# 5. zadaća iz Kvantne fizike

28. siječnja 2009.

## Zadatak 1

Varijacijskom metodom izračunati energije i valne funkcije osnovnog i prvog pobuđenog stanja vodikovog atoma. Koristiti probne valne funkcije:

$$\psi_{1s} = A e^{-\beta r} \quad , \quad \psi_{2s} = B\left(1+\gamma \frac{r}{a}\right) e^{-\frac{\alpha r}{a}}$$

Napomena: Za pobuđeno stanje koristiti ortogonalnost svojstvenih stanja $\langle \psi_{1s} | \psi_{2s} \rangle = 0$.

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Ortogonalnost $\psi_{1s}$ i $\psi_{2s}$ stanja...

$$\langle \psi_{1s} | \psi_{2s} \rangle = 0 =$$

$$= A^* B \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\varphi \left(1+\gamma\frac{r}{a}\right) e^{-(\beta+\frac{\alpha}{a})r} =$$

$$= 4\pi A^* B \left[ \int_0^{\infty} dr\, r^2 e^{-(\beta+\frac{\alpha}{a})r} + \frac{\gamma}{a} \int_0^{\infty} dr\, r^3 e^{-(\beta+\frac{\alpha}{a})r} \right] =$$

$$= 4\pi A^* B \left[ \frac{2a^3}{(\alpha+a\beta)^3} + \frac{\gamma}{a} \frac{6a^4}{(\alpha+a\beta)^4} \right] =$$

$$= \frac{8\pi A^* B a^3}{(\alpha+a\beta)^4} \left[ \alpha + a\beta + 3\gamma \right] = 0$$

$$\Rightarrow \quad \gamma = -\frac{1}{3}\left(\alpha + a\beta\right)$$

$$\Rightarrow \quad \psi_{2s} = B\left(1 - \frac{1}{3}\left(\frac{\alpha}{a} + \beta\right) r\right) e^{-\alpha \frac{r}{a}}$$

Normalizacija . . .

$$\langle \psi_{1s} | \psi_{1s} \rangle = 1 = 4\pi |A|^2 \int_0^\infty dr\, r^2 e^{-2\beta r} = \frac{\pi |A|^2}{\beta^3}$$

$$\longrightarrow \quad A = \left(\frac{\beta^3}{\pi}\right)^{1/2}$$

$$\longrightarrow \quad \underline{\psi_{1s} = \left(\frac{\beta^3}{\pi}\right)^{1/2} e^{-\beta r}}$$

$$\langle \psi_{2s} | \psi_{2s} \rangle = 1 = 4\pi |B|^2 \int_0^r dr\, r^2 \left(1 - \frac{1}{3}\left(\frac{\alpha}{a} + \beta\right) r\right)^2 e^{-2\alpha \frac{r}{a}} =$$

$$= 4\pi |B|^2 \left[ \int_0^\infty dr\, r^2 e^{-2\alpha \frac{r}{a}} - \frac{2}{3}\left(\frac{\alpha}{a} + \beta\right) \int_0^\infty dr\, r^3 e^{-2\alpha \frac{r}{a}} + \right.$$

$$\left. + \frac{1}{9}\left(\frac{\alpha}{a} + \beta\right)^2 \int_0^\infty dr\, r^4 e^{-2\alpha \frac{r}{a}} \right] =$$

$$= 4\pi |B|^2 \left[ \frac{a^3}{4\alpha^3} - \frac{2}{\cancel{3}}\left(\frac{\alpha}{a} + \beta\right) \times \frac{\cancel{3}}{8} \frac{a^4}{\alpha^4} + \frac{1}{9}\left(\frac{\alpha}{a} + \beta\right)^2 \times \frac{3}{4} \frac{a^5}{\alpha^5} \right] =$$

