

Force and torque of a string on a pulley

Thiago R. de Oliveira and Nivaldo A. Lemos

Instituto de Física - Universidade Federal Fluminense

Av. Litorânea, S/N, Boa Viagem, Niterói, CEP.24210-340, Rio de Janeiro - Brazil

tro@if.uff.br, nivaldo@if.uff.br

July 3, 2017

Abstract

Every university introductory physics course considers the problem of Atwood's machine taking into account the mass of the pulley. In the usual treatment the tensions at the two ends of the string are offhandedly taken to act on the pulley and be responsible for its rotation. However such a free-body diagram of the forces on the pulley is not *a priori* justified, inducing students to construct wrong hypotheses such as that the string transfers its tension to the pulley or that some symmetry is in operation. We reexamine this problem by integrating the contact forces between each element of the string and the pulley and show that although the pulley does behave as if the tensions were acting on it, this comes only as the end result of a detailed analysis. We also address the question of how much friction is needed to prevent the string from slipping over the pulley. Finally, we deal with the case in which the string is on the verge of sliding and show that this will never happen unless certain conditions are met by the coefficient of friction and the masses involved.

1 Introduction

A crucial step in solving a problem in mechanics by applying Newton's laws to interacting bodies is to identify the individual forces that act on each object. We dare say that the physics ends there and the rest is only manipulation of equations. Although this may be too strong a statement, it has been recognized in recent years that many students have difficulties with this step and that textbooks and instructors should give more attention to the correct identification of the forces on each body [1]. One type of problem in which most, if not all, textbooks fail to correctly identify the forces on each object are the ones containing strings and massive pulleys. These problems are treated in any university elementary physics course that addresses the rotational dynamics of rigid bodies about a fixed axis. A staple problem is Atwood's machine [2] with a pulley whose mass M is not negligible in comparison with m_1 and m_2 , depicted in Fig. 1. As usual, we assume that the string is inextensible, its mass is negligible and it does not slide on the pulley, which requires enough static friction between the string and the pulley. On the other hand, we assume that the pulley is mounted on a frictionless axle.

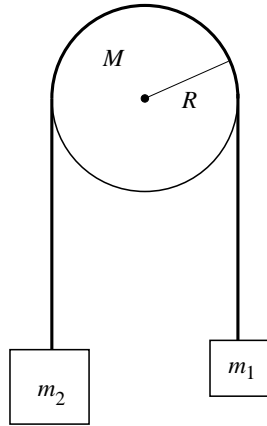


Figure 1: Two masses attached to a massless string that goes around a massive pulley.

For the sake of definiteness let us assume that $m_2 > m_1$. The standard free-body diagram of forces on the hanging masses and the pulley is shown in Fig. 2. Since $W_1 = m_1g$ and $W_2 = m_2g$, Newton's second law applied to the masses gives

$$T_1 - m_1g = m_1a, \tag{1.1a}$$

$$m_2g - T_2 = m_2a. \tag{1.1b}$$

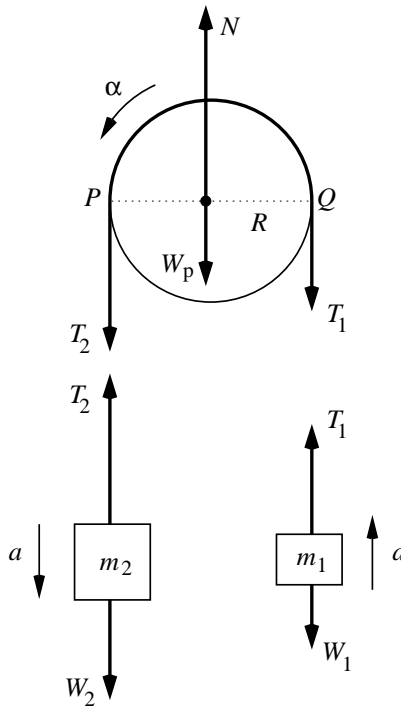


Figure 2: Standard free-body diagram of forces on the masses and the pulley.

