Differential forms

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M is a smooth m-manifold with a metric g_{ab} which has s negative eigenvalues. Metric determinant is denoted by $g = \det(g_{\mu\nu})$.

- Base of p-forms via **wedge product**,

$$\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} = \sum_{\pi \in S_p} \mathrm{sgn}(\pi)\, \mathrm{d}x^{\mu_{\pi(1)}} \otimes \cdots \otimes \mathrm{d}x^{\mu_{\pi(p)}}$$

- General p-form $\omega \in \Omega^p$,

$$\boldsymbol{\omega} = \frac{1}{p!}\, \omega_{\mu_1 \ldots \mu_p}\, \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$$

- Unless otherwise stated, we assume:

$$f \in \Omega^0\,;\ \boldsymbol{\alpha}, \boldsymbol{\omega} \in \Omega^p\,;\ \boldsymbol{\beta} \in \Omega^q\,;\ \boldsymbol{\gamma} \in \Omega^{p-1}\,;\ X, Y \in TM$$

- **Generalized Kronecker delta**

$$\delta^{a_1 \cdots a_n}_{b_1 \cdots b_n} := n!\, \delta_{b_1}{}^{[a_1}\, \delta_{b_2}{}^{a_2} \cdots \delta_{b_n}{}^{a_n]}$$

$$\det(A) := n!\, A^1{}_{[1} A^2{}_2 \cdots A^n{}_{n]} = A^1{}_{\mu_1} \cdots A^n{}_{\mu_n}\, \delta^{\mu_1 \cdots \mu_n}_{1 \cdots n}$$

$$n!\, A^1{}_{[\mu_1} A^2{}_{\mu_2} \cdots A^n{}_{\mu_n]} = \det(A)\, \delta^{1 \cdots n}_{\mu_1 \cdots \mu_n}$$

$$\delta^{a_1 \cdots a_k a_{k+1} \cdots a_n}_{a_1 \cdots a_k b_{k+1} \cdots b_n} = \binom{m-n+k}{k} k!\, \delta^{a_{k+1} \cdots a_n}_{b_{k+1} \cdots b_n}$$

- **Volume form** $\boldsymbol{\epsilon} \in \Omega^m$, $\boldsymbol{\epsilon} := \sqrt{|g|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$

$$\epsilon_{\mu_1 \cdots \mu_m} = \sqrt{|g|}\, \delta^{1 \cdots m}_{\mu_1 \cdots \mu_m}\,, \quad \epsilon^{\mu_1 \cdots \mu_m} = \frac{(-1)^s}{\sqrt{|g|}}\, \delta^{\mu_1 \cdots \mu_m}_{1 \cdots m}$$

$$\epsilon^{a_1 \cdots a_k a_{k+1} \cdots a_m}\, \epsilon_{a_1 \cdots a_k b_{k+1} \cdots b_m} = (-1)^s\, k!\, \delta^{a_{k+1} \cdots a_m}_{b_{k+1} \cdots b_m}$$

Operators on differential forms

- **Contraction with vector** $i_X : \Omega^p \to \Omega^{p-1}$

$$i_X f = 0\,, \quad (i_X \boldsymbol{\omega})(X_1, \ldots, X_{p-1}) = \boldsymbol{\omega}(X, X_1, \ldots, X_{p-1})$$

- **Hodge dual** $\star : \Omega^p \to \Omega^{m-p}$

$$\star\omega = \frac{\omega_{\mu_1 \ldots \mu_p}}{p!(m-p)!}\, \epsilon^{\mu_1 \ldots \mu_p}{}_{\mu_{p+1} \ldots \mu_m} \mathrm{d}x^{\mu_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{\mu_m}$$

$$\star 1 = \boldsymbol{\epsilon}\,, \quad \star\boldsymbol{\epsilon} = (-1)^s\,, \quad i_X \boldsymbol{\epsilon} = \star X$$

- **Exterior derivative** $\mathrm{d} : \Omega^p \to \Omega^{p+1}$

$$\mathrm{d}\boldsymbol{\omega} = \frac{1}{p!}\, \partial_\sigma \omega_{\mu_1 \ldots \mu_p}\, \mathrm{d}x^\sigma \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$$

- **Lie derivative** $\pounds_X : \Omega^p \to \Omega^p$

$$\pounds_X \boldsymbol{\omega} = (i_X \mathrm{d} + \mathrm{d} i_X)\boldsymbol{\omega}$$

