-Boltzmannov zakon

$$\rightarrow \boxed{\frac{U}{V} = \sigma T^4} \qquad \sigma = \frac{8\pi^5k^4}{15c^3h^3}$$

DIGRESIJA

Kako raspisati sumu po k?

$$\sum_k \rightarrow \frac{V}{(2\pi)^D} \int d^D k$$

$$\frac{V}{(2\pi)^D} \int k^{D-1}\, dk \cdot (\text{kutni dio})$$

njega samo zapišete kao konstantu, inače:

$$S_n(R) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} R^n$$

Površina sfere u n-dimenzija

$1D \rightarrow 1$

$2D \rightarrow 2\pi k$

$3D \rightarrow 4\pi k^2$

U nekom 2D sustavu bozonska disperzija dana je s

$\omega_k = \omega_0 (e^{vk} - 1)$. Odredite:

a) gustoću stanja $g(\omega)$

b) pronađite limes gustoće stanja $g(\omega)$ za $k \to 0$ i $k \to \infty$

a) $$g(\omega) = \frac{(2s+1)V}{(2\pi)^2} \int_0^\infty 2\pi k \,\delta(\omega - \omega_k)\, dk$$

$$= \frac{3V\, \cancel{2\pi}}{(2\pi)^{\cancel{2}}} \int_0^\infty k\, \delta \underbrace{\left(\omega - \omega_0 (e^{vk} - 1)\right)}_{z} dk$$

$$0 = \omega - \omega_0 (e^{vk} - 1)$$

$$0 = \omega - \omega_0 e^{vk} + \omega_0$$

$$\omega_0 e^{vk} = \omega + \omega_0 \quad | : \omega_0$$

$$e^{vk} = \left(\frac{\omega}{\omega_0} + 1\right) \quad | \ln$$

$$vk = \ln\left(\frac{\omega}{\omega_0} + 1\right)$$

$$k_0 = \frac{1}{v} \ln\left(\frac{\omega}{\omega_0} + 1\right)$$

$$\frac{\partial z}{\partial k} = -\omega_0 v e^{vk}$$

$$\left|\frac{\partial z}{\partial k}\right| = \omega_0 v e^{vk}$$

$$e^{vk_0} = e^{v \frac{1}{v} \ln\left(\frac{\omega}{\omega_0} + 1\right)}$$

$$e^{vk_0} = \left(\frac{\omega}{\omega_0} + 1\right)$$

$$g(\omega) = \frac{3V}{2\pi} \frac{k_0}{\omega_0 v e^{vk_0}} = \frac{3V}{2\pi} \frac{\frac{1}{v} \ln\left(\frac{\omega}{\omega_0} + 1\right)}{\omega_0 v \left(\frac{\omega}{\omega_0} + 1\right)}$$

$$g(\omega) = \frac{3V}{2\pi} \frac{\ln\left(\frac{\omega}{\omega_0} + 1\right)}{v^2 (\omega + \omega_0)}$$

b) $k \to 0$

$$\omega_k = \omega_0 (e^{\nu k} - 1)$$

↑ razvijamo ovo

$$\omega_k = \omega_0 \left(\cancel{1} + \nu k + \frac{(\nu k)^2}{2} - \cancel{1} \right)$$

↑ njega ćemo isto zanemariti jer je mali

$$\omega_k \simeq \omega_0 \nu k$$

$$g(\omega) = \frac{(2s+1)V}{(2\pi)^2} \int_0^\infty 2\pi k \, \delta(\omega - \omega_0 \nu k) \, dk$$

$$= \frac{3V}{2\pi} \int_0^\infty k \, (\underbrace{\omega - \omega_0 \nu k}_{z}) \, dk$$

$$\omega - \omega_0 \nu k = 0 \qquad \frac{\partial z}{\partial k} = -\omega_0 \nu$$

$$\omega = \omega_0 \nu k \quad |: \omega_0 \nu \qquad \left| \frac{\partial z}{\partial k} \right| = \omega_0 \nu$$

$$\boxed{k_0 = \frac{\omega}{\omega_0 \nu}}$$

$$= \frac{3V}{2\pi} \frac{\frac{\omega}{\omega_0 \nu}}{\omega_0 \nu} \quad \to \quad \boxed{g(\omega) = \frac{3V}{2\pi} \frac{\omega}{(\omega_0 \nu)^2}}$$

