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Dependence of lifetimes of chaotic transients in the weakly dissipative Duffing oscillator on numerical precision and external noise

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Abstract

A numerical experiment was performed for a distribution of lifetimes of chaotic transients in the weakly dissipative single-well Duffing oscillator. It is shown that the results of computations with a uniform distribution of precision of numerical algorithm at a fixed initial condition cannot be distinguished from the well-known exponential distribution of lifetimes computed with an uniform distribution of initial conditions at fixed numerical precision. The same exponential distribution of lifetimes is also obtained if a noise term is added to the driving force in the fixed precision case.

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Keywords: Chaotic transients; Duffing oscillator; Numerical precision; Noise; Mean lifetimes

1. Introduction

The single-well Duffing equation

$$\ddot{x} + \gamma\dot{x} + x + x^3 = F \sin \omega t \tag{1}$$

was previously investigated for strong dissipation ($\gamma \geqslant 0.1$) [1,2] and in the absence of dissipation ($\gamma = 0$) [3]. Here we investigate the Duffing equation (1) in a case of weak dissipation, $\gamma = 0.001$. The case of Duffing equation with weak dissipation is interesting since it is associated with fairly long chaotic transients, and thus is appropriate for the study of lifetimes of chaotic transients.

In contrast to the well-known cases where chaos is a steady state [4], many nonlinear dynamical systems exhibit transient chaotic behavior [5–16]; they behave chaotically

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during some transient time interval before the trajectory locks onto an attractor and the transience is ended. This lifetime sensitively depends on initial conditions. Kaplan, Yorke and Yorke have shown that choosing sufficiently many initial conditions, the exponential distribution of lifetimes of chaotic transients is obtained [5].

Grebogi et al. have discussed the chaotic transients accompanying the crisis [6]. As a control parameter is varied, a chaotic attractor suddenly changes at the crisis point into a chaotic repeller showing transient chaos [6–11]. For values of the control parameter beyond the crisis point the attractors are nonchaotic. Trajectories starting close to a chaotic repeller or a saddle exhibit long, transiently chaotic motion [6–11].

Investigations of chaotic dynamics have been so far mostly directed either to Hamiltonian systems or to systems with strong dissipation. However, as pointed out by Izrailev et al. [13], there is a large region of physical applications for systems with weak dissipation. As models for the numerical analysis of weakly dissipative systems some two-dimensional maps have been investigated, the perturbed Chirikov maps and the dissipative Fermi maps [13–16]. It was found that a transiently chaotic trajectory randomly wanders in some parts of the phase space which were occupied by Hamiltonian chaos in the absence of dissipation and then suddenly starts to approach some periodic orbit. For a random or uniform distribution of initial conditions the lifetimes were found to be distributed exponentially [13–16], similarly as in the case of one-dimensional maps [7].

In this paper, numerical experiments for transient lifetimes are performed for the weakly dissipative Duffing oscillator: both for an ensemble of initial conditions at a fixed numerical accuracy (as commonly done) and for a distribution of numerical precisions at a fixed initial conditions (as introduced in this paper). In the second step, the effect of an additional noise term in the external force on the lifetimes of transients is investigated.

2. Results and discussion

In the present calculations, the value of the driving frequency $\omega = 0.95$ is taken from the previous calculations for strong dissipation [2] and the driving amplitude F was treated as a control parameter, and the strength of weak dissipation is $\gamma = 0.001$. Increasing the value of F, the first appearance of a chaotic attractor is via the inverse crisis at the crisis point $F = F_c = 20$. For the values of F below the crisis point F_c all attractors are nonchaotic, but in the interval between the values $F = F_t = 9.5$ and $F = F_c = 20$ the chaotic transient is present. We note that the point F_t of the first appearance of a chaotic transient is near the critical value for the onset of Hamiltonian chaos in conservative system ($\gamma = 0$ in Eq.(1)).