- **Coderivative** δ (or $\mathrm{d}^\dagger$) $: \Omega^p \to \Omega^{p-1}$

$$\delta f = 0\,, \quad \delta\boldsymbol{\omega} = (-1)^{m(p+1)+s} \star \mathrm{d} \star \boldsymbol{\omega}$$

- **Laplace–Beltrami operator (Laplacian)** $\Delta : \Omega^p \to \Omega^p$

$$\Delta = \mathrm{d}\delta + \delta\mathrm{d} = (\mathrm{d} + \delta)^2$$

Abstract index notation

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})_{a_1 \ldots a_p b_1 \ldots b_q} = \frac{(p+q)!}{p!q!}\, \alpha_{[a_1 \ldots a_p} \beta_{b_1 \ldots b_q]}$$

$$(\star\boldsymbol{\omega})_{a_{p+1} \ldots a_m} = \frac{1}{p!}\, \omega_{a_1 \ldots a_p} \epsilon^{a_1 \ldots a_p}{}_{a_{p+1} \ldots a_m}$$

$$(i_X \boldsymbol{\omega})_{a_1 \ldots a_{p-1}} = X^b \omega_{b a_1 \ldots a_{p-1}}$$

$$(\mathrm{d}\boldsymbol{\omega})_{a_1 \ldots a_{p+1}} = (p+1) \nabla_{[a_1} \omega_{a_2 \ldots a_{p+1}]}$$

$$(\delta\boldsymbol{\omega})_{a_1 \ldots a_{p-1}} = \nabla^b \omega_{b a_1 \ldots a_{p-1}}$$

Thm. $\star\star = (-1)^{p(m-p)+s}\,, \quad i_X^2 = \mathrm{d}^2 = \delta^2 = 0$

Thm. $\pounds_X (f\boldsymbol{\epsilon}) = \delta(f\mathbf{X})\, \boldsymbol{\epsilon}$

Thm. $\delta(f\boldsymbol{\alpha}) = i_X \boldsymbol{\alpha} + f\delta\boldsymbol{\alpha}$ with $X^a = \nabla^a f$

Thm. $i_X \mathrm{d}\star\boldsymbol{\alpha} = \star(\mathbf{X} \wedge \delta\boldsymbol{\alpha})$

Leibniz

$$i_X(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (i_X \boldsymbol{\alpha}) \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge (i_X \boldsymbol{\beta})$$

$$\mathrm{d}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (\mathrm{d}\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge (\mathrm{d}\boldsymbol{\beta})$$

$$\pounds_X(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (\pounds_X \boldsymbol{\alpha}) \wedge \boldsymbol{\beta} + \boldsymbol{\alpha} \wedge (\pounds_X \boldsymbol{\beta})$$

Commuting

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = (-1)^{pq} \boldsymbol{\beta} \wedge \boldsymbol{\alpha}$$

$$i_X \star\boldsymbol{\alpha} = \star(\boldsymbol{\alpha} \wedge \mathbf{X})\,, \quad i_X \boldsymbol{\alpha} = (-1)^{m(p+1)+s} \star(\mathbf{X} \wedge \star\boldsymbol{\alpha})$$

$$[\pounds_X, \pounds_Y] = \pounds_{[X,Y]}\,, \quad [\pounds_X, i_Y] = i_{[X,Y]}\,, \quad [\pounds_X, \mathrm{d}] = 0$$

$$\pounds_X \star\boldsymbol{\alpha} - \star\pounds_X \boldsymbol{\alpha} = (\delta\mathbf{X}) \star\boldsymbol{\alpha} + \star\widehat{\boldsymbol{\alpha}} \qquad \text{with}$$

$$\widehat{\alpha}_{a_1 \ldots a_p} := p(\pounds_X g^{bc}) g_{b[a_1} \alpha_{|c|a_2 \ldots a_p]}$$

$$\mathrm{d}\star = (-1)^{p+1} \star \delta\,, \quad \delta\star = (-1)^p \star \mathrm{d}$$

$$\star\Delta = \Delta\star\,, \quad \mathrm{d}\Delta = \Delta\mathrm{d}\,, \quad \delta\Delta = \Delta\delta$$

Inner products

$$(\boldsymbol{\alpha} \mid \boldsymbol{\omega}) := \frac{1}{p!}\, \alpha_{a_1 \ldots a_p} \omega^{a_1 \ldots a_p}$$

Normalization of the volume form, $(\boldsymbol{\epsilon} \mid \boldsymbol{\epsilon}) = (-1)^s$.

