Characteristic classes

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1 Introduction

The aim of this paper is to provide an introduction into the theory of characteristic classes and its application in theoretical physics. However, first, a brief introduction into cohomology theory and fibre bundles is presented. Our interest is the study of topological properties of manifolds and its fibre bundles, that is the properties of these objects as a whole, which are then related to local properties coming from differential geometry. We mainly follow [1] and [2] with the help of the other references in certain sections as specifically noted.

1.1 Basic topological definitions

We begin by introducing a few definitions we will need in this exposition.

Definition 1.1 *Define the following.*

(a) A **relation** *R defined in a set X is a subset of X^2: $R \subseteq X^2$. If a point $(a, b) \in X^2$ is in R then we may write $a\,R\,b$.*

(b) A relation is called an **equivalence relation** $\sim$ *if it is:*

(i) reflective: $a \sim a$,

(ii) symmetric: if $a \sim b$, then $b \sim a$,

(iii) transitive: if $a \sim b$ and $b \sim c$, then $a \sim c$.

Definition 1.2 *Let a set X have a defined equivalence relation $\sim$. Then, we have a partition of X into mutually disjoint subsets called* **equivalence classes**. *An equivalence class $[a]$ is then:*

$$[a] = \{x \in X \,|\, x \sim a\}.$$

It can be shown that if $a \sim b$, then $[a] = [b]$, that is, an equivalence class does not depend on its **representative**. *The family of all classes is called the* **quotient space**, *$X/\sim$.*

Definition 1.3 *Define the following.*

(a) A topological space X is **connected** *if it cannot be written as $X = X_1 \cup X_2$ for X_1 and X_2 both disjoint and nonempty.*

(b) A topological space X is **arcwise connected** *if, for any points $x, y \in X$, there exists a continuous map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

(c) A **loop** *in a topological space X is a continuous map $f : [0, 1] \to X$ such that $f(0) = f(1)$.*

(d) A topological space X is **simply connected** *if any loop in X can be continuously shrunken to a point.*

Definition 1.4 *Let X and $\tilde{X}$ be connected topological spaces. $\tilde{X}$ or $(\tilde{X}, p)$ is called a* **covering space** *if $p : X \to \tilde{X}$ is a continuous map such that:*

(i) p is a surjection,

(ii) $\forall x \in X \mid \exists U \subseteq X$, $x \in U$ connected and open, such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each one homeomorphically mapped onto U by p.

If $\tilde{X}$ is simply connected then it is called the **universal covering space** *of X.*

1.2 Topological invariants

In topology the equivalence of two spaces is defined by continuous deformations; if one can be continuously deformed into the other it is said they are *topologically equivalent.* Therefore we define the notion of homeomorphisms.

Definition 1.5 *Let X_1 and X_2 be topological spaces. A map $f : X_1 \to X_2$ is a* **homeomorphism** *if it is continuous and has a continuous inverse $f^{-1} : X_2 \to X_1$. It can be shown that a homeomorphism is an equivalence relation. If such a map exists, X_1 and X_2 are said to be* **homeomorphic** *or topologically equivalent.*

We are now interested in characterizing the equivalence classes of homeomorphisms. It is highly nontrivial to explicitly construct homeomorphisms between topological spaces in general, therefore we construct quantities conserved under homeomorphisms called **topological invariants** which if different indicate the spaces in question are not homeomorphic. The complete set of topological invariants, if known, would specify the equivalence classes fully. One of the most popular topological invariants is the Euler characteristic which we define now.

Definition 1.6 *Let X be a subset of $\mathbb{R}^3$, which is homeomorphic to a polyhedron K. Then the* **Euler characteristic** *$\chi(X)$ is defined by:*

$$\chi(X) \equiv v - e + f,$$

where v, e and f are the number of vertices, edges and faces, respectively.

2 Homology and Cohomology

As we have seen, topological invariants are extremely important in topologically classifying spaces. We now build on the idea of Euler characteristics by introducing the homology group from which it will be trivial to read out the Euler characteristic. On the other hand we have manifolds and its fibre bundles on which differential forms are studied. In analogy with homology we define the cohomology of forms. The relation between homology and the cohomology of forms comes from de Rham's theorem stated later. Additionally to [1] and [2], in this section [3] is used.

