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SENSITIVE DEPENDENCE OF LIFETIMES OF CHAOTIC TRANSIENT ON NUMERICAL ACCURACY FOR A MODEL WITH DRY FRICTION AND FREQUENCY DEPENDENT DRIVING AMPLITUDE

V. PAAR and N. PAVIN

Department of Physics, Faculty of Science, University of Zagreb, Zagreb, Croatia

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Chaotic transients are investigated for a one-dimensional model, which is a driven nonlinear system having frequency dependent driving amplitude, with weak dissipation consisting of Coulomb and quadratic friction. It is shown that lifetimes of chaotic transients are sensitively dependent on numerical accuracy when the initial condition is fixed and approximately follow the exponential decay law, which cannot be distinguished from the well-known exponential decay law for dependence on initial conditions when the numerical precision is fixed.

1. Introduction

Investigations of sustained chaotic behavior present one of the most interesting topics of contemporary science.[1] Much attention has been also paid to the phenomenon of transient chaotic motion when the system behaves chaotically during some transient time interval but ultimately falls onto a periodic attractor.[2–18] It was found that for the computations with a distribution of many initial conditions the lifetimes of chaotic transients follow the exponential decay law.[3,4,8,9,15,17,18] Grebogi, Ott and Yorke have associated this transient chaos with the crisis.[5–8] As a control parameter is varied, a chaotic attractor suddenly changes at the crisis point into a chaotic repeller showing transient chaos. For values of the control parameter beyond the crisis point the attractors are nonchaotic. Trajectories starting close to a chaotic repeller or a saddle exhibit long, transiently chaotic motion.[5–12]

Investigations of chaotic dynamics have been, so far, directed mostly to Hamiltonian systems or to systems with strong dissipation. However, as pointed out by Izrailev *et al*,[15] there is a large region of physical applications for systems with weak dissipation. As models for the numerical analysis of weakly dissipative systems some two-dimensional maps have been investigated, the perturbed Chirikov maps and the dissipative Fermi maps.[15–18] It was found that the transiently chaotic trajectory randomly wanders in some parts of the phase space which were occupied by Hamiltonian chaos in the absence of dissipation, and then suddenly starts to

approach some periodic orbit. For a random or uniform distribution of initial conditions the lifetimes of the transients were found to be distributed exponentially,[15–18] similarly as in the case of the one-dimensional maps.[9]

In this paper we introduce a generalization of weak dissipation Duffing model, which presents a one-dimensional model of a simple robot. A bifurcation diagram of this robot model is studied in dependence on the translation length as a control parameter. The first appearance of transient chaos with increasing value of control parameter is identified and the associated distribution of its lifetimes investigated.

2. Bifurcation Diagram of a Robot Model

In this paper we investigate a generalization of Duffing oscillator, described by the equation of motion:

$$\ddot{x} - \gamma_2 \operatorname{sgn}(\dot{x}) - \beta_2 \dot{x}^2 \operatorname{sgn}(\dot{x}) - \delta_{21} x - \delta_{23} x^3 = L \frac{\omega^2}{2\pi} \cos(\omega t) - \xi_{21} e^{\lambda_{21} t} - \xi_{22} e^{\lambda_{22} t} . \quad (1)$$

This equation of motion corresponds to a one-dimensional model of a simple robot.[19,20]

The second and third term on the l.h.s. of Eq. (1), with parameters γ_2 and β_2, correspond to the Coulomb and quadratic damping, respectively. Such a form of friction was proposed by Hemp.[21] The parameters $\delta_{21} < 0$ and $\delta_{23} < 0$ are the coefficients of rigidity which correspond to a hard spring. The first term on the r.h.s. of Eq. (1) is the periodic driving force with driving frequency ω and driving amplitude $L\frac{\omega^2}{2\pi}$, where L denotes the translation length. A characteristic feature of this driving is that the driving amplitude itself depends on the driving frequency. The last two terms on the r.h.s. of Eq. (1) present the feedback drive having strengths

$$\xi_{21} = \frac{a_2}{c_2} \lambda_{21}^2 , \quad (2)$$

$$\xi_{22} = \frac{b_2}{c_2} \lambda_{22}^2 , \quad (3)$$

where

$$a_2 = \lambda_{22} \cdot z_2(0) - z_4(0) , \quad (4)$$

$$b_2 = z_4(0) - \lambda_{21} \cdot z_2(0) , \quad (5)$$

$$c_2 = \lambda_{22} - \lambda_{21} , \quad (6)$$

and the parameters λ_{21}, λ_{22} ($\lambda_{21} \neq \lambda_{22}$) denote the desired roots of the system in the regime of closed regulation loops. As the consequence of these two feedback terms the response of the system is non-stationary. In this case it is not possible to strictly define a chaotic system. However, these terms are significant only at

a short initial stage of the motion, while with increasing time they very quickly become irrelevant. In the present investigation, where we consider the dynamics of the system after one hundred periods of the driving force, the response is practically stationary.