Thm. $(\boldsymbol{\alpha} \mid \boldsymbol{\omega})\, \boldsymbol{\epsilon} = \boldsymbol{\alpha} \wedge \star\boldsymbol{\omega} = \boldsymbol{\omega} \wedge \star\boldsymbol{\alpha} = (-1)^s\, (\star\boldsymbol{\alpha} \mid \star\boldsymbol{\omega})\, \boldsymbol{\epsilon}$

Thm. $(\mathbf{X} \wedge \boldsymbol{\gamma} \mid \boldsymbol{\alpha}) = (\boldsymbol{\gamma} \mid i_X \boldsymbol{\alpha})$

Thm. $(\mathbf{X} \mid \mathbf{Y})(\boldsymbol{\alpha} \mid \boldsymbol{\alpha}) = (-1)^s\, (i_X \star\boldsymbol{\alpha} \mid i_Y \star\boldsymbol{\alpha}) + (i_X \boldsymbol{\alpha} \mid i_Y \boldsymbol{\alpha})$

$$\langle \boldsymbol{\alpha}, \boldsymbol{\omega} \rangle := \int_M \boldsymbol{\alpha} \wedge \star\boldsymbol{\omega}$$

Thm. If $s = 0$ then $\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle \geq 0$ and $\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle = 0$ iff $\boldsymbol{\alpha} = 0$.

Thm. If $s = 0$ then $-\Delta$ is a positive operator in a sense that

$$\langle \boldsymbol{\omega}, -\Delta\boldsymbol{\omega} \rangle = \langle \mathrm{d}\boldsymbol{\omega}, \mathrm{d}\boldsymbol{\omega} \rangle + \langle \delta\boldsymbol{\omega}, \delta\boldsymbol{\omega} \rangle \geq 0$$

Consequently $\Delta\boldsymbol{\omega} = 0$ iff $\mathrm{d}\boldsymbol{\omega} = 0$ and $\delta\boldsymbol{\omega} = 0$.

Pullback

For a C^∞ map $\phi : S \to M$, we define $\phi^* : \Omega^p(M) \to \Omega^p(S)$,

$$(\phi^*\boldsymbol{\omega})(X_1,\dots,X_p) := \boldsymbol{\omega}(\phi_* X_1,\dots,\phi_* X_p)$$

$$(\phi^*\boldsymbol{\omega})_{\sigma_1\dots\sigma_p}|_s = \frac{\partial(x^{\mu_1}\circ\phi)}{\partial y^{\sigma_1}}\Big|_s \cdots \frac{\partial(x^{\mu_p}\circ\phi)}{\partial y^{\sigma_p}}\Big|_s \omega_{\mu_1\dots\mu_p}|_{\phi(s)}$$

$$\phi^*(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) = (\phi^*\boldsymbol{\alpha})\wedge(\phi^*\boldsymbol{\beta})\ , \quad \phi^*\mathrm{d} = \mathrm{d}\phi^*$$

N.B. $\phi^*(\star\omega) \neq \star(\phi^*\omega)$

Stokes's theorem. Let M be a smooth orientable m-manifold boundary ∂M and inclusion $\imath : \partial M \to M$. Then for any compactly supported $\boldsymbol{\omega} \in \Omega^{m-1}(M)$,

$$\int_M \mathrm{d}\boldsymbol{\omega} = \int_{\partial M} \imath^*\boldsymbol{\omega}$$

Corollary $\langle \mathrm{d}\boldsymbol{\gamma},\boldsymbol{\alpha}\rangle + \langle\boldsymbol{\gamma},\delta\boldsymbol{\alpha}\rangle = \int_{\partial M}\boldsymbol{\gamma}\wedge\star\boldsymbol{\alpha}$