$$= 4\pi |B|^2 \left[ \cancel{\frac{a^3}{4\alpha^3}} - \cancel{\frac{a^3}{4\alpha^3}} - \frac{\beta}{4} \frac{a^4}{\alpha^4} + \frac{a^3}{12\alpha^3} + \frac{\beta}{6} \frac{a^4}{\alpha^4} + \frac{\beta^2}{12} \frac{a^5}{\alpha^5} \right] =$$

$$= \pi |B|^2 \frac{a^3}{3\alpha^3} \left[ 1 - \beta \frac{a}{\alpha} + \beta^2 \frac{a^2}{\alpha^2} \right] =$$

$$= \pi |B|^2 \frac{a^3}{3\alpha^5} \left[ \alpha^2 - \beta \alpha a + \beta^2 a^2 \right]$$

$$\rightarrow \quad B = \left( \frac{3\alpha^5}{\pi a^3 (\alpha^2 - \beta\alpha a + \beta^2 a^2)} \right)^{1/2}$$

$$\rightarrow \quad \psi_{2s} = \left( \frac{3\alpha^5}{\pi a^3 (\alpha^2 - \beta\alpha a + \beta^2 a^2)} \right) \left( 1 - \frac{1}{3}\left( \frac{\alpha}{a} + \beta \right) r \right) e^{-\alpha \frac{r}{a}}$$

$$\langle \psi_{1s} | H | \psi_{1s} \rangle = \langle \psi_{1s} | \frac{p^2}{2m} | \psi_{1s} \rangle - \langle \psi_{1s} | \frac{e^2}{r} | \psi_{1s} \rangle$$

$$\langle \psi_{1s} | \frac{p^2}{2m} | \psi_{1s} \rangle = -\frac{\hbar^2}{2m} \int dV \, \psi_{1s}^* \nabla^2 \psi_{1s} =$$

$$= -\frac{\hbar^2}{2m} 4\pi \frac{\beta^3}{\pi} \int_0^\infty dr \, \cancel{r^2} e^{-\beta r} \frac{1}{\cancel{r^2}} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) e^{-\beta r} =$$

$$= -\frac{2\hbar^2 \beta^3}{m} \int_0^\infty dr \left( -2\beta r \, e^{-2\beta r} + \beta^2 r^2 e^{-2\beta r} \right) =$$

$$= -\frac{2\hbar^2\beta^3}{m} \left( -2\beta \frac{1}{4\beta^2} + \beta^2 \frac{1}{4\beta^3} \right) = \frac{\hbar^2 \beta^2}{2m}$$

$$\langle \psi_{1s} | \frac{e^2}{r} | \psi_{1s} \rangle = 4\cancel{\pi} \frac{\beta^3}{\cancel{\pi}} e^2 \int_0^\infty dr \, r \, e^{-2\beta r} =$$

$$= 4\beta^3 e^2 \frac{1}{4\beta^2} = e^2 \beta$$

$$\Rightarrow \langle \psi_{1s} | H | \psi_{1s} \rangle = \frac{\hbar^2 \beta^2}{2m} - e^2 \beta$$

Varijacijski princip:

$$E_{1s} \leq \langle \psi_{1s} | H | \psi_{1s} \rangle$$

Tražimo minimum te granice po $\beta$ ...

$$\frac{d}{d\beta} \langle \psi_{1s} | H | \psi_{1s} \rangle = \frac{\hbar^2 \beta}{m} - e^2 = 0$$

$$\Rightarrow \underline{\underline{\beta = \frac{me^2}{\hbar^2} = \frac{1}{a}}}$$ a - Bohrov radijus

$$\Rightarrow \underline{\underline{E_{1s} \leq -\frac{me^4}{2\hbar^2}}} = \frac{-\hbar^2}{2ma^2}$$ ~ ovdje vrijedi jednakost jer smo pogodili točnu valnu funkciju

$$\underline{\underline{\psi_{1s} = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}}}$$

Pošto točno znamo osnovno stanje, možemo raditi varijacijski postupak za prvo pobuđeno stanje sa probnom valnom funkcijom ortogonalnom na $\psi_{1s}$.