The system (1) asymptotically resembles a single-well (i.e. a hard spring) Duffing oscillator[23–26] with three modifications: the linear damping is replaced by the Coulomb plus quadratic friction, the driving amplitude is frequency-dependent, and the nonstationary feedback terms are added. Another important point should also be stressed: Duffing oscillator was investigated previously for the case of sizeable damping, while the parametrization in the robot model corresponds to a weak damping.

In this paper we have performed calculations for the Eq. (1) using a set of model parameters fixed at the values $\omega = 2\pi$, $\lambda_{21} = -10.5$, $\lambda_{22} = -11.5$, $z_2(0) = 0.01$, $z_4(0) = 0.01$, $\beta_2 = -5.18 \cdot 10^{-6}$, $\gamma_2 = -0.00298$, $\delta_{21} = -0.7611$, $\delta_{23} = -0.0127$. These values were chosen on the basis of estimates deduced for a robot model.[19,20] The initial conditions are taken $x_0 = 0$, $\dot{x}_0 = 0$. The remaining model parameter, the translation length parameter L, was treated as a control parameter. Here we perform numerical simulations employing Eq. (1), without restricting the range of values of parameter L, with the aim to investigate the onset of chaotic transients.

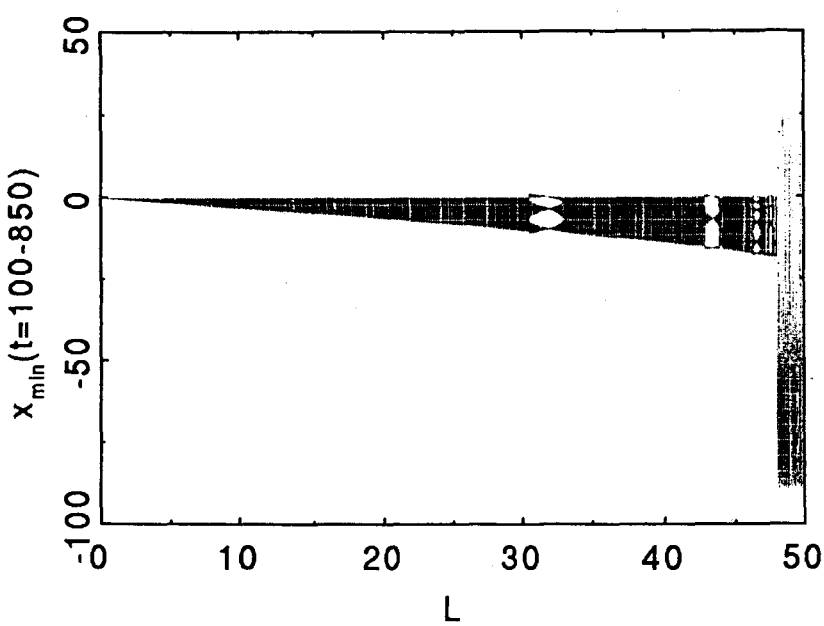

Fig. 1. Bifurcation diagram for the equation (1) in the range of control parameter from $L = 0$ to $L = 50$. This interval of control parameter L is covered by 625 equidistant points and the trajectory is calculated for each point with a single initial condition $x_0 = 0$, $\dot{x}_0 = 0$. Diagram displays local minima of x, denoted by $x_{\min}$, in the time interval between $t = 100$ and $t = 850$ (time is expressed in the units of the period of external driving). Up to the value of control parameter $L \approx 48$ the motion is transiently quasiperiodic (with a periodic asymptote) or mode locked. At $L \approx 48$ the bifurcation diagram displays the onset of transient chaos.

In the range of control parameter values $L \leq 1$, the solutions of Eq. (1) are of the period-1 with respect to driving. In fact, we find a very weak quasiperiodic transient which, for engineering purposes, should be practically indistinguishable from the period-1 solution. In this case the system practically acts as a passive system: it responds to the driving frequency ω as pure periodic oscillations of the same frequency.