In the previous investigations of transient chaos in various dynamical systems the lifetime was studied for random or uniform distributions of initial conditions. On the contrary, in this paper, we investigate the lifetimes of chaotic transients for a distribution of a precisions of a numerical algorithm used for a numerical integration of

differential equations but keeping the initial condition fixed ($x_0 = 2$, $\dot{x}_0 = 0$). We performed the calculations for a uniform distribution (on the logarithmic scale) of numerical precisions in the interval between 10^{-6} and 10^{-10}. For each numerical precision, the differential equation (1) was integrated using the Runge–Kutta–Merson method [18]. The lifetime of the chaotic transient was determined from a calculated trajectory using the method in accordance with procedure [9,10] that the lifetime is equal to the escape time for the orbit to leave the restraining region Γ. (Γ is a region in the phase space from which open balls of small radius, centered at the points of periodic attractors, are deleted.) In Fig. 1 the lifetimes calculated in this way are presented versus numerical precision of the numerical algorithm. These results show an extreme sensitivity of lifetimes on numerical precision. It should be noted that there is no sign of convergence of lifetimes with increasing precision at the $\gamma = 0.001$ level of weak dissipation.

The same results are displayed in Fig. 2 by plotting $N(\tau)$, the number of surviving chaotic orbits in the restraining region Γ after time τ, in dependence on τ (full line). The result displays approximately an exponential decay of the number of survivors:

$$N(\tau) \sim \exp\left[-\frac{\tau}{\langle\tau\rangle}\right]. \tag{2}$$

The number of orbits which remain in the restraining region Γ decreases exponentially with time τ at the rate $1/\langle\tau\rangle_{np}$, where $\langle\tau\rangle_{np} = 2.14 \times 10^3$ is the corresponding mean lifetime. The subscript *np* is added to denote the association with a distribution of numerical precisions for a fixed initial condition.

In Fig. 3 we present the period of attractor which is an asymptote of transient chaos in Fig. 1, in dependence on numerical precision. The resulting sensitive dependence is

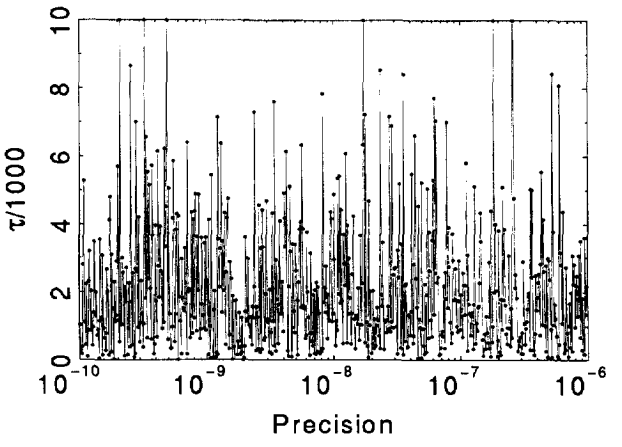

Fig. 1. Lifetimes τ (expressed in the periods of driving force) of chaotic transients in weakly dissipative Duffing oscillator vs numerical precision of the D02BAF NAG routine [18] (Runge–Kutta–Merson method). Calculation is performed up to the time of 10^4 periods of driving force. Parameter values in Eq. (1) are $F = 18$, $\omega = 0.95$, $\gamma = 0.001$, and initial conditions $x_0 = 2$, $\dot{x}_0 = 0$.

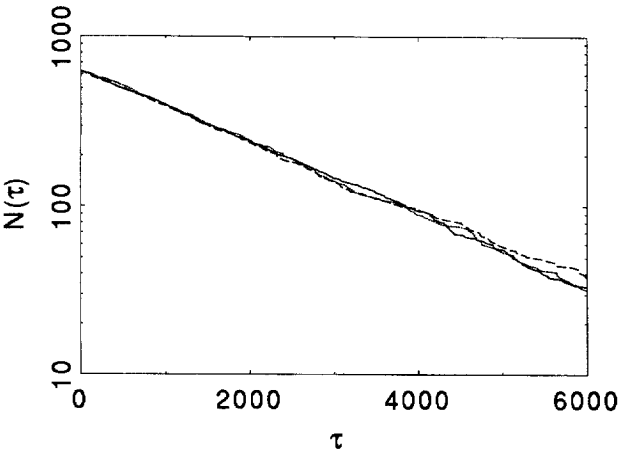

Fig. 2. Number of trajectories $N(\tau)$ with lifetime greater than τ vs. τ. The vertical scale is logarithmic. For parameter values and initial conditions see caption to Fig. 1. Full line: computation with a distribution of numerical precisions, at a fixed initial condition $x_0 = 2$, $\dot{x}_0 = 0$. Dashed line: computation with a distribution of initial conditions (around the point $x_0 = 2$, $\dot{x}_0 = 0$), at a fixed numerical precision of 10^{-8}. Dot-dashed line: computation at fixed initial condition $x_0 = 2$, $\dot{x}_0 = 0$ and fixed numerical precision of 10^{-9} for the case when a noise term is added to the driving force in Duffing oscillator Eq. (1).