$$\psi_{2s} = \left( \frac{3\alpha^5}{\pi a^3 (\alpha^2 - \alpha + 1)} \right)^{\frac{1}{2}} \left( 1 - \frac{\alpha + 1}{3a} r \right) e^{-\alpha \frac{r}{a}}$$

$$\langle \Psi_{2s} | H | \Psi_{2s} \rangle = \langle \Psi_{2s} | \frac{p^2}{2m} | \Psi_{2s} \rangle - \langle \Psi_{2s} | \frac{e^2}{r} | \Psi_{2s} \rangle$$

$$\frac{\partial}{\partial r} \Psi_{2s} = \left( \frac{3\alpha^5}{\pi a^3 (\alpha^2 - \alpha + 1)} \right)^{1/2} \left( - \frac{4\alpha + 1}{3a} + \frac{\alpha(\alpha+1)}{3a^2} r \right) e^{-\alpha \frac{r}{a}}$$

$$\frac{\partial^2}{\partial r^2} \Psi_{2s} = \left( \frac{3\alpha^5}{\pi a^3 (\alpha^2 - \alpha + 1)} \right)^{1/2} \left( \frac{\alpha(5\alpha+2)}{3a^2} - \frac{\alpha^2(\alpha+1)}{3a^3} r \right) e^{-\alpha \frac{r}{a}}$$

$$\langle \Psi_{2s} | \frac{p^2}{2m} | \Psi_{2s} \rangle =$$

$$= - \frac{\hbar^2}{2m} 4\pi \int_0^\infty dr\, r^2 \Psi_{2s} \left( \frac{\partial^2 \Psi_{2s}}{\partial r^2} + \frac{2}{r} \frac{\partial \Psi_{2s}}{\partial r} \right) \qquad \circledast^{\circ}$$

$$\int_0^\infty dr\, r^2 \sin\vartheta\, \Psi_{2s} \frac{\partial^2 \Psi_{2s}}{\partial r^2} =$$

$$= \frac{3\alpha^5}{\pi a^3 (\alpha^2 - \alpha + 1)} \int_0^\infty dr\, r^2 \left( 1 - \frac{\alpha+1}{3a} r \right) \left( \frac{\alpha(5\alpha+2)}{3a^2} - \frac{\alpha^2(\alpha+1)}{3a^3} r \right) e^{-2\alpha \frac{r}{a}}$$

$$= \frac{3\alpha^5}{\pi a^3 (\alpha^2 - \alpha + 1)} \left[ \frac{\alpha(5\alpha+2)}{3a^2} \int_0^\infty dr\, r^2 e^{-2\alpha \frac{r}{a}} - \right.$$

$$- \frac{2\alpha(\alpha+1)(4\alpha+1)}{9a^3} \int_0^\infty dr\, r^3 e^{-2\alpha \frac{r}{a}} +$$

$$\left. + \frac{\alpha^2(\alpha+1)^2}{9a^4} \int_0^\infty dr\, r^4 e^{-2\alpha \frac{r}{a}} \right] =$$

$$= \frac{3\alpha^5}{\pi a^3(\alpha^2-\alpha+1)} \Bigg[ \frac{\alpha(5\alpha+2)}{3a^2} \frac{a^3}{4\alpha^3} -$$

$$- \frac{2\alpha(\alpha+1)(4\alpha+1)}{9a^3} \frac{3a^4}{8\alpha^4} + \frac{\alpha^2(\alpha+1)^2}{9a^4} \frac{3a^5}{4\alpha^5} \Bigg] =$$

$$= \frac{3\alpha^5}{\pi a^3(\alpha^2-\alpha+1)} \Bigg[ \frac{a}{12\alpha^3}\left(5\alpha^2+2\alpha\right) - \frac{a}{12\alpha^3}\left(4\alpha^2+5\alpha+1\right) +$$

$$+ \frac{a}{12\alpha^3}\left(\alpha^2+2\alpha+1\right) \Bigg] =$$

(*)[1]

