Total vs partial derivative

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One small detail, which is often a source of confusion among the students (and possibly some not-so-young colleagues), is difference between the total derivative, denoted by the upright d, and partial derivatives, denoted by the fancy symbol ∂.

Basic notational convention

Let's start with the proper definition of this notation. For example, if we have a differentiable real function $f : \mathbb{R} \to \mathbb{R}$, then its derivative at some point $x_0 \in \mathbb{R}$ is denoted by

$$\frac{\mathrm{d}f}{\mathrm{d}x}\,(x_0)\,.$$

On the other hand, suppose we have a higher dimensional domain, an Euclidean space $\mathbb{R}^m$ with $m \geq 2$, coordinates $\{x^1, \dots, x^m\}$ and a differentiable function $F : \mathbb{R}^m \to \mathbb{R}$. Then its partial derivatives with respect to these coordinates at some point $p \in \mathbb{R}^m$ are denoted by

$$\frac{\partial F}{\partial x^i}\,(p)\,,$$

with $i \in \{1, \dots, m\}$. Note that the usage of ∂ instead of d in the latter case just serves to emphasize that F is in fact a function of several variables.

Setting an example

We shall now take very simple example from the classical mechanics to illustrate the main point that we would like to communicate. Suppose that we are looking at some point particle, moving through the Euclidean space $\mathbb{R}^3$. Instead of drawing its trajectory as a curve in space $\mathbb{R}^3$, we might add the time coordinate in our picture, draw the point's "world line" through the spacetime $\mathbb{R}^4$. As is often the case, we may denote the coordinates on $\mathbb{R}^4$ with $\{x^0 = t, x^1, x^2, x^3\}$. Just to keep everything tidy we shall introduce a formal set of maps $r^k : \mathbb{R}^4 \to \mathbb{R}$ for each $k \in \{0, 1, 2, 3\}$, such that for any point $p(x^0, x^1, x^2, x^3)$ we have $r^k(p) = x^k$. In other words, each map r^k is just "reading off" the coordinates of the given point p.

The world line of our point particle is a differentiable map $\gamma : \mathbb{R} \to \mathbb{R}^4$. The variable on the domain of γ will be denoted as λ, and referred to as the *parameter* of the curve. If we want to add the constraint that this particle cannot travel back and forth through time, we may assume in addition that the map $r^0 \circ \gamma$ is monotone, but this doesn't affect our discussion.

How to degrade notation

Now, let us introduce a differentiable map $A : \mathbb{R}^4 \to \mathbb{R}$, which shall represent some physical observable in our spacetime. We are interested how this observable changes *along the world line* of our particle. The chain rule at some point $\lambda = \lambda_0$ then reads

$$\frac{\mathrm{d}(A \circ \gamma)}{\mathrm{d}\lambda}\,(\lambda_0) = \sum_{k=0}^{3} \frac{\partial A}{\partial x^k}\,(\gamma(\lambda_0))\, \frac{\mathrm{d}(r^k \circ \gamma)}{\mathrm{d}\lambda}\,(\lambda_0)\,.$$

Note that we are very careful here with the notation used for the derivatives. For example, compositions $A \circ \gamma : \mathbb{R} \to \mathbb{R}$ and $r^k \circ \gamma : \mathbb{R} \to \mathbb{R}$ are functions of one variable, hence d is the appropriate choice, whereas A is a function of several variables, so that we use ∂.

A special, but most often used parametrization is the one corresponding to $(r^0 \circ \gamma)(\lambda) = \lambda$. In this case, the partial derivatives on the right hand side can be separated on time and space derivatives,

$$\frac{\mathrm{d}(A \circ \gamma)}{\mathrm{d}\lambda}\,(\lambda_0) = \frac{\partial A}{\partial t}\,(\gamma(\lambda_0)) + \sum_{k=1}^{3} \frac{\partial A}{\partial x^k}\,(\gamma(\lambda_0))\, \frac{\mathrm{d}(r^k \circ \gamma)}{\mathrm{d}\lambda}\,(\lambda_0)\,.$$

So far so good. What people mostly do at this point is that they write t instead of λ since we have already made the choice that $r^0 \circ \gamma : \mathbb{R} \to \mathbb{R}$ is the identity map (i.e. the t component of γ is exactly λ). Thus, this equality is written as

$$\frac{\mathrm{d}(A \circ \gamma)}{\mathrm{d}t}\,(t_0) = \frac{\partial A}{\partial t}\,(\gamma(t_0)) + \sum_{k=1}^{3} \frac{\partial A}{\partial x^k}\,(\gamma(t_0))\, \frac{\mathrm{d}(r^k \circ \gamma)}{\mathrm{d}t}\,(t_0)\,.$$

Then, to make things even more "muddled", some additional notational surgery is often performed at this point:

- composition $A \circ \gamma$ is abbreviated as "A" (causing a doppelgänger confusion with the function $A : \mathbb{R}^4 \to \mathbb{R}$),
- function $x^k = r^k \circ \gamma$ is introduced (causing a doppelgänger confusion with the coordinate x^k), and finally
- arguments of the functions are left over.

All this results in the equation which now takes the following form

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + \sum_{k=1}^{3} \frac{\partial A}{\partial x^k}\, \frac{\mathrm{d}x^k}{\mathrm{d}t}\,.$$

Presented with just this results the readers are (rightfully so) baffled by the difference between the total derivative on the left hand side of the equation, $\mathrm{d}A/\mathrm{d}t$, and the partial derivative on the right hand side of the equation, $\partial A/\partial t$.

Pedagogically disastrous "explanation" often presented at this point is that $\partial A/\partial t$ captures the "explicit dependence" of A with respect to t, whereas $\mathrm{d}A/\mathrm{d}t$ captures the "total change" of A with respect to t, whatever that might mean.

Resolution

We can now see clearly what is going on here. First of all, instead of one function "A" we have in fact two completely different functions in this story, $A : \mathbb{R}^4 \to \mathbb{R}$ and $A \circ \gamma : \mathbb{R} \to \mathbb{R}$. Furthermore, the derivative $\mathrm{d}/\mathrm{d}t$ on the left hand side is with respect to the *parameter* of the curve (just look at the original, properly written form of the equation), which happens to be equal to the *coordinate* t, due to our specific (convenient) choice of parametrization. On the other hand, the derivative $\partial/\partial t$ on the right hand side of the equation is with respect to one of the *coordinates* (namely, t) in the domain of the function $A : \mathbb{R}^4 \to \mathbb{R}$.

Coda

Abbreviations of the tedious, formal mathematical expressions are never an issue, given that the notation has been properly explained. A great deal of confusion can be avoided simply by explicit definitions of all the maps, together with their domains and codomains, involved in the equations.