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RELATION BETWEEN UNCERTAINTY EXPONENT AND MEAN LIFETIME OF CHAOTIC TRANSIENT FOR MAP ON ANNULUS

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For a map on the annulus, it is tested numerically that, within errors of the calculation of 0.4%, the inverse of the mean lifetime of chaotic transient is equal to the product of the uncertainty exponent and the Lyapunov exponent. The second-order term in the Taylor series expansion for inverse lifetime has no effect within the precision of the calculation.

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Transient chaos, connected with an underlying strange repeller, is characterized by finite mean lifetime τ.[1–6] The length of the time interval in which the trajectory is chaotic depends upon initial conditions and computer precision.[1–7] When sufficiently many initial conditions are chosen, one can find the corresponding distribution of lifetimes of transient chaos, which is exponential, except for a part of distribution corresponding to very short lifetimes. If one sprinkles a large number of initial conditions with a uniform distribution in some phase space region containing the repeller, the number $N(t)$ of orbits still in the chaotic transient phase of their orbit after a time t decays exponentially with t:

$$\frac{N(t)}{N_0} \sim e^{-t/\tau} \,, \tag{1}$$

where τ is the mean lifetime of chaotic transient.

The dependence of τ on a system parameter has been investigated as this parameter passes through its crisis value.[8]

On the other hand, it is important to consider the extent to which uncertainty in initial conditions leads to uncertainty in the final state. To characterize this property, an exponent α has been introduced.[9–11] If one covers the phase space by boxes of linear size ϵ and counts the number $N_u(\epsilon)$ of boxes from which initial conditions can asymptote to more than one attractor, the fraction of phase space

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with uncertain final state is expressed as:

$$f = \frac{N_u(\epsilon)}{N(\epsilon)}\,, \tag{2}$$

where $N(\epsilon)$ is the number of boxes needed to cover the whole region of interest. This fraction is expected to scale with the initial uncertainty ϵ as:

$$f \sim \epsilon^{\alpha}\,, \tag{3}$$

where α is the uncertainty exponent. For a simple basin boundary, there is $\alpha = 1$, while for a fractal boundary, $\alpha < 1$.

An upper bound on α for concrete calculations can be given in terms of the fractal dimension of the repeller, and from the relation[1] between the mean lifetime τ and the uncertainty exponent, α was obtained:

$$\frac{1}{\tau} \geq \alpha\lambda\,. \tag{4}$$

In this paper, we shall test numerically the expression (4) in the case of a map on annulus.[8,9] The fractal basin boundaries of a well-known two-dimensional map on the annulus

$$\begin{aligned} \theta_{n+1} &= \theta_n + a\sin 2\theta_n - b\sin 4\theta_n - x_n \sin\theta_n \\ x_{n+1} &= -J_0 \cos\theta_n \end{aligned} \tag{5}$$

were studied in Refs. 8 and 9. The fixed points of this map are at $(\theta,\, x) = (0,\, -J_0)$ and $(\pi,\, J_0)$, and they are attractors if $|1 + 2a - 4b| < 1$. Previously, numerical study of this map has been performed and computer-generated pictures of the corresponding basins of attraction have revealed the Cantor set structure of the basin boundary, i.e. an infinitely fine-scaled structure.[8,9]

In this article, we are testing the inequality relation (4) for the annulus map (5). In this calculation, we use the parameterization $J_0 = 0.05$, $a = 1.32$, $b = 0.90$. The corresponding basins of attraction are displayed in Fig. 1. For each initial condition, from a grid of 500×500 initial conditions, the orbit was followed until its entry into the $\epsilon = 10^{-2}$ neighborhood of one of the attractors. At a position of an initial condition leading to the attractor at $(\theta,\, x) = (0,\, -J_0)$, a black point was plotted, while the position leading to the attractor at $(\theta,\, x) = (\pi,\, J_0)$ was left blank. This picture of the basins of attraction is similar as in Ref. 8, which corresponds to a different value of the parameter J_0.

Now, we calculate the values of the mean lifetime τ, the uncertainty exponent α and the Lyapunov exponent λ, in order to test the inequality relation (4).