$$= \frac{3\alpha^5}{\pi a^3(\alpha^2-\alpha+1)} \frac{a}{12\alpha^3}\left(2\alpha^2-\alpha\right) = \underline{\underline{\frac{\alpha^3(2\alpha-1)}{4\pi a^2(\alpha^2-\alpha+1)}}}$$

$$\int_0^\infty dr\, r^2 \psi_{2s} \frac{2}{r} \frac{\partial \psi_{2s}}{\partial r} = 2\int_0^\infty dr\, r\, \psi_{2s} \frac{\partial \psi_{2s}}{\partial r} =$$

$$= \frac{6\alpha^5}{\pi a^3(\alpha^2-\alpha+1)} \int_0^\infty dr\, r\left(1-\frac{\alpha+1}{3a}r\right)\left(-\frac{4\alpha+1}{3a}+\frac{\alpha(\alpha+1)}{3a^2}r\right)e^{-2\alpha\frac{r}{a}} =$$

$$= \frac{6\alpha^5}{\pi a^3(\alpha^2-\alpha+1)} \Bigg[ -\frac{4\alpha+1}{3a}\int_0^\infty dr\, r\, e^{-2\alpha\frac{r}{a}}$$

$$+ \frac{(\alpha+1)(7\alpha+1)}{9a^2}\int_0^\infty dr\, r^2 e^{-2\alpha\frac{r}{a}} - \frac{\alpha(\alpha+1)^2}{9a^3}\int_0^\infty dr\, r^3 e^{-2\alpha\frac{r}{a}} \Bigg] =$$

$$= \frac{6\alpha^5}{\pi a^3(\alpha^2-\alpha+1)}\left[-\frac{4\alpha+1}{3a}\frac{a^2}{4\alpha^2}+\frac{(\alpha+1)(7\alpha+1)}{9a^2}\frac{a^3}{4\alpha^3}-\right.$$

$$\left.-\frac{\alpha(\alpha+1)^2}{9a^3}\frac{3a^4}{8\alpha^4}\right] =$$

$$= \frac{6\alpha^5}{\pi a^3(\alpha^2-\alpha+1)}\left[-\frac{a}{72\alpha^3}(24\alpha^2+6\alpha)+\right.$$

$$\left.+\frac{a}{72\alpha^3}(14\alpha^2+16\alpha+2)-\frac{a}{72\alpha^3}(3\alpha^2+6\alpha+3)\right] =$$

$$= \frac{6\alpha^5}{\pi a^3(\alpha^2-\alpha+1)}\frac{a}{72\alpha^3}(-13\alpha^2+4\alpha-1) =$$

$$= \frac{-\alpha^2(13\alpha^2-4\alpha+1)}{12\pi a^2(\alpha^2-\alpha+1)} \qquad (*)^2$$

Uvrstimo $(*)^1$ i $(*)^2$ u $(*)^0$:

$$\langle \psi_{2s}|\frac{p^2}{2m}|\psi_{2s}\rangle =$$

$$= -\frac{2\pi\hbar^2}{m}\left(\frac{\alpha^3(2\alpha-1)}{4\pi a^2(\alpha^2-\alpha+1)}-\frac{\alpha^2(13\alpha^2-4\alpha+1)}{12\pi a^2(\alpha^2-\alpha+1)}\right) =$$

$$= \frac{\hbar^2\alpha^2(7\alpha^2-\alpha+1)}{6ma^2(\alpha^2-\alpha+1)} \qquad (*)^3$$

$$\langle \psi_{2s} | \frac{e^2}{r} | \psi_{2s} \rangle = 4\pi e^2 \int_0^\infty dr\, r\, \psi_{2s}^{\ 2} =$$

$$= \frac{12\alpha^5 e^2}{a^3(\alpha^2-\alpha+1)} \int_0^\infty dr\, r \left(1 - \frac{\alpha+1}{3a} r\right)^2 e^{-2\alpha \frac{r}{a}} =$$

