ZFC in a nutshell

Coffee & Chalk Press

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Ernst F.F. **Z**ermelo	(1871 – 1953)
Adolf A.H. **F**raenkel	(1891 – 1965)
Axiom of **C**hoice	(1904 –)

1. The Axiom of Extensionality

a set is completely characterized by its elements alone

If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

$$\forall X\,\forall Y\,\big(\forall a\,(a \in X \leftrightarrow a \in Y) \to X = Y\big)$$

By an axiom of predicate calculus we have the converse $\forall X\,\forall Y\big(X = Y \to \forall a(a \in X \leftrightarrow a \in Y)\big)$.

2. The Axiom Schema of Separation

a recipe to construct a set

If φ is a formula (with parameter p) then for any X and p there exists a set S that contains all those $a \in X$ for which φ holds.

$$\forall X\,\exists S\,\forall a\,(a \in S \leftrightarrow a \in X \wedge \varphi(a,p))$$

3. The Axiom of Union

an union of sets, to merge them all

For any F, there exists a set U (usually denoted by $\bigcup F$) such that $x \in U$ iff $x \in A$ for some $A \in F$.

$$\forall F\,\exists U\,\forall x\,\big(x \in U \leftrightarrow \exists A\,(A \in F \wedge x \in A)\big)$$

4. The Axiom of Power Set

the set of all subsets

For any X, there exists a set P (usually denoted by $\mathscr{P}(X)$ or 2^X), such that $A \in P$ iff $A \subseteq X$.

$$\forall X\,\exists P\,\forall A\,\big(A \in P \leftrightarrow \forall a(a \in A \to a \in X)\big)$$

The origin of the name "power set" is related to the fact that for any set X we have $|2^X| = 2^{|X|}$.

5. The Axiom of Infinity

a blueprint for the set of natural numbers

An inductive set exists.

$$\exists S\,\big(\emptyset \in S \wedge (\forall x \in S)\; x \cup \{x\} \in S\big)$$

where the empty set $\emptyset$ is defined below.

6. The Axiom Schema of Replacement

image of a set is a set

Let $\varphi(x,y,p)$ be a formula (with some parameter p) such that for every x there is a unique y for which $\varphi(x,y,p)$ holds. Then for every set X there is a set Y which collects all the y's as x ranges over X.

$$\forall x\,\forall y\,\forall z\,\big((\varphi(x,y,p) \wedge \varphi(x,z,p) \to y = z) \to$$
$$\to \forall X\,\exists Y\,\forall y\,(y \in Y \leftrightarrow (\exists x \in X)\,\varphi(x,y,p))\big)$$

7. The Axiom of Foundation

sets are forbidden to devour themselves

All sets are well-founded: Every non-empty set S contains an element A such that S and A are disjoint sets.

$$\forall S\,\big(\exists X(X \in S) \to \exists A\,(A \in X \wedge \nexists B\,(B \in A \wedge B \in S))\big)$$

This implies, for example, that no set is an element of itself.

8. The Axiom of Choice

abandon all explicitness ye who enter here

Given any set A of pairwise disjoint non-empty sets, there exists at least one set C that contains exactly one element in common with each of the sets in A.

$$\forall A\,(\varphi(A) \to (\exists C)\,\psi(A,C))$$

$$\begin{aligned}
\varphi(A) &\equiv \forall a\,(a \in A \to \exists x\,(x \in a)) \wedge \\
&\wedge \forall a\,\forall b\,\big(a \in A \wedge b \in A \wedge \exists x\,(x \in a \wedge x \in b) \to a = b\big) \\
\psi(A,C) &\equiv \forall a\,(a \in A \to \exists x\,(x \in a \wedge x \in C)) \wedge \\
&\wedge \forall a\,\big(a \in A \to \forall x\,\forall y\,(x \in a \wedge x \in C \wedge y \in a \wedge y \in C \to x = y)\big)
\end{aligned}$$

Redundant axioms

Very often, for practical reasons, several other "axioms" are included in the list.

0. The Axiom of Existence

let there be nothing!

There exists a set E which has no elements.

$$\exists E\,\forall X\,(X \notin E)$$

► The axiom of infinity implies that there exists at least one set, S. Then the axiom schema of separation implies the existence of the empty set $E = \{A \in S \mid A \neq A\}$ which, by the axiom of extensionality, is unique. This set is known as the *empty set* and is usually denoted by $\emptyset$.

2½. The Axiom of Pairing

a pairs of sets, and nothing else

For any A and B, there is C such that $X \in C$ iff $X = A$ or $X = B$,

$$\forall A\,\forall B\,\exists C\,\forall X\,\big(X \in C \leftrightarrow X = A \vee X = B\big)$$

► From the existence of the empty set $\emptyset$ and the axiom of power set (applied twice) there are sets $\mathscr{P}(\emptyset) = \{\emptyset\}$ and $\mathscr{P}(\mathscr{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$. The idea is to replace the elements of the set $\mathscr{P}(\mathscr{P}(\emptyset))$ with given sets A and B. Using the axiom schema of replacement with the formula

$$\varphi(X,Y,A,B) \equiv (X = \emptyset \wedge Y = A) \vee (X = \{\emptyset\} \wedge Y = B)$$

we define the set

$$\{A,B\} \equiv \{\,Y \mid \exists\,X \in \mathscr{P}(\mathscr{P}(\emptyset))\,:\,\varphi(X,Y,A,B)\}$$

This allows us, for example, to define the union of two sets,

$$A \cup B \equiv \bigcup\{A,B\}\,.$$