The motion of two identical masses connected by an ideal string symmetrically placed over a corner

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Abstract

We introduce a new example of a system which slides up an inclined plane, while its center of mass moves down. The system consists of two identical masses connected by an ideal string symmetrically placed over a corner. This system is similar to the double-cone rolling up the inclined V-shaped rails. The double-cone's motion, while relatively easy to demonstrate, is rather difficult to analyze. Our example is easy to follow and it doesn't require subtle understanding of the 3-d geometry.

1 Introduction

A double-cone that rolls up an incline is a very well know classroom demonstration [1, 2, 3, 4]. It shows that the cone moves to lower its center of gravity even while its end points ascend on V-shaped rails. Despite its pedagogical appeal, there have been very few quantitative studies of the dynamics of this demonstration. This probably is due to the difficulty in identifying the points of contact between the double-cone and the rails [4]. Here we provide an example in which a string tied to two hanging masses slides up an incline plane. Similarly, as the string moves upward, the center of gravity of the system actually goes down. However, unlike the double-cone, this system is comparatively easy to investigate and accessible to undergraduate students.

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2 The system

We consider two identical balls A and B of mass m, connected by a massless ideal string of length l which is symmetrically placed over the corner of a frictionless table, as shown in figure (1). Let us mark with P the middle of the string. Initially the system is at rest, with the masses at the table's level, separated by the distance l. We first investigate the motion of the system for a horizontal table. As the system is released and the point P advances

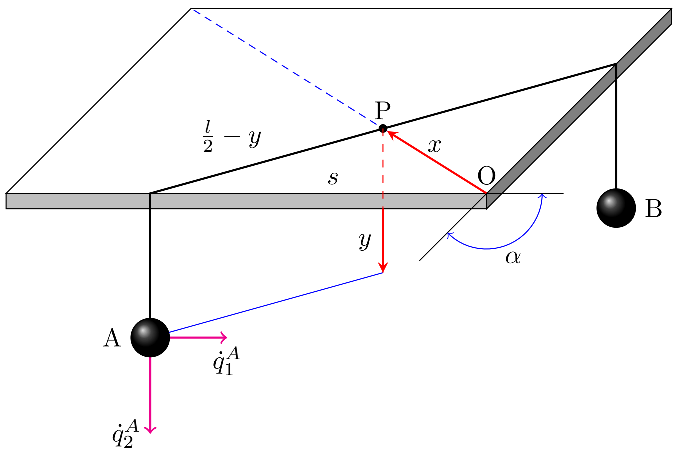

Figure 1: Two identical masses A and B connected by an ideal string of length l, falling symmetrically over the corner of a horizontal table.

towards the corner, we will be interested in finding the acceleration of P as a function of time and the angle α at the corner O. For this we want to find the motion of the two masses A and B, and most importantly, the motion of their center of mass.

The central point P moves along the bisector of angle α, which we take to be the x axis with the origin at O. The center of mass of the system moves both along the x axis towards O, and down along the y axis. Initially, the center of mass of the system is at $y = 0$. The figure depicts an intermediate state where A and B have already descended by an amount y as P moved along x towards the corner O. The mass A moves along the side s of the table and vertically along y. B moves symmetrically on the other side of the table. The generalized velocities of A are $\dot{q}_1^A = \dot{s}$ and $\dot{q}_2^A = \dot{y}$. From the geometry of the problem we have

$$\tan\frac{\alpha}{2} = \frac{l/2 - y}{x} \ , \qquad \cos\frac{\alpha}{2} = \frac{x}{s} \ . \tag{1}$$

Hence

$$\dot{q}_1^A = \dot{s} = \frac{\dot{x}}{\cos\frac{\alpha}{2}} \ , \qquad \dot{q}_2^A = \dot{y} = -\,\dot{x}\tan\frac{\alpha}{2} \ . \tag{2}$$