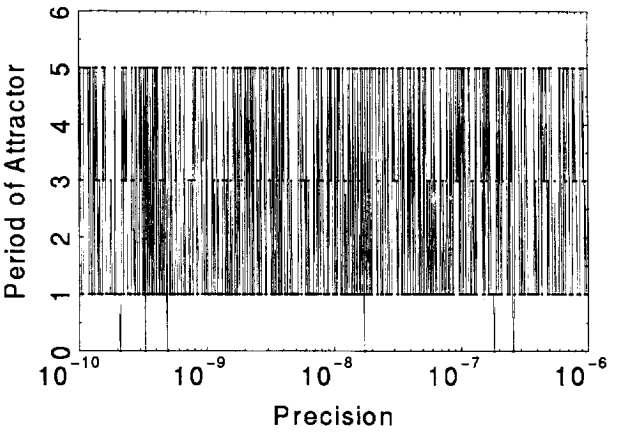

Fig. 3. Period of the periodic attractor (expressed in the periods of driving force), which is the asymptote of the chaotic transient from Fig. 1, in dependence on the numerical precision. The grid of numerical precisions is the same as in Fig. 1. Labels on the vertical axis denote the period of attractor, except for the label 0 which denotes that the system is still in a transient state after 10^4 periods of the driving force.

in accordance with the sensitive dependence of lifetimes of chaotic transient shown in Fig. 1.

This result is in accordance with Poincare sections shown in Figs. 4(a)–(d) computed for four different values of numerical precision for which the chaotic transient ends up in four different attractors: one of period-1, two of period-3, and one of period-5.

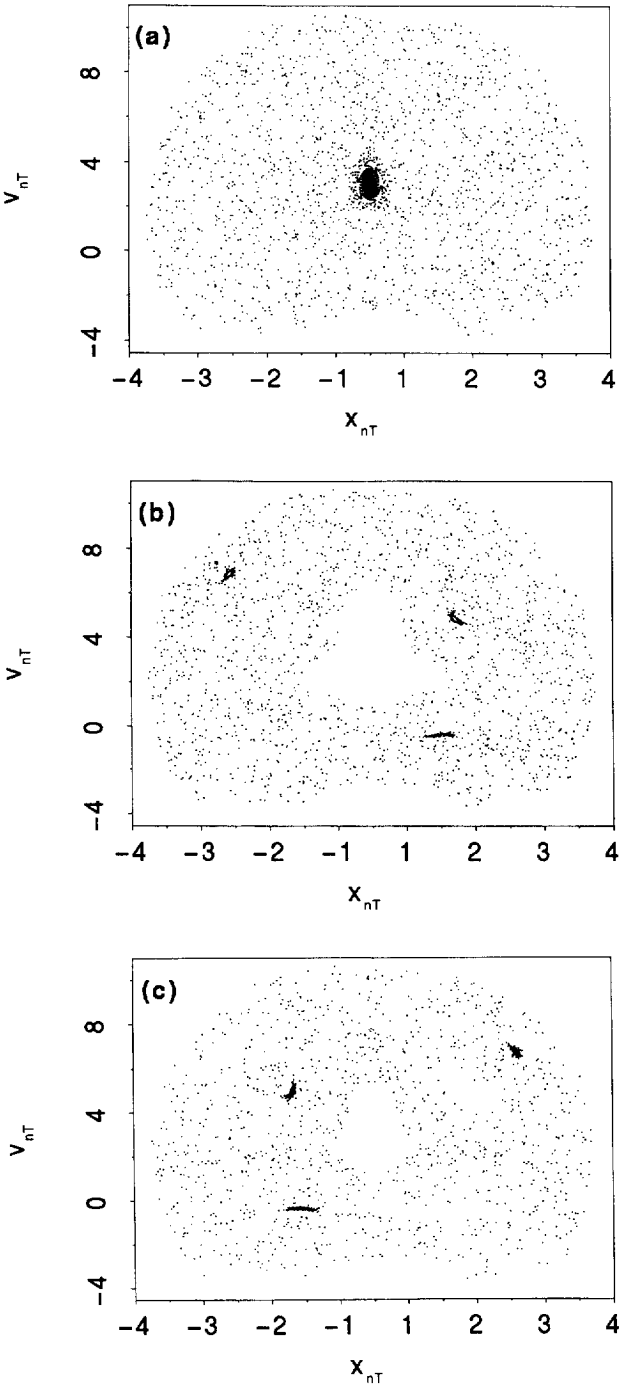

Fig. 4. Poincare sections computed for four different numerical precisions for which chaotic transient ends up in four different attractors: (a) in the period-1 attractor, (b, c) in one of the two period-3 attractors, and (d) in the period-5 attractor. The parameters F, ω, γ and initial conditions x_0, $\dot{x}_0$ are the same as in Fig. 1.