$$= \frac{12\alpha^5 e^2}{a^3(\alpha^2-\alpha+1)} \left[ \int_0^\infty dr\, r\, e^{-2\alpha\frac{r}{a}} - \right.$$

$$\left. - \frac{2(\alpha+1)}{3a} \int_0^\infty dr\, r^2 e^{-2\alpha \frac{r}{a}} + \frac{(\alpha+1)^2}{9a^2} \int_0^\infty dr\, r^3 e^{-2\alpha\frac{r}{a}} \right] =$$

$$= \frac{12\alpha^5 e^2}{a^3(\alpha^2-\alpha+1)} \left[ \frac{a^2}{4\alpha^2} - \frac{2(\alpha+1)}{3a}\frac{a^3}{4\alpha^3} + \frac{(\alpha+1)^2}{9a^2}\frac{3a^4}{8\alpha^4} \right]$$

$$= \frac{12\alpha^5 e^2}{a^3(\alpha^2-\alpha+1)} \frac{a^2}{24\alpha^4} (3\alpha^2 - 2\alpha + 1) =$$

$$= \frac{e^2 \alpha (3\alpha^2 - 2\alpha + 1)}{2a(\alpha^2-\alpha+1)} = \underline{\underline{\frac{\hbar^2 \alpha (3\alpha^2-2\alpha+1)}{2ma^2(\alpha^2-\alpha+1)}}} \qquad \circledast^4$$

Iz $\circledast^3$ i $\circledast^4$ slijedi:

$$E_{2s}(\alpha) = \langle \psi_{2s} | H | \psi_{2s} \rangle =$$

$$= \langle \psi_{2s} | \frac{p^2}{2m} | \psi_{2s} \rangle - \langle \psi_{2s} | \frac{e^2}{r} | \psi_{2s} \rangle =$$

$$= \frac{\hbar^2 \alpha^2 (7\alpha^2 - \alpha + 1)}{6ma^2(\alpha^2 - \alpha + 1)} - \frac{\hbar^2 \alpha (3\alpha^2 - 2\alpha + 1)}{2ma^2(\alpha^2 - \alpha + 1)} =$$

$$= \frac{\hbar^2 \alpha}{6ma^2(\alpha^2 - \alpha + 1)} \left( 7\alpha^3 - \alpha^2 + \alpha - 9\alpha^2 + 6\alpha - 3 \right) =$$

$$= \frac{\hbar^2}{2ma^2} \frac{\alpha (7\alpha^3 - 10\alpha^2 + 7\alpha - 3)}{3(\alpha^2 - \alpha + 1)}$$

Sada treba naći $\alpha$ koji zadovoljava

$\frac{d}{d\alpha} E_{2s}(\alpha) = 0$.

Račun je poprilično dugačak pa ćemo raditi u Mathematici:

$$\frac{d}{d\alpha} E_{2s}(\alpha) = \frac{\hbar^2}{2ma^2} \left[ -1 + \frac{14\alpha}{3} - \frac{1+\alpha}{(\alpha^2 - \alpha + 1)^2} + \frac{1}{\alpha^2 - \alpha + 1} \right]$$

Gornja funkcija ima nekoliko nultočaka. Pokazuje se da za nultočku $\underline{\underline{\alpha = \frac{1}{2}}}$ funkcija $E_{2s}(\alpha)$ ima globalni minimum.

To vodi na gornju granicu za energiju:

$$E_{2s} \leq -\frac{1}{4}\frac{\hbar^2}{2ma^2}$$

Međutim, to je točna energija prvog pobuđenog stanja sa $l=0$ i $m=0$.

To je očito zato što smo pogodili pravi funkcijski oblik varijacijske funkcije, za koju dobijemo:

$$\psi_{2s}(r) = \frac{1}{\sqrt{8\pi a^3}}\left(1-\frac{r}{2a}\right)e^{-r/2a}$$

što je upravo točna valna funkcija za 2s stanje.