As to the pulley, the standard claim [3] is that T_2 and T_1 are forces exerted by the string *on the pulley* at the points P and Q , respectively. The weight of the pulley and the sustaining

force exerted by the axle produce no torque about the rotation axis. Therefore, the net torque on the pulley is $\tau = T_2R - T_1R$. Applying the so-called Newton's second law for rotational motion, $\tau = I\alpha$, to the pulley one has

$$(T_2 - T_1)R = I\alpha. \quad (1.2)$$

Since the string does not slide on the pulley, there holds the constraint $a = \alpha R$. With the use of this constraint and the assumption that the pulley is a homogeneous disk, whose pertinent moment of inertia is $I = MR^2/2$, equation (1.2) becomes

$$T_2 - T_1 = \frac{M}{2}a. \quad (1.3)$$

By simply summing equations (1.1a), (1.1b) and (1.3) one gets the acceleration, and with a little more elementary algebra one finds the tensions. The result is

$$a = \frac{m_2 - m_1}{m_1 + m_2 + M/2}g, \quad T_1 = \frac{2m_2 + M/2}{m_1 + m_2 + M/2}m_1g, \quad T_2 = \frac{2m_1 + M/2}{m_1 + m_2 + M/2}m_2g. \quad (1.4)$$

This standard solution to the problem of the motion of Atwood's machine with a massive pulley, as well as the solutions to similar problems that can be found in so many textbooks, is open to a serious physical objection. The forces T_2 and T_1 are *not* forces on the pulley, but forces on the points P and Q of the string exerted by the hanging parts of the string at each side of the pulley. This gives rise to the problem of justifying the above results obtained on the basis of a physically unwarranted identification of the forces on the pulley. Note that putting directly the forces T_2 and T_1 on the pulley might reinforce a common misconception among students that all strings do is convey forces from one object to another. Here the problem is even more subtle since we do consider the string as massless, which usually entails that the tension is constant all along the string. There is also an interesting question that, as far as we can tell, is never asked in the textbooks: What is the magnitude of the friction force that prevents the string from slipping on the pulley?

The proper physical analysis consists in taking into account that each element of the string exerts a force on the part of the pulley with which it is in contact [4]. The determination of the net force and the net torque on the pulley requires an integration of the infinitesimal forces and torques exerted on the pulley by each element of the string that touches the pulley. Here one might be tempted to justify the usual treatment by the seemingly reasonable conjecture that, except for the forces at points P and Q of the pulley, the vector sum of all forces exerted by the the string on the other points of the pulley cancel each other owing to an apparent symmetry. This argument turns out to be wrong, and no such symmetry exists.

2 Forces on an element of the string

We follow the approach used in the analysis of the related problem of determining the effect of the friction force on a rope wrapped around a capstan [5, 6].

For the sake of definiteness we insist on the assumption that $m_2 > m_1$, which implies $T_2 > T_1$. Figure 3 shows the forces on a piece of the string that subtends the small angle $\Delta\theta$. \mathbf{F}_1 and \mathbf{F}_2 are the tensions at the ends of the string element; \mathbf{f} and \mathbf{n} are, respectively, the tangential (friction)

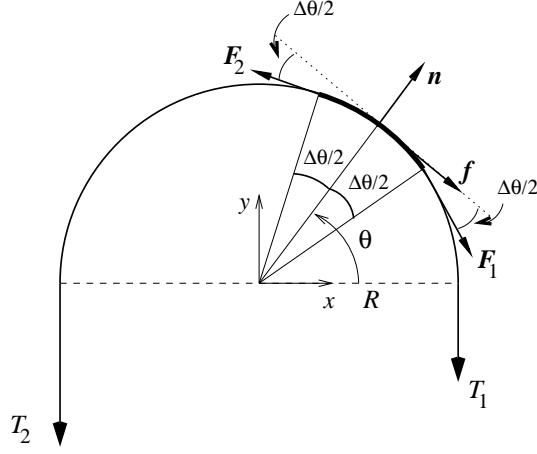


Figure 3: Forces on a piece of the string that subtends the small angle $\Delta\theta$.

and normal forces exerted by the pulley. Since the string is massless, Newton's second law entails that the vector sum of the forces shown in Figure 3 is zero:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{f} + \mathbf{n} = 0. \quad (2.5)$$

Let

$$\hat{\mathbf{r}} = \cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{y}}, \quad \hat{\boldsymbol{\theta}} = -\sin\theta \hat{\mathbf{x}} + \cos\theta \hat{\mathbf{y}} \quad (2.6)$$

respectively be the outward normal and tangential (oriented toward increasing θ) unit vectors at the point of the pulley with angular coordinate θ . For the the friction force and the normal force of the pulley *on the piece of string* we write

$$\mathbf{f} = -fR\Delta\theta \hat{\boldsymbol{\theta}}, \quad \mathbf{n} = nR\Delta\theta \hat{\mathbf{r}} \quad (2.7)$$

where f and n are positive and have dimension of force per unit length.