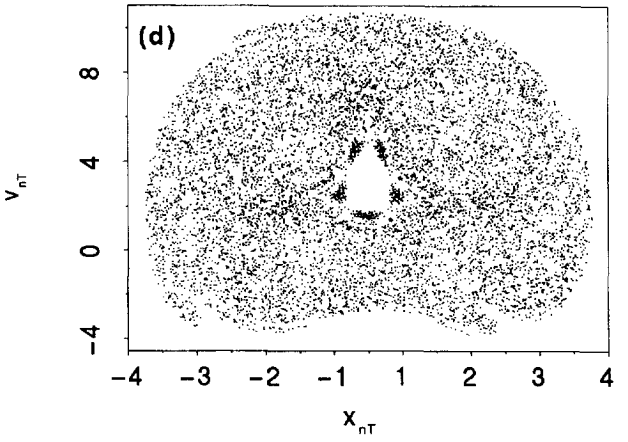

Fig. 4. Continued.

Therefrom it is also seen that there is nothing special about the illustrative choice of initial conditions $x_0 = 2$, $\dot{x}_0 = 0$ used in the present calculations of lifetimes.

Assuming that the probability of entering an attractor increases with the size of the attractors basin, one would guess from Fig. 3 that the period-3 attractor holds the smallest-size basin of the three periods shown. Namely, the probability of entering the period-3 attractors is smaller than the probability of entering the period-1 or period-5 attractors. This is indeed in accordance with the pattern of basins of attraction shown in Fig. 5(a) (two period-3 attractors labeled 3_1 and 3_2, and one period-5 attractor).

In the next step we have compared the above results for lifetimes of chaotic transients to the results of computations at fixed numerical precision. For comparison, two types of numerical experiments at fixed numerical precision were investigated:

(i) computations with a distribution of initial conditions;

(ii) computations at a fixed initial condition, with a noise term added to the driving force in the Duffing equation (1).

Computations of the type (i) were performed for an ensemble of the initial points uniformly distributed on a 25×25 grid within the square $1.999875 \leqslant x_0 \leqslant 2.000125$, $-0.000125 \leqslant \dot{x}_0 \leqslant 0.000125$ in the restraining region Γ, where $x_0 = 2$, $\dot{x}_0 = 0$. The corresponding lifetimes are presented in Fig. 2 for a precision of 10^{-8} (dashed line). The mean lifetime deduced from this distribution is $\langle\tau\rangle_{ic} = 2.10 \times 10^3$. (Here the subscript ic denotes that the mean lifetime is associated with a distribution of initial conditions). Thus, we obtain for the mean lifetimes

$$\langle\tau\rangle_{np} \approx \langle\tau\rangle_{ic}, \tag{3}$$

i.e. the resulting histogram of lifetimes for a numerical experiment with a distribution of numerical precisions at fixed initial condition cannot be distinguished from the numerical experiment with a distribution of initial conditions at fixed numerical precision. When a differential equation is integrated numerically, significant deviations of