## Zadatak 2

Promotrite česticu koja se nalazi u 1D potencijalu $V(x) = \lambda x^4$. Koristeći varijacijsku metodu sa probnom funkcijom

$$\psi = \mathcal{N} e^{-\alpha x^2},$$

aproksimirajte energiju osnovnog stanja. Usporedite dobivenu vrijednost sa egzaktnim rješenjem $E_0 = 1.06 \frac{\hbar^2}{2m} k^{1/3}$, gdje je $k = \frac{2m\lambda}{\hbar^2}$.

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Normalizacija...

$$1 = |\mathcal{N}|^2 \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} = |\mathcal{N}|^2 \sqrt{\frac{\pi}{2\alpha}} \longrightarrow \mathcal{N} = \left(\frac{2\alpha}{\pi}\right)^{1/4}$$

Varijacijska funkcija $\underline{\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2}}$

$$E_0 \leq \langle \psi | H | \psi \rangle =$$

$$= \int_{-\infty}^{\infty} dx \left( -\frac{\hbar^2}{2m} \psi^* \frac{d^2\psi}{dx^2} + \lambda \psi^* x^4 \psi \right)$$

$$= \left(\frac{2\alpha}{\pi}\right)^{1/2} \left( \frac{\alpha\hbar^2}{m} \int_{-\infty}^{\infty} dx\, e^{-2\alpha x^2} - \frac{2\alpha^2\hbar^2}{m} \int_{-\infty}^{\infty} dx\, x^2 e^{-2\alpha x^2} + \lambda \int_{-\infty}^{\infty} dx\, x^4 e^{-2\alpha x^2} \right) =$$

$$= \cancel{\left(\frac{2\alpha}{\pi}\right)^{1/2}} \left( \frac{\alpha\hbar^2}{m} \cancel{\left(\frac{\pi}{2\alpha}\right)^{1/2}} - \frac{\alpha\hbar^2}{2m} \cancel{\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}}} + \frac{3\lambda}{16\alpha^2} \cancel{\left(\frac{\pi}{2\alpha}\right)^{\frac{1}{2}}} \right) =$$

$$= \underline{\frac{\alpha\hbar^2}{2m} + \frac{3\lambda}{16\alpha^2}}$$

Tražimo minimum granice po $\alpha$ ...

$$\frac{d}{d\alpha}\langle\psi|H|\psi\rangle = \frac{\hbar^2}{2m} - \frac{3\lambda}{8\alpha^3} = 0$$

$$\Rightarrow \quad \alpha = \left(\frac{3\lambda m}{4\hbar^2}\right)^{1/3}$$

$$\Rightarrow \quad E_0 \leq \frac{\hbar^2}{2m}\left(\frac{3\lambda m}{4\hbar^2}\right)^{1/3} + \frac{3\lambda}{16}\left(\frac{4\hbar^2}{3\lambda m}\right)^{2/3} =$$

$$= \frac{\hbar^2}{2m}\left(\frac{2m\lambda}{\hbar^2}\right)^{1/3}\left[\left(\frac{3}{8}\right)^{1/3} + \frac{1}{2}\left(\frac{3}{8}\right)^{1/3}\right] =$$

$$= 1.08\,\frac{\hbar^2}{2m}\,k^{1/3}$$

Gornja granica koju smo dobili je vrlo blizu točne energije osnovnog stanja $1.06\,\frac{\hbar^2}{2m}k^{1/3}$.