2.1 Preliminaries from group theory

Before we can delve into homology, a few definitions and properties from group theory are necessary.

Definition 2.1 *Let G be a group and H be a subgroup of G. If $g, g' \in G$ we introduce an equivalence relation $\sim$ such that $g \sim g'$ if $g' = gh$, $h \in H$. The equivalence class $[g] = gH = \{gh \,|\, h \in H\}$ is called a* **left coset** *of H. The* **quotient** *space G/H is a* **group** *if and only if H is a normal subgroup of G, that is if*

$$gHg^{-1} = H, \quad \forall g \in G.$$

Definition 2.2 *Let $f : G_1 \to G_2$ be a homomorphism. Define the:*

(a) **kernel** *of f:*

$$\ker f = \{x \in G_1 \,|\, f(x) = 0\},$$

a subgroup of G_1;

(b) **image** *of f:*

$$\operatorname{im} f = \{x = f(g_1) \,|\, f(G_1) \subseteq G_2, \forall g_1 \in G_1\},$$

a subgroup of G_2.

Theorem 2.1 (Fundamental theorem of homomorphisms) *Let $f : G_1 \to G_2$ be a homomorphism. Then,*

$$G_1/\ker f \cong \operatorname{im} f.$$

Definition 2.3 *Let G be a group. G is called* **finitely generated** *if every element of G can be written as a group operation combination of finitely many elements of a finite set $S \subseteq G$, with elements of S the* **generators** *of the group.*

Definition 2.4 *If an Abelian group G is finitely generated by r linearly independent elements, G is called a* **free Abelian group** *of rank r.*

Definition 2.5 *A* **cyclic group** *G is an Abelian group generated by one element x:*

$$G = \{0, \pm x, \pm 2x, \dots\}.$$

If $nx \neq 0$, $\forall n \in \mathbb{Z} - \{0\}$, G is an **infinite** *cyclic group, otherwise it is a* **finite** *cyclic group. Any infinite cyclic group is isomorphic to $\mathbb{Z}$ and any finite cyclic group is isomorphic to $\mathbb{Z}_k$ for some $1 < k \in \mathbb{N}$, otherwise trivial.*

2.2 Chains, cycles and boundaries

Just as when we considered the Euler characteristic we must now construct polyhedrons homeomorphic to the topological space (manifold) we want to classify. To do this we define the building blocks of a polyhedron – simplexes in Euclidean space first and then generalize to arbitrary manifolds.

Definition 2.6 *Let points $p_0, \dots, p_r$ be geometrically independent points in $\mathbb{R}^m$ with $m \geq r$. An* **r-simplex** *$\sigma_r = \langle p_0 \dots p_r \rangle$ is then:*

$$\sigma_r = \left\{ x \in \mathbb{R}^m \,\middle|\, x = \sum_{i=0}^{r} c_i p_i,\, c_i \geq 0,\, \sum_{i=0}^{r} c_i = 1 \right\}.$$

An **oriented** *r-simplex $(p_0 \dots p_r)$ is an equivalence class of even permutations of $\{p_0, \dots, p_r\}$ in $\langle p_0 \dots p_r \rangle$. Therefore,*

$$(p_{i_0} \dots p_{i_r}) = \operatorname{sgn} P \; (p_0 \dots p_r),$$

where $\operatorname{sgn} P$ is the sign of the permutation of $\{p_{i_0}, \dots, p_{i_r}\}$.

Definition 2.7 *Let K be a set of a finite number of simplexes in $\mathbb{R}^m$. K is a* **simplicial complex** *if the following are satisfied:*

(i) if $\sigma \in K$ and $\sigma' \leq \sigma$ then $\sigma' \in K$;

(ii) if $\sigma, \sigma' \in K$ then either $\sigma \cap \sigma' = \varnothing$ or both $\sigma \cap \sigma' \leq \sigma$ and $\sigma \cap \sigma' \leq \sigma'$.

It is time to define chains and its subgroups: cycles and boundaries. These are the key ingredients needed in defining the homology group later.