Since the kinetic energy of B is the same as that of A, the total kinetic energy of the system is given by

$$\begin{aligned} T &= \frac{m}{2}\left((\dot{q}_1^A)^2 + (\dot{q}_2^A)^2 + (\dot{q}_1^B)^2 + (\dot{q}_2^B)^2\right) \ , \\ &= m\,(\dot{s}^2 + \dot{y}^2) \ , \\ &= m\dot{x}^2\,\frac{1+\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} \ . \end{aligned} \tag{3}$$

Similarly, the potential energy is

$$\begin{aligned} V &= mg(q_2^A + q_2^B) \ , \\ &= -2mgy \ , \\ &= 2mgx\tan\frac{\alpha}{2} - mgl \ , \end{aligned} \tag{4}$$

where we chose the potential energy to be zero at the table's level. The Lagrangian of the system, $L = T - V$, becomes

$$L(x,\dot{x}) = m\dot{x}^2\,\frac{1+\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} - 2mgx\tan\frac{\alpha}{2} + mgl \ . \tag{5}$$

The corresponding Euler-Lagrange equation reads

$$2m\ddot{x}\,\frac{1+\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}} + 2mg\tan\frac{\alpha}{2} = 0 \ , \tag{6}$$

which yields

$$\ddot{x} + g\,\frac{\sin\alpha}{3-\cos\alpha} = 0 \ . \tag{7}$$

At this point, we are well equipped to describe the motion of the two masses. Let's consider only A, because B will move symmetrically. As the system evolves, A moves along s with the acceleration $\ddot{s}$ and along y with the acceleration $\ddot{y}$. From (2) and (7) we have

$$\ddot{s} = \frac{\ddot{x}}{\cos\frac{\alpha}{2}} = -g\,\frac{\sin\alpha}{(3-\cos\alpha)\cos\frac{\alpha}{2}} \tag{8}$$

and respectively

$$\ddot{y} = -\ddot{x}\tan\frac{\alpha}{2} = g\,\frac{\sin\alpha\,\tan\frac{\alpha}{2}}{3-\cos\alpha} \ . \tag{9}$$

The total acceleration of A will be $a_A = \sqrt{\ddot{s}^2 + \ddot{y}^2}$. Substituting the above results we obtain:

$$a_A = g\sqrt{\frac{1-\cos\alpha}{3-\cos\alpha}}\ . \tag{10}$$

Because B moves symmetrically on the other side of the table, its acceleration will be the same $a_B = a_A$.

The center of mass of the system moves simultaneously along x and y axes, as depicted in figure (2). Initially the two masses are at the table's level: $y = 0$. From the geometry of the system, at this point x is at its maximum value $x_{max} = (l/2)\cot(\alpha/2)$. At the end of the motion, when P arrives at

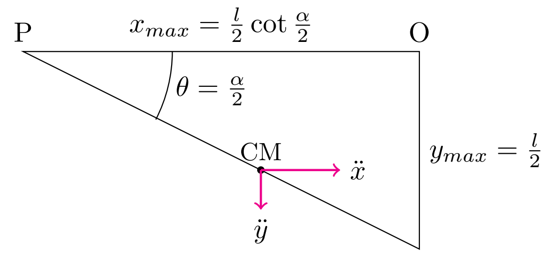

Figure 2: The motion of the center of mass of the system

point O, y reaches its maximum value $y_{max} = l/2$, which corresponds to $x = 0$. It follows that the center of mass descends on a oblique path at an angle of $\alpha/2$. Using eqs. (7) and (9), we obtain

$$a_{CM} = \sqrt{\ddot{x}^2 + \ddot{y}^2} = |\ddot{x}|\ \sec\frac{\alpha}{2} = g\,\frac{2\sin\frac{\alpha}{2}}{3-\cos\alpha} \ = \ g\ \frac{\sin\frac{\alpha}{2}}{1+\sin^2\frac{\alpha}{2}}\ . \tag{11}$$

So far we have been considering the motion of the string on a horizontal plane. Would the string slide up if we had an inclined plane instead? We find that the answer to that question is yes. For any given corner angle α, we can tilt the table upward with an arbitrary angle ϕ by raising the point O. We find that the point P still moves towards O provided ϕ is less than a maximum value ϕ_{max} to be determined. This upward motion of P is only apparently paradoxical, because the center of mass of the system still goes down. To determine ϕ_{max}, we refer the reader to figure (3). The point P moves upward on the plane only if the center of mass of the system continues to go down.