## Zadatak 3

Pomoću WKB pristupa odrediti valnu funkciju asimptotski za $x \to \infty$ za sustav opisan Hamiltonijanom $H = -\frac{d^2}{dx^2} + x^2 + x^4$

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Imamo gotovu formulu za WKB valnu funkciju daleko od klasičnih točaka obrata:

$$(*) \quad \psi(x) = \frac{A}{\sqrt{|K(x)|}} \exp\left[\int_{x_0}^{x} K(x')dx'\right] + \frac{B}{\sqrt{|K(x)|}} \exp\left[-\int_{x_0}^{x} K(x')dx'\right]$$

Za ovaj bezdimenzionalni slučaj je:

$$K(x) = \sqrt{V(x) - \varepsilon} \quad , \text{ gdje je } V(x) = x^2 + x^4$$

Za $x \to \infty$ je:

$$K(x) = \sqrt{x^2 + x^4 - \varepsilon} = x^2 \sqrt{1 + \frac{1}{x^2} - \frac{\varepsilon}{x^4}} \approx x^2$$

$$\int_{x_0}^{x} K(x')dx' \approx \int_{x_0}^{x} x'^2 dx' = \frac{x^3}{3} - c$$

U $(*)$ moramo uzeti $A=0$ zbog konačnosti $\psi(x \to \infty)$

$$\Rightarrow \psi(x \to \infty) \sim \frac{1}{x} \exp\left(-\frac{x^3}{3}\right)$$

## Zadatak 4

Izračunati poluklasičnu aproksimaciju transmisijskog koeficijenta kroz barijeru oblika:

$$U(x) = \begin{cases} 0 & , \ |x| > a \\ U_0\left(1 - \frac{|x|}{a}\right) & , \ |x| < a \end{cases}$$

Odrediti pod kojim uvjetima je poluklasična aproksimacija valjana.

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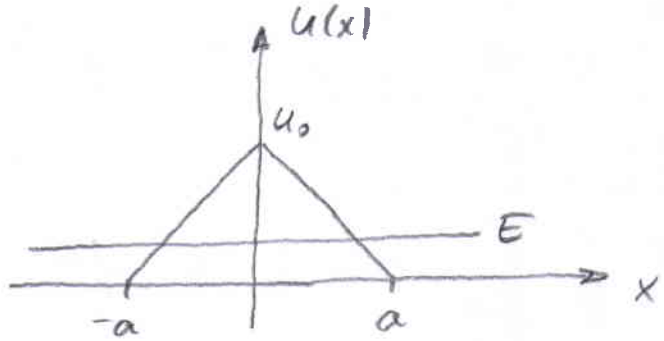
Koeficijent transmisije u WKB aproksimaciji:

$$T = \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2} dx\, p(x)\right] \quad , \quad p(x) = \sqrt{2m\left(U(x) - E\right)}$$

$x_1, x_2$ – klasične točke obrata.




Točke obrata:

$$E = U(x_1) = U_0\left(1 + \frac{x_1}{a}\right) \longrightarrow x_1 = -a\left(1 - \frac{E}{U_0}\right) = -x_0$$

$$E = U(x_2) = U_0\left(1 - \frac{x_2}{a}\right) \longrightarrow x_2 = a\left(1 - \frac{E}{U_0}\right) = x_0$$

$$\int_{-x_0}^{x_0} dx \sqrt{U(x)-E} = 2\int_0^{x_0} dx \sqrt{U(x)-E} =$$

$$= 2\int_0^{x_0} dx \sqrt{U_0\left(1-\frac{x}{a}\right)-E} = 2\int_0^{x_0} dx \sqrt{-\frac{U_0}{a}x + (U_0-E)} =$$

$$= 2 \cdot \frac{2}{3}\left(-\frac{a}{U_0}\right)\left(-\frac{U_0}{a}x + (U_0-E)\right)^{3/2}\Bigg|_0^{x_0} =$$

$$= \frac{4a}{3}\,\frac{(U_0-E)^{3/2}}{U_0}$$

$$\Longrightarrow \quad T = \exp\left[-\frac{8a\sqrt{m}}{3\hbar}\,\frac{(U_0-E)^{3/2}}{U_0}\right]$$

WKB aproksimacija vrijedi kada je energija upadne čestice puno manja od tipične visine barijere, u što se možemo uvjeriti ako primjetimo da $T \to 1$ za $E \to U_0$, što je krivo.