In Fig. 2, we display the way in which the fraction of uncertain initial conditions f, given by Eq. (2), scales as the initial condition error ϵ is reduced. In the case of the annulus map (5), the log–log plot exhibits linear dependence of $\log f$ on $\log\epsilon$, in accordance with previous calculations.[8,9] Here, a set of 10^6 initial conditions was uniformly distributed over the interval $(0,\, \pi)$ for θ_0, at $x_0 = 0$. The calculated values are shown in Fig. 2 by open circles and the solid line represents a straight

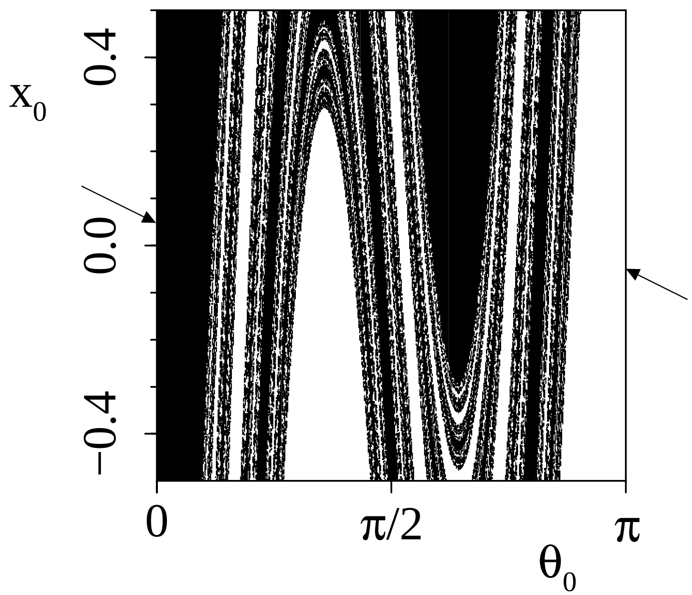

Fig. 1. Basins of attraction for the map on annulus (5) (see text). Attractors are indicated by arrows.

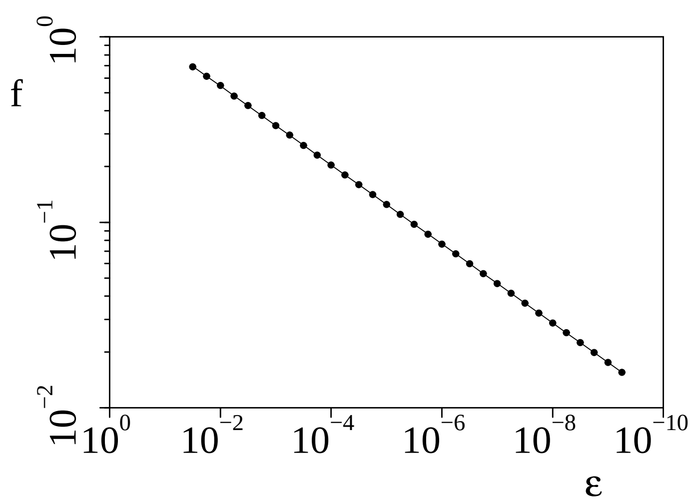

Fig. 2. Log–log plot for dependence of the fraction of uncertain orbits f on ϵ (open circles) and the fit of Eq. (3) (solid line) (see text).

line fit, corresponding to the exponential scaling (3). The fit of (3) to the calculated points is excellent, giving the corresponding value of the uncertainty exponent:

$$\alpha = 0.2134 \pm 0.0006\,. \tag{6}$$

In the next step, we have computed the mean lifetimes for the same set of initial conditions uniformly distributed over the interval $(0, \pi)$ for θ_0, at $x_0 = 0$. For each initial condition, the lifetime was determined as the time needed for the orbit to enter into an $\epsilon = 10^{-2}$ neighborhood of one or the other attractor. In Fig. 3, we display in the lin–log scale a distribution of lifetimes calculated in such a way for our set of 10^6 initial conditions. At a small time interval at the beginning, we observe a plateau in the distribution, which is omitted in the calculation of the mean lifetime. After this initial interval of time, the distribution of lifetimes closely follows the exponential time dependence, as seen from an excellent fit of the exponential function (1) (solid line) to the calculated points (open circles). From there, we obtain the mean lifetime of chaotic transient, τ. Its inverse value, i.e. the decay rate is:

$$\frac{1}{\tau} = 0.2895 \pm 0.0005 \tag{7}$$

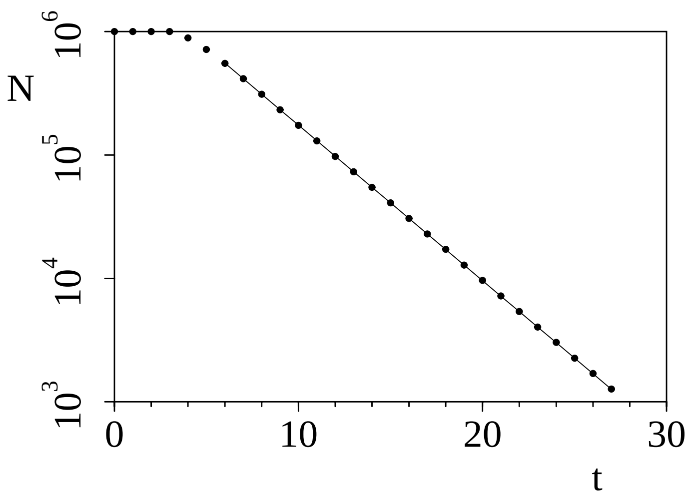

Fig. 3. Lin–log plot of lifetimes of chaotic transients in dependence on time (open circles) and a fit of Eq. (1) (solid line) (see text).