Thus, when the inclination ϕ reaches the maximum value ϕ_{max}, the center of mass moves horizontally as the point P slides up the plane. From figure

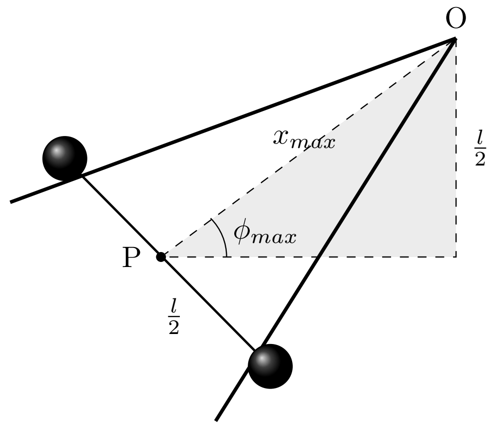

Figure 3: When $\phi = \phi_{max}$ the center of mass moves on the horizontal dashed line.

(3), we see that this happens when

$$\frac{l}{2} = x_{max} \sin\phi_{max} \ ,$$

or

$$\frac{l}{2} = \frac{l}{2} \cot\frac{\alpha}{2} \ \sin\phi_{max} \ .$$

That is

$$\phi_{max} = \sin^{-1}\left(\tan\frac{\alpha}{2}\right) \ . \tag{12}$$

Hence, we see that for small values of α, we have $\phi_{max} \simeq \frac{\alpha}{2}$, and in general $\phi_{max} > \frac{\alpha}{2}$. In figure (4 a), we see that for $\alpha = \pi/2$, for a vertical plane, the center of mass stays in place as the point P climbs towards point O.

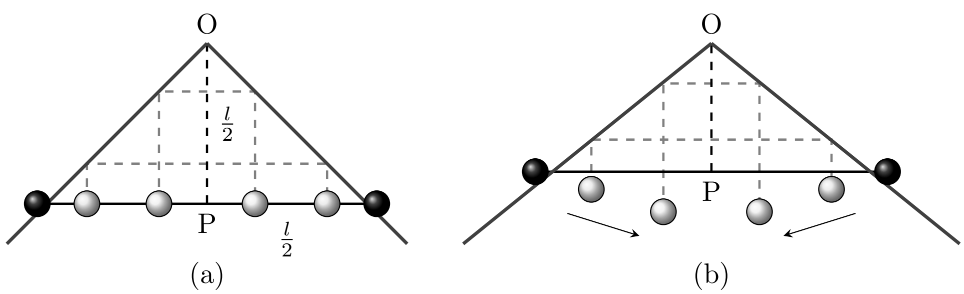

Figure 4: The vertical case.

In addition, eq. (12) suggests that for a plane with a corner angle $\alpha > \pi/2$, the point P will move up for any inclination of the plane, even vertical. For $\alpha > \pi/2$, the center of mass goes down as point P climbs up a vertical board as shown in figure (4 b).

3 Conclusions

We analyzed a system that consists of two identical masses connected by an ideal string placed symmetrically over a corner of a frictionless table. On a horizontal table, the string moves towards the corner for any value of the corner angle α. If the table is tilted upward, we find that the string still moves towards the corner provided the angle is less than a critical value. This system reminds of the double-cone rolling up the inclined V-shaped rails. The double-cone's motion, while relatively easy to demonstrate, is rather difficult to analyze. The example considered here is simple to understand, and it doesn't need subtleties of the 3-d geometry required for the involved analysis of the double-cone problem. We find that the corner problem is not without an intrigue. If the corner angle is greater than $\pi/2$, then the string will slide up and slip out of the plane, even for a vertical plane.

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