In Fig. 1 we present the bifurcation diagram computed for the system (1) versus the control parameter L in the time interval between $t = 100$ and $t = 850$ (expressed in the units of period of external driving which is $T = \frac{2\pi}{\omega} = 1$ s). Up to $L \approx 30$ the motion is quasiperiodic, which asymptotes to the period-1 solution. With a further increase of L there appear windows of periodicity. Pronounced windows appear in the order of periods 5, 4, 7, 11, 17 etc. (Computations are performed at equidistant points at a distance $\Delta L = 0.08$. Thus, very narrow windows squeezed between the two neighboring points of the grid are not shown in Fig. 1). The chaotic transient appears first at $L \approx 48$ and persists up to very high values of control parameter ($L \approx L_c \approx 3 \cdot 10^4$), interrupted by narrow intervals of periodic windows. A particularly pronounced periodic window is of period-3 in the range between $L \approx 70$ and $L \approx 80$. First at a very high critical value of control parameter, $L \approx L_c$, the transient chaos turns into a sustained chaos. At this point of an inverse crisis, a chaotic transient is converted into a chaotic attractor. However, these values of the control parameter L lie out of the range of values significant for engineering purposes.

3. Dependence of Lifetimes of Transient Chaos on the Numerical Precision

In the second step of our calculation we have investigated the lifetimes of chaotic transient which appears for Eq. (1) when the control parameter L increases beyond the critical value $L = 48$. Taking a fixed precision of numerical algorithm, and a uniform distribution of initial conditions, we have obtained the result that the lifetimes of chaotic transient are distributed exponentially. This result is in accordance with previous investigations of several continuous and discrete nonlinear dynamical systems.[3,4,8,9,14,15]

In addition to this well known initial condition dependence of lifetimes we have investigated the dependence of lifetimes of transient chaos on the accuracy of numerical algorithm used to integrate the equation of motion. In Fig. 2, this dependence is shown for the computation with a control parameter value $L = 48.0789$: the lifetimes of transient chaos are presented versus the accuracy of numerical algorithm (Runge–Kutta–Merson method from[22]) in the range of accuracy between 10^{-10} and 10^{-6}. The results show an extreme sensitivity of lifetimes to the numerical accuracy.

Let $N(\tau)$ be the number of surviving chaotic trajectories after time τ. The values of $N(\tau)$ are presented versus τ in Fig. 3. For a short initial time interval from $\tau = 0$ to $\tau \approx 100$ s we observe a plateau of roughly constant $N(\tau)$, i.e. chaotic

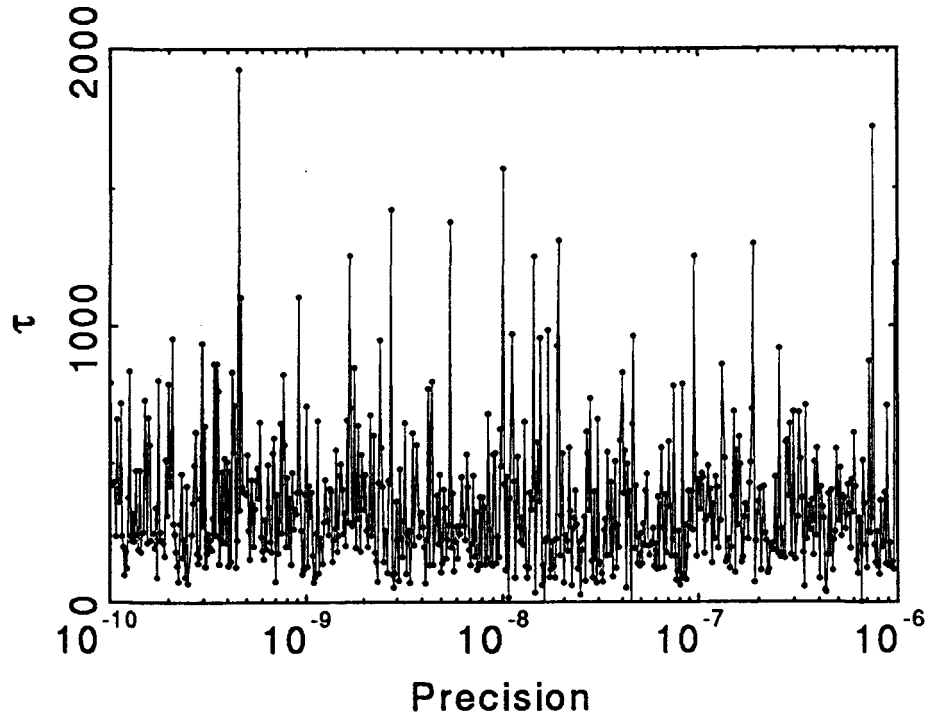

Fig. 2. Lifetimes τ (in units of the period of driving force) of chaotic transient for $L = 48.0789$ versus numerical accuracy in D02BAF NAG routine (Runge-Kutta-Merson method).[22]