The tangential and normal components of equation (2.5) are

$$F_2 \cos\left(\frac{\Delta\theta}{2}\right) - F_1 \cos\left(\frac{\Delta\theta}{2}\right) - fR\Delta\theta = 0, \quad (2.8a)$$

$$-F_1 \sin\left(\frac{\Delta\theta}{2}\right) - F_2 \sin\left(\frac{\Delta\theta}{2}\right) + nR\Delta\theta = 0. \quad (2.8b)$$

In equations (2.8) we have

$$F_2 = |\mathbf{F}_2| = T\left(\theta + \frac{\Delta\theta}{2}\right), \quad F_1 = |\mathbf{F}_1| = T\left(\theta - \frac{\Delta\theta}{2}\right), \quad (2.9)$$

where $T(\theta)$ is the tension at the point of the string with angular coordinate θ with, of course,

$$T(0) = T_1, \quad T(\pi) = T_2. \quad (2.10)$$

Therefore Eqs.(2.8) become

$$T\left(\theta + \frac{\Delta\theta}{2}\right) \cos\left(\frac{\Delta\theta}{2}\right) - T\left(\theta - \frac{\Delta\theta}{2}\right) \cos\left(\frac{\Delta\theta}{2}\right) - fR\Delta\theta = 0, \quad (2.11a)$$

$$-T\left(\theta - \frac{\Delta\theta}{2}\right) \sin\left(\frac{\Delta\theta}{2}\right) - T\left(\theta + \frac{\Delta\theta}{2}\right) \sin\left(\frac{\Delta\theta}{2}\right) + nR\Delta\theta = 0. \quad (2.11b)$$

Now we divide each of the two last equations by $\Delta\theta$ and let $\Delta\theta \rightarrow 0$ to obtain

$$fR = \frac{dT}{d\theta} \quad (2.12)$$

and also

$$nR = T(\theta). \quad (2.13)$$

Equation (2.12) shows that friction causes the tension in the string to be variable, even though the string is massless.

3 Force on the pulley

We are now in a position to compute the friction force and the total force exerted by the string on the pulley.

The net friction force exerted by the pulley *on the the string* is given by

$$\mathbf{F}_f^{string} = \int_0^\pi (-fR \hat{\boldsymbol{\theta}}) d\theta = \hat{\mathbf{x}} \int_0^\pi \frac{dT}{d\theta} \sin \theta d\theta - \hat{\mathbf{y}} \int_0^\pi \frac{dT}{d\theta} \cos \theta d\theta \quad (3.14)$$

where we have used (2.6), (2.7) and (2.12). Integrations by parts yield

$$\int_0^\pi \frac{dT}{d\theta} \sin \theta d\theta = T(\theta) \sin \theta \Big|_0^\pi - \int_0^\pi T(\theta) \cos \theta d\theta = - \int_0^\pi T(\theta) \cos \theta d\theta \quad (3.15)$$

and

$$\int_0^\pi \frac{dT}{d\theta} \cos \theta d\theta = T(\theta) \cos \theta \Big|_0^\pi + \int_0^\pi T(\theta) \sin \theta d\theta = -(T_1 + T_2) + \int_0^\pi T(\theta) \sin \theta d\theta. \quad (3.16)$$

With these results Eq. (3.14) becomes

$$\mathbf{F}_f^{string} = (T_1 + T_2) \hat{\mathbf{y}} - \int_0^\pi T(\theta) (\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}) d\theta. \quad (3.17)$$

One could think that by symmetry the x -component in (3.17) would be zero, but this is not true because $T(\theta) \neq T(\pi - \theta)$. We also note that the friction force on the string cannot be determined unless the tension is known as a function of θ . Thus, in general, the question ‘‘What is the magnitude of the friction force that prevents the string from slipping over the pulley?’’ does not have a definite answer. As will be seen shortly, however, $T(\theta)$ can be found explicitly if the string is on the verge of sliding on the pulley.