A convincing evidence that we are really dealing with transiently chaotic motion is shown in Fig. 4, where a chaotic repeller of the system (5) is presented. This drawing of the chaotic repeller is obtained by performing a calculation for 10^4 initial conditions uniformly distributed over the interval $(0, \pi)$ for θ_0, at $x_0 = 0$. In analogy to the method from Ref. 12, the first 10 and the last 10 iterations were deleted, so that the drawing displays points in the neighborhood of the chaotic repeller.

The exponential distribution of lifetimes corresponds to the orbits spending some time in the neighborhood of the chaotic repeller and entering afterwards into the neighborhood of one of the attractors. For the same orbit, one can consider the uncertainty that it enters into only one of the two attractors. It is evident

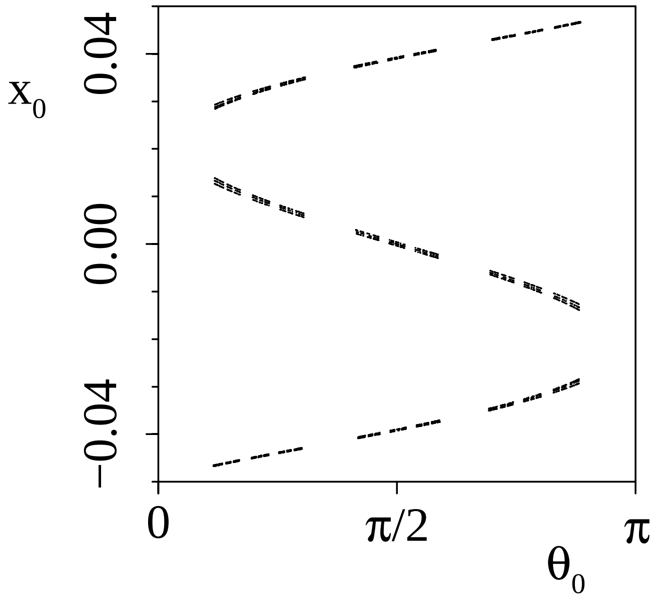

Fig. 4. Chaotic repeller of the system (5) (see text).

that two initially close-lying trajectories will diverge more from each other if they spend more time in the neighborhood of the chaotic repeller, i.e. the uncertainty of orbits will increase with increase of mean lifetime. Thus, the uncertainty exponent α should be smaller (meaning larger uncertainty) if the mean lifetime τ is longer. The relation (4) is in accordance with this qualitative observation.

In order to test the relation (4), in the last step of our calculation, we compute the value of the Lyapunov exponent λ. In analogy to the method from Ref. 12, this calculation is performed in such a way that in the orbits with a lifetime larger than 35, the first 10 and the last 10 iterations were deleted. In this way, we obtain the value of the Lyapunov exponent:

$$\lambda = 1.352 \pm 0.002\,. \tag{8}$$

Using the values (6) and (8) for α and λ, we obtain:

$$\alpha\lambda = (0.2134 \pm 0.0006) \times (1.352 \pm 0.002) = 0.2885 \pm 0.0012\,. \tag{9}$$

This value is equal to the value of the inverse lifetime (7) within the limits of precision of the calculation, i.e. with the precision of 0.4%. Thus, our calculation for the map on annulus (5) shows that, within the precision of 0.4%, the inequality relation (4) reduces to a more stringent relation:

$$\frac{1}{\tau} = \alpha\lambda\,. \tag{10}$$

We note that in the case of a piecewise linear function, an approximate relation was derived in Ref. 12 for the long-lived transients:

$$\frac{1}{\tau} \approx (1 - D^{(0)})\lambda\,, \tag{11}$$

taking only the first term in the Taylor series expansion, where $D^{(0)}$ denotes the capacity dimension of the repeller. Although this expression was derived only for a very special class of models, it was proposed that it holds more generally.[12] On the other hand, the uncertainty exponent α was related to the dimension of the basin boundary by Ref. 10:

$$\alpha = D - d\,,$$

where D is the dimension of the phase space and d is the dimension of the basin boundary. In analogy, taking $\alpha = 1 - D^{(0)}$, Eq. (11) can be written as:

$$\frac{1}{\tau} \approx \alpha\lambda\,. \tag{12}$$

Performing for the map (5) the calculation of the second-order term in the Taylor series expansion from Ref. 12, we obtain a contribution of 9×10^{-4}, which is 0.3% of the calculated value (9). We see that this term has no effect within the precision of our calculation.

It will be interesting to test this relation by further increasing the precision of the calculation and by performing similar tests for other nonlinear systems with chaotic transients.

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