$k \to \infty$

$$\omega_k = \omega_0 (e^{\nu k} - \cancel{1})$$

ovo je zanemarivo

$$g(\omega) = \frac{(2s+1)V}{(2\pi)^{\cancel{2}}} \int_0^\infty \cancel{2\pi} k \, \delta(\omega - \omega_0 e^{\nu k}) = \frac{3V}{2\pi} \int_0^\infty k \, \delta(\underbrace{\omega - \omega_0 e^{\nu k}}_{z}) \, dk$$

$$0 = \omega - \omega_0 e^{\nu k}$$

$$\omega = \omega_0 e^{\nu k} \quad |: \omega_0 \qquad \frac{\partial z}{\partial k} = -\omega_0 \nu e^{\nu k} \qquad \left| \frac{\partial z}{\partial k} \right| = \omega_0 \nu e^{\nu k}$$

$$e^{\nu k} = \frac{\omega}{\omega_0} \quad | \ln \quad | \frac{1}{\nu}$$

$$\boxed{k_0 = \frac{1}{\nu} \ln \frac{\omega}{\omega_0}} \qquad g(\omega) = \frac{3V}{2\pi} \frac{k_0}{\omega_0 \nu e^{\nu k_0}} = \frac{3V}{2\pi} \frac{k_0}{\cancel{\omega_0} \nu \frac{\omega}{\cancel{\omega_0}}} = \frac{3V}{2\pi} \frac{\frac{1}{\nu} \ln \frac{\omega}{\omega_0}}{\omega \nu}$$

$$e^{\nu k_0} = e^{\cancel{\nu} \frac{1}{\cancel{\nu}} \ln \frac{\omega}{\omega_0}} = \frac{\omega}{\omega_0} \qquad \boxed{g(\omega) = \frac{3V}{2\pi} \frac{\ln \frac{\omega}{\omega_0}}{\omega \nu^2}}$$

Energija elektrona u nekom D-dim kristalu dana je s:

$E = (a + bk^n)^p$

Izračunajte gustoću stanja $g(E)$ u ovisnosti o D, n, p

$$g(\varepsilon) = \frac{V(2s+1)}{(2\pi)^D} \int_0^\infty \delta \underbrace{(\varepsilon - \varepsilon_k)}_{z(k)} \underbrace{k^{D-1}}_{f(k)} dk \cdot A$$

konstanta A je došla od integracije po kutu i nije $f(D,n,p)$

$$g(\varepsilon) = \frac{2VA}{(2\pi)^D} \int_0^\infty \delta(\varepsilon - (a+bk^n)^p) k^{D-1} \, dk$$

1) Nultočke: $z(k_0) = 0$

$$\varepsilon - (a + bk_0^n)^p = 0 \qquad \varepsilon^{1/p} = a + bk_0^n$$

$$\varepsilon^{1/p} - a = bk_0^n$$

$$k_0^n = \frac{1}{b}(\varepsilon^{1/p} - a) \,\Big|\, {}^{1/n}$$

$$k_0 = \left[\frac{1}{b}(\varepsilon^{1/p} - a)\right]^{1/n}$$

2) $$\frac{\partial z}{\partial k} = -p(a + bk^n)^{p-1} bnk^{n-1}$$

$$\frac{\partial z}{\partial k}\Big|_{k_0} = -p\left(a + b\frac{1}{b}(\varepsilon^{1/p} - a)\right)^{p-1} \cdot bn\left(\frac{1}{b}(\varepsilon^{1/p} - a)\right)^{\frac{n-1}{n}}$$

$$g(\varepsilon) = 2\frac{VA}{(2\pi)^D} \frac{\left[\frac{1}{b}(\varepsilon^{1/p} - a)\right]^{\frac{D-1}{n}}}{\left| -\frac{pn}{b^{1/n}} \varepsilon^{\frac{p-1}{p}} (\varepsilon^{1/p} - a)^{\frac{n-1}{n}} \right|} \qquad p, n > 0$$

$$g(\varepsilon) = \frac{2VA}{(2\pi)^D} \frac{\left[\frac{1}{b}(\varepsilon^{1/p} - a)\right]^{\frac{D-1}{n}}}{\frac{pn}{b^{1/n}} \varepsilon^{\frac{p-1}{p}} (\varepsilon^{1/p} - a)^{\frac{n-1}{n}}}$$