The net normal force exerted by the pulley *on the the string* is given by

$$\mathbf{F}_n^{string} = \int_0^\pi nR \hat{\mathbf{r}} d\theta = \int_0^\pi T(\theta) (\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}) d\theta, \quad (3.18)$$

where equations (2.6), (2.7) and (2.13) have been used. From (3.17) and (3.18) it follows at once that

$$\mathbf{F}_f^{string} + \mathbf{F}_n^{string} - (T_1 + T_2) \hat{\mathbf{y}} = 0. \quad (3.19)$$

Note that $-(T_1 + T_2)\hat{\mathbf{y}}$ is the total force on the part of the string in contact with the pulley exerted by the hanging pieces of the string at each side of the pulley. Thus the last result is correct: the total force on the part of the string in contact with the pulley is zero because the string is massless.

By Newton's third law, the total force exerted by the string *on the pulley* is

$$\mathbf{F}_{pulley} = -(\mathbf{F}_f^{string} + \mathbf{F}_n^{string}) = -(T_1 + T_2)\hat{\mathbf{y}}. \quad (3.20)$$

Therefore, although T_2 and T_1 **are not** forces on the pulley, everything happens as if they actually were forces applied by the string at the points P and Q of the pulley shown in Fig. 2, and as if the forces exerted by the string on the other points of the pulley cancelled each other owing to an apparent (but nonexistent) symmetry.

4 Torque on the pulley

The torque exerted by the string on an element *of the pulley* that subtends an angle $d\theta$ is

$$d\boldsymbol{\tau}_{pulley} = R\hat{\mathbf{r}} \times (-\mathbf{f}) = R\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} f R d\theta = f R^2 \hat{\mathbf{z}} d\theta \quad (4.21)$$

where (2.7) has been used. By making use of (2.12) we are led to

$$d\boldsymbol{\tau}_{pulley} = \hat{\mathbf{z}} R \frac{dT}{d\theta} d\theta. \quad (4.22)$$

Therefore,

$$\boldsymbol{\tau}_{pulley} = \hat{\mathbf{z}} R \int_0^\pi \frac{dT}{d\theta} d\theta = R(T_2 - T_1)\hat{\mathbf{z}}. \quad (4.23)$$

Once again, this is the torque on the pulley obtained by the *a priori* physically unwarranted assumption that T_2 and T_1 are forces on the pulley and that the torques applied by the string on the pulley at points other than the points P and Q shown in Fig. 2 cancel each other owing to an apparent (but nonexistent) symmetry. It should be noted that one can calculate directly the total torque on the pulley without first finding the net frictional and normal forces as has been done here [4].

5 String on the verge of sliding

Let us suppose that the string is on the verge of sliding on the pulley. If μ is the coefficient of static friction between the string and the pulley we have

$$f = \mu n. \quad (5.24)$$

Combining this equation with (2.12) and (2.13) we find

$$\frac{dT}{d\theta} = \mu T. \quad (5.25)$$

It follows that

$$T(\theta) = T_1 e^{\mu\theta} \quad (5.26)$$

inasmuch as $T(0) = T_1$. Since $T_2 = T(\pi) = T_1 e^{\mu\pi}$, the friction coefficient is determined:

$$\mu = \frac{1}{\pi} \ln\left(\frac{T_2}{T_1}\right). \quad (5.27)$$

By the way, the exponential growth of the tension explains why, if a rope is wound several times around a capstan, it takes an enormous force to make the rope slide on the capstan by pulling one end against a tiny force at the other end [5, 6].

Now the friction force *on the pulley* can be explicitly computed. From (3.17) and (5.26) we have

$$\begin{aligned} \mathbf{F}_f^{pulley} &= -\mathbf{F}_f^{string} = -(T_1 + T_2)\hat{\mathbf{y}} + \int_0^\pi T(\theta)(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}})d\theta \\ &= -(T_1 + T_2)\hat{\mathbf{y}} + \hat{\mathbf{x}}T_1 \int_0^\pi e^{\mu\theta} \cos\theta d\theta + \hat{\mathbf{y}}T_1 \int_0^\pi e^{\mu\theta} \sin\theta d\theta. \end{aligned} \quad (5.28)$$

The integrals are elementary and can also be found in any table:

$$\begin{aligned} \int_0^\pi e^{\mu\theta} \cos\theta d\theta &= \frac{e^{\mu\theta}}{1 + \mu^2}(\sin\theta + \mu\cos\theta) \Big|_0^\pi = -\frac{\mu}{1 + \mu^2}(1 + e^{\mu\pi}); \\ \int_0^\pi e^{\mu\theta} \sin\theta d\theta &= \frac{e^{\mu\theta}}{1 + \mu^2}(\mu\sin\theta - \cos\theta) \Big|_0^\pi = \frac{1}{1 + \mu^2}(1 + e^{\mu\pi}). \end{aligned} \quad (5.29)$$