Green's identity for $\boldsymbol{\alpha},\boldsymbol{\beta} \in \Omega^p(M)$

$$\int_M (\Delta\boldsymbol{\alpha}\wedge\star\boldsymbol{\beta} - \Delta\boldsymbol{\beta}\wedge\star\boldsymbol{\alpha}) = \int_{\partial M}(\delta\boldsymbol{\alpha}\wedge\star\boldsymbol{\beta} - \delta\boldsymbol{\beta}\wedge\star\boldsymbol{\alpha} + \boldsymbol{\beta}\wedge\star\mathrm{d}\boldsymbol{\alpha} - \boldsymbol{\alpha}\wedge\star\mathrm{d}\boldsymbol{\beta})$$

Frobenius's theorem. Let M be a smooth m-manifold and $\mathcal{D}$ a k-dim distribution (subbundle of TM), defined by $(m-k)$ linearly independent 1-forms $\{\boldsymbol{\theta}^{(1)},\dots,\boldsymbol{\theta}^{(m-k)}\}$, so that at each point $p \in M$ we have $\mathcal{D}|_p = \bigcap_i \mathrm{Ker}(\boldsymbol{\theta}^{(i)}|_p)$. Then the following conditions are equivalent

- $\mathcal{D}$ is involutive, $(\forall\, X, Y \in \mathcal{D}) :\ [X,Y] \in \mathcal{D}$
- $\mathcal{D}$ is totally integrable: for any $p \in M$ there is a nbh O_p and functions $h^i_j, f_j : O_p \to \mathbb{R}$, such that

$$\boldsymbol{\theta}^{(i)} = \sum_{j=1}^{m-k} h^i_j\,\mathrm{d}f_j$$

- the condition

$$\boldsymbol{\theta}^{(1)}\wedge\dots\wedge\boldsymbol{\theta}^{(m-k)}\wedge\mathrm{d}\boldsymbol{\theta}^{(i)} = 0$$

  holds for any $i \in \{1,\dots,m-k\}$.

De Rham cohomology

closed forms $Z^r(M) = \{\boldsymbol{\omega}\in\Omega^r \mid \mathrm{d}\boldsymbol{\omega} = 0\}$

exact forms $B^r(M) = \{\boldsymbol{\omega}\in\Omega^r \mid \exists\,\boldsymbol{\gamma}\in\Omega^{r-1} :\ \boldsymbol{\omega} = \mathrm{d}\boldsymbol{\gamma}\}$

$$H^r_{\mathrm{dR}}(M) = Z^r(M)/B^r(M)$$

Euler-Poincaré characteristic $\chi(M) = \sum_{r=0}^m (-1)^r b^r$ with **Betti numbers** $b^r = \dim H^r_{\mathrm{dR}}(M)$. For $n \geq 1$ the de Rham cohomology groups of spheres $\mathbb{S}^n$ are $H^r_{\mathrm{dR}}(\mathbb{S}^n) = (\delta_{r0}+\delta_{rn})\mathbb{R}$.

Thm. If M is a connected smooth manifold with finite fundamental group, then $H^r_{\mathrm{dR}}(M) = 0$. **Poincaré lemma.** Every closed form on a contractible open set is exact.

Hodge decomposition theorem. On any closed orientable Riemannian manifold M, for any $\boldsymbol{\omega}\in\Omega^p(M)$ there is a unique global decomposition

$$\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha} + \delta\boldsymbol{\beta} + \boldsymbol{\chi}$$

with $\boldsymbol{\alpha}\in\Omega^{p-1}$, $\boldsymbol{\beta}\in\Omega^{p+1}$ and $\boldsymbol{\chi}\in\Omega^p$ is harmonic, $\Delta\boldsymbol{\chi} = 0$.

Maxwell's equations

$(m,s) = (4,1)$

$$\mathrm{d}\mathbf{F} = 4\pi\star\mathbf{j}_\mathrm{m}\ , \quad \mathrm{d}\star\mathbf{F} = 4\pi\star\mathbf{j}_\mathrm{e}$$

Electro/magnetic decomposition via X^a with $N = X_a X^a$:

$$\mathbf{E} = -i_X\mathbf{F}\ , \quad \mathbf{B} = i_X\star\mathbf{F}$$

$$-N\mathbf{F} = \mathbf{X}\wedge\mathbf{E} + \star(\mathbf{X}\wedge\mathbf{B})$$
$$N(\mathbf{F}\mid\mathbf{F}) = (\mathbf{E}\mid\mathbf{E}) - (\mathbf{B}\mid\mathbf{B})$$
$$N(\mathbf{F}\mid\star\mathbf{F}) = -2\,(\mathbf{E}\mid\mathbf{B})$$