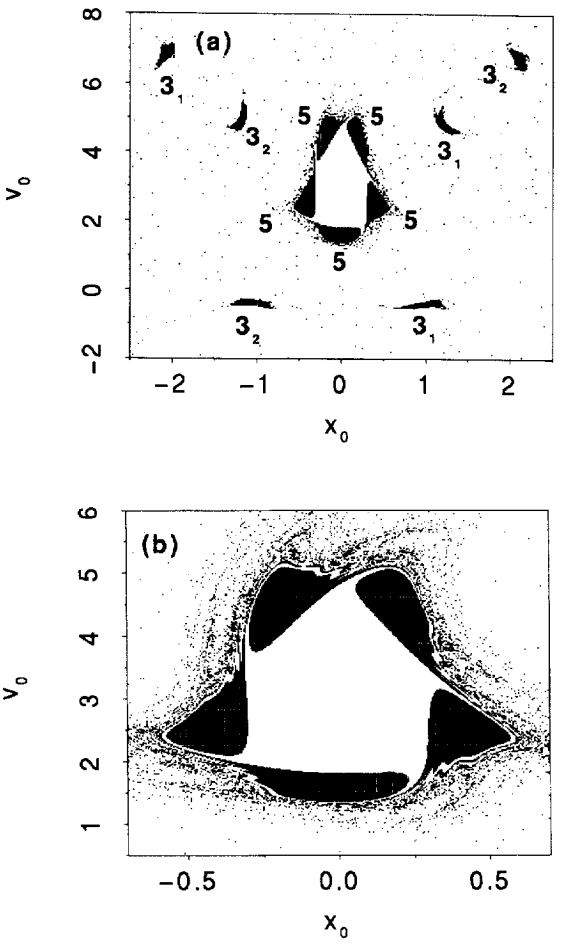

Fig. 5. (a) Basins of attraction for two attractors of period-3 and one of period-5. Computation was performed for initial points uniformly distributed on a 300×300 grid within the square $-2.5 < x_0 < 2.5$, $-2 < \dot{x}_0 < 8$. Each orbit corresponding to these initial conditions was followed until it enters the ε-surrounding of an attractor ($\varepsilon = 0.05$). If an orbit enters the ε-surrounding of an attractor during the time of less than 500 periods of the external driving, a small closed circle is drawn at the position of the corresponding initial point. Dark regions obtained in this way correspond to two attractors of period-3 (denoted by 3_1 and 3_2) and one attractor of period-5 (denoted by 5). (b) Magnification of the period-5 attractor with initial points uniformly distributed on a 300 × 300 grid within the square $-0.7 < x_0 < 0.7$, $0.5 < \dot{x}_0 < 6$.

computed ("noisy") trajectory from the true trajectory may appear for transiently chaotic case, similarly as for the steady-state chaos [19]. In the latter case, this feature is due to discretization errors of the mathematical algorithm and the rounding errors. These errors could be interpreted [20] as a great number of small perturbations, each causing a jump in trajectory. When such a short perturbation ends, the system is again governed by the same differential equation but starting from the changed initial values and travelling along a neighboring trajectory [20].

Suppose, as an extreme simplification, that the simulation made only one numerical error, on the first step, and that the equation was integrated exactly (with no error) for the remainder of the trajectory. The effect of a distribution of numerical precisions is that one step away from the sole initial condition, there is a distribution of different points (because of different numerical errors made), which could be considered as initial conditions for the rest of the trajectory. The effect of the temporary exponential divergence of trajectories seen in transient chaos is that the lifetimes of the transients fall into an exponential distribution, for essentially the same reason as if one started out with uniformly distributed initial conditions.

In a further step, we have performed the computations by adding on the r.h.s. of Eq. (1), the noise term $\alpha\eta(t)$, which has a random value for each integration step: in the integration step for a time interval between t and $t + \Delta t$, the noise is a constant force $\alpha\eta(t)$, where $\eta(t)$ is a random number in the interval [0,1] and the parameter α has a small value of 10^{-6}. In this computation the numerical precision was held fixed at the level of 10^{-9} and the initial condition had a fixed value $x_0 = 2$, $\dot{x}_0 = 0$. The resulting distribution of lifetimes is shown by the dot-dashed line in Fig. 2. As seen, this distribution, i.e. the corresponding mean lifetime $\langle\tau\rangle_{noise}$ is practically indistinguishable from the two previous numerical experiments,

$$\langle\tau\rangle_{noise} \approx \langle\tau\rangle_{ic}. \tag{4}$$

A similar result is obtained if the level of noise is varied, for example, by increasing the magnitude of noise to $\alpha = 10^{-4}$ and $\alpha = 10^{-2}$.

3. Conclusion

Lifetimes of chaotic transients were investigated numerically for a weakly dissipative Duffing system. It is shown that the sensitive dependence on the precision of the numerical algorithm, when the initial condition is kept fixed, results in an exponential distribution of lifetimes of chaotic transients, which is practically not distinguishable from the well-known exponential distribution obtained in computations with a distribution of initial conditions, when the precision of numerical algorithm is kept fixed. A similar distribution with the same mean lifetime is obtained if in each integration step a noise is added to the driving force, while keeping both the numerical precision and the initial condition fixed. In all three cases the same scaling is obtained. These results are in accordance with the expectation that the individual transiently chaotic trajectories depend sensitively on all details while the ensemble averages should be independent of these details.

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