Therefore,

$$\mathbf{F}_f^{pulley} = -(T_1 + T_2)\hat{\mathbf{y}} - \frac{\mu T_1}{1 + \mu^2}(1 + e^{\mu\pi})\hat{\mathbf{x}} + \frac{T_1}{1 + \mu^2}(1 + e^{\mu\pi})\hat{\mathbf{y}}. \quad (5.30)$$

With the help of (5.27) and a little algebra this last result can be cast in the following form:

$$\mathbf{F}_f^{pulley} = -\frac{\mu}{1 + \mu^2}(T_1 + T_2)\hat{\mathbf{x}} - \frac{\mu^2}{1 + \mu^2}(T_1 + T_2)\hat{\mathbf{y}}. \quad (5.31)$$

Now we have a definite answer to our previous question: the magnitude of the friction force that prevents slippage of the string over the pulley is

$$F_f^{pulley} = \frac{\mu}{\sqrt{1 + \mu^2}}(T_1 + T_2). \quad (5.32)$$

From (1.4) and (5.27) it follows that

$$\frac{T_2}{T_1} = \frac{(4m_1 + M)m_2}{(4m_2 + M)m_1} = e^{\mu\pi}. \quad (5.33)$$

Solving this equation for M we find

$$M = \frac{4m_1 m_2 (e^{\mu\pi} - 1)}{m_2 - m_1 e^{\mu\pi}}. \quad (5.34)$$

Note that if $m_2 \leq m_1 e^{\mu\pi}$ then the string will never slip relative to the pulley no matter how large the pulley mass is. On the other hand, solving (5.33) for m_2 we get

$$m_2 = \frac{M m_1 e^{\mu\pi}}{M - 4m_1 (e^{\mu\pi} - 1)}. \quad (5.35)$$

If $M \leq 4m_1(e^{\mu\pi} - 1)$ there is no positive solution for m_2 . Therefore, two necessary conditions for the string to be on the verge of slipping over the pulley are

$$m_2 > m_1 e^{\mu\pi} \quad \text{and} \quad M > 4m_1(e^{\mu\pi} - 1). \quad (5.36)$$

The first requirement is expected from the force amplification effect brought about by a rope wrapped around a capstan [5, 6]. It would be the only necessary condition if the pulley could not rotate or, equivalently, if its mass were infinite. The second requirement is not so obvious and stems from the fact that the pulley can freely turn on its axle.

So as to have an idea of the order of magnitude of the masses that are required for the string to be on the verge of sliding, let us assume that $m_1 = 1$ kg, $m_2 = 3$ kg and $\mu = 0.3$. Then (5.34) gives $M \approx 43$ kg, an appreciably large mass for the pulley.

6 Concluding Remarks

We have argued that the usual textbook solution to a classic problem in rotational dynamics, Atwood's machine, relies on a faulty identification of forces on each object, since the tensions in the hanging parts of the string are identified as forces on the pulley. This may seem a small detail, but we believe it is an important one, since correctly identifying the forces on each of a system of interacting bodies is a fundamental step in solving a mechanics problem by means of Newton's laws. Such a shortcut also reinforces a common misconception between students to the effect that strings merely convey forces without affecting them.

We have presented a consistent treatment of the problem by considering the normal and friction forces between each element of the string and the pulley, and have shown that the contact force exerted on the pulley by the entire segment of the string that touches the pulley gives rise to the net force and the net torque usually assumed without a convincing justification in the standard treatment given in the textbooks.

Although presenting the full mathematical treatment of the problem, as done here, is beyond the scope of an introductory physics course, we believe that attention should be called to the fact that the force on the pulley arises from contact forces exerted by the string and that a careful analysis gives the conjectured result (4.23) for the torque on the pulley. Another possibility is to consider the pulley together with the segment of the string that touches it as a single object with the same moment of inertia as that of the pulley, since the string is massless. As far as the angular acceleration of this object is concerned, one is allowed to disregard the internal forces between the aforesaid segment of the string and the pulley, and now T_1 and T_2 are actually external forces responsible for the net external torque on the pulley-string segment system [4].

Acknowledgements

TRO is supported by the National Institute for Science and Technology of Quantum Information (INCT-IQ) and also thanks the National Council for Scientific and Technological Development (CNPq), Brazil.

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