Maxwell's equations via $\delta = -\star\mathrm{d}\star$ and $\boldsymbol{\omega} = -\star(\mathbf{X}\wedge\mathrm{d}\mathbf{X})$,

$$-\delta\mathbf{E} = \frac{(\mathbf{E}\mid i_X\mathrm{d}\mathbf{X})}{N} - \frac{(\mathbf{B}\mid\boldsymbol{\omega})}{N} + 4\pi(\mathbf{X}\mid\mathbf{j}_\mathrm{e})$$
$$-\delta\mathbf{B} = \frac{(\mathbf{B}\mid i_X\mathrm{d}\mathbf{X})}{N} + \frac{(\mathbf{E}\mid\boldsymbol{\omega})}{N} + 4\pi(\mathbf{X}\mid\mathbf{j}_\mathrm{m})$$
$$-\mathrm{d}\mathbf{E} = \pounds_X\mathbf{F} + 4\pi\star(\mathbf{X}\wedge\mathbf{j}_\mathrm{m})$$
$$\mathrm{d}\mathbf{B} = \pounds_X\star\mathbf{F} + 4\pi\star(\mathbf{X}\wedge\mathbf{j}_\mathrm{e})$$

Killing vectors

Thm. For a Killing vector K^a with $N = K_a K^a$ we have

$$\pounds_K\mathbf{K} = 0\ , \quad \delta\mathbf{K} = 0\ , \quad \mathrm{d}\star\mathrm{d}\mathbf{K} = 2\star\mathbf{R}(K)\ , \quad \mathrm{d}N = -i_K\mathrm{d}\mathbf{K}$$

$$\pounds_K\boldsymbol{\alpha} = \delta(\mathbf{K}\wedge\boldsymbol{\alpha}) + \mathbf{K}\wedge\delta\boldsymbol{\alpha}\ , \quad \pounds_K\star\boldsymbol{\alpha} = \star\pounds_K\boldsymbol{\alpha}$$

Thm. $(m,s) = (4,1)$. Let (M, g_{ab}) be a Lorentzian 4-manifold with the Killing vector field K^a. Then the twist 1-form ω_a, defined as

$$\boldsymbol{\omega} := -\star(\mathbf{K}\wedge\mathrm{d}\mathbf{K})\ , \quad \omega_a = \epsilon_{abcd}K^b\nabla^c K^d$$

satisfies the following relations

$$\mathrm{d}\boldsymbol{\omega} = -2\star(\mathbf{K}\wedge\mathbf{R}(K))\ , \quad N\delta\boldsymbol{\omega} = 2\,(\boldsymbol{\omega}\mid\mathrm{d}N)$$

$$(\boldsymbol{\omega}\mid\boldsymbol{\omega}) = (\mathrm{d}N\mid\mathrm{d}N) - N\Delta N - 2NR(K,K)$$

$$N\mathrm{d}\mathbf{K} = \star(\mathbf{K}\wedge\boldsymbol{\omega}) - \mathbf{K}\wedge\mathrm{d}N$$
$$N(\mathrm{d}\mathbf{K}\mid\mathrm{d}\mathbf{K}) = (\mathrm{d}N\mid\mathrm{d}N) - (\boldsymbol{\omega}\mid\boldsymbol{\omega})$$

$$\delta\left(\frac{\mathbf{K}}{N}\right) = 0\ , \quad \delta\left(\frac{\boldsymbol{\omega}}{N^2}\right) = 0$$
$$\delta\left(\frac{\mathrm{d}N}{N}\right) = -\frac{(\boldsymbol{\omega}\mid\boldsymbol{\omega}) + 2NR(K,K)}{N^2}$$
$$\delta\left(\frac{\mathbf{E}}{N}\right) = \frac{(\boldsymbol{\omega}\mid\mathbf{B})}{N^2} - \frac{4\pi}{N}(\mathbf{K}\mid\mathbf{j}_\mathrm{e})$$
$$\delta\left(\frac{\mathbf{B}}{N}\right) = -\frac{(\boldsymbol{\omega}\mid\mathbf{E})}{N^2} - \frac{4\pi}{N}(\mathbf{K}\mid\mathbf{j}_\mathrm{m})$$