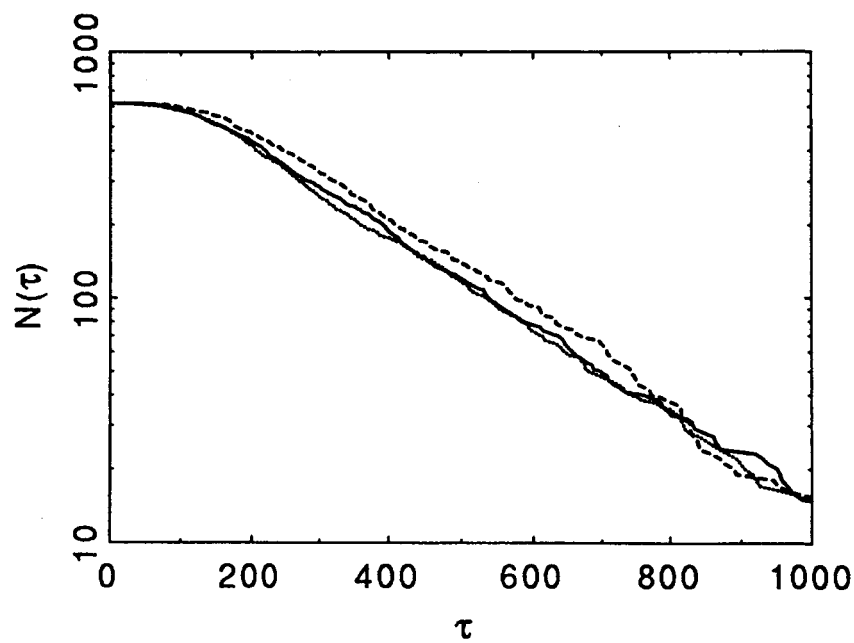

Fig. 3. Number of trajectories with lifetime greater than τ, versus τ. Full line corresponds to the results shown in Fig. 2, i.e. to a distribution of numerical accuracies with a fixed initial condition. Dot and dashed lines correspond to the calculations for two distributions of initial conditions (around the point $x_0 = 0$, $\dot{x}_0 = 0$) with fixed numerical precisions of 10^{-6} and 10^{-10}, respectively.

transients practically do not decay during the first 100 periods of the driving force. With increasing time the transients begin to decay and already for $\tau \geq 200$ s they roughly follow the exponential decay law

$$N(\tau) = N_0 \exp\left(-\frac{\tau}{\langle\tau\rangle}\right). \tag{7}$$

The number of orbits which remain chaotic decreases exponentially with time τ at the rate $1/\langle\tau\rangle$, where $\langle\tau\rangle$ is the corresponding mean lifetime. By fitting the expression (7) to the calculated lifetimes in the time interval between $\tau = 200$ s and $\tau = 1000$ s we obtain a value of the mean lifetime associated with numerical precision, $\langle\tau\rangle = 231$ s.

For comparison, we present the results of the computations with a distribution of initial conditions, using a fixed numerical accuracy. Initial points have been taken on a 25×25 grid within the square $-10^{-8} \leq x_0 \leq 10^{-8}$, $-10^{-8} \leq \dot{x}_0 \leq 10^{-8}$. The corresponding lifetimes are presented in Fig. 3 for the accuracy of 10^{-6} (dot line) and 10^{-10} (dashed line). Fitting exponential decay law (7) to these lifetimes we obtain the corresponding mean lifetimes $\langle\tau\rangle = 246$ s and $\langle\tau\rangle = 234$ s, respectively. Thus, we find that the value of mean lifetime associated with a fixed initial condition and a uniform distribution of numerical accuracies is similar to the value of mean lifetime associated with a fixed numerical accuracy and a uniform distribution of initial conditions. This result cannot be considered as surprising. Suppose, although this is not the case, that the simulation made only one numerical error, on the first step, and that the equation was integrated exactly (with no error) for the remainder of the trajectory. The effect of a distribution of numerical precisions is that one step away from the sole initial condition, there is a distribution of different points (because of different numerical errors made), which could be considered as initial conditions for the rest of the trajectory. The effect of the temporary exponential divergence of trajectories seen in transient chaos in that the life-times of the transients fall into an exponential distribution, for essentially the same reason as if one started out with uniformly distributed initial conditions.

4. Conclusions

A generalization of weakly dissipated Duffing oscillator was investigated, which corresponds to a simple one-dimensional model of a robot. This system is characterized by a Coulomb plus quadratic damping and by an asymptotically stationary response. Bifurcation diagram was studied, with the translation length L as a control parameter. The crisis point was determined, where a chaotic transient appears. It was shown that the lifetimes of individual trajectories depend strongly and irregularly on the precision of numerical algorithm, when the initial condition is kept fixed. This results in the exponential distribution of lifetimes of chaotic transients. This is the same scaling as obtained in the calculations with a uniform distribution of initial conditions, when the precision of numerical algorithm is kept fixed. Our

result is in accordance with the expectation that, if the transient is chaotic, then the individual trajectories must depend sensitively on all details, including the accuracy of the integration algorithm.

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