Definition 2.8 *The* **r-chain group** *$C_r(K)$ of a simplicial complex K is a free Abelian group generated by I_r oriented r-simplexes in K. $C_r(K)$ is defined to be: $\{0\}$, $\forall r > \dim K$.[1] Elements $c \in C_r(K)$ are called* **r-chains***:*

$$c = \sum_{i=0}^{I_r} c_i \sigma_{r,i}, \quad c_i \in \mathbb{Z}.$$

[1] The element 0 is understood as the identity element of an additive group, so $\{0\}$ is the trivial group.

Definition 2.9 *If $\sigma_r = (p_o \dots p_r)$ is an oriented r-simplex, then we define the* **boundary** $\partial_r \sigma_r$ *of* σ_r *as an* $(r-1)$*-chain:*

$$\partial_r \sigma_r \equiv \sum_{i=0}^{r} (-1)^i (p_0 \dots p_{i-1} p_{i+1} \dots p_r), \quad \forall r > 0,$$

$$\partial_0 p_0 \equiv 0.$$

Definition 2.10 *Define the* **boundary operator** *homomorphism* $\partial_r : C_r(K) \to C_{r-1}(K)$ *as a linear operator:*

$$\partial_r c = \sum_i c_i \partial_r \sigma_{r,i} \in C_{r-1}(K), \quad c = \sum_i c_i \sigma_{r,i} \in C_r(K).$$

The boundary operator is nilpotent, that is,

$$\partial_r \circ \partial_{r+1} = 0.$$

Therefore the boundary operator defines an exact sequence called the **chain complex** $C(K)$*:*

$$0 \to C_r(K) \xrightarrow{\partial_r} C_{r-1}(K) \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0.$$

Moving from Euclidean space to general m-dimensional manifolds, we have.

Definition 2.11 *An* **r-simplex on a m-dimensional manifold** *(also called a singular r-simplex) M is the equivalence class of pairs $s_r = [(\sigma_r, \Theta)]$, where σ_r is the Euclidean r-simplex and*

$$\Theta : \mathcal{O} \to M, \quad \sigma_r \subseteq \mathcal{O} \subseteq \mathbb{R}^r$$

a smooth map over an open neighbourhood $\mathcal{O}$ of σ_r,[2] *defined by the equivalence relation:*

$$(\sigma_r, \Theta) \sim (\chi(\sigma_r), \Theta') \quad \Leftrightarrow \quad \Theta = \Theta' \circ \chi,$$

for some affine transformation χ of the Euclidean simplex.

Definition 2.12 *A linear combination of r-simplices in M with real coefficients is an* **r-chain on an m-dimensional manifold** *(or singular r-chain) c. The space of all chains c is the (singular)* **r-chain group** *$C_r(M)$ on M.*

Definition 2.13 *The* **boundary operator on a manifold**[3] $\partial : C_r(M) \to C_{r-1}(M)$ *is defined by the Θ map as:*

$$\partial c = \partial \left(\sum_i c_i s_{r,i} \right) \equiv \sum_i c_i \partial s_{r,i},$$

$$\partial s_r = \partial(\Theta(\sigma_r)) \equiv \Theta(\partial \sigma_r).$$

As in the Euclidean case it too is nilpotent:

$$\partial^2 = 0.$$

Definition 2.14 *Let M be an m-dimensional manifold and $c \in C_r(M)$, then if*

$$\partial c = 0,$$

c is called an (singular) **r-cycle**. *The set of all cycles of a given r-chain is denoted $Z_r(M)$.*

$$Z_r(M) \equiv \ker \partial_r = \{c \in C_r(M) \,|\, \partial c = 0\}$$

is a subgroup of $C_r(M)$ and is called the (singular) **r-cycle group**.

Definition 2.15 *Let $c \in C_r(M)$, then if there exists some $d \in C_{r+1}(M)$ such that:*

$$c = \partial d,$$

c is called an (singular) **r-boundary**. *The set of all boundaries of a given r-chain*

$$\begin{aligned} B_r(M) &\equiv \operatorname{im} \partial_{r+1} \\ &= \{c \in C_r(M) \,|\, \exists d \in C_{r+1}(M);\ \partial d = c\} \end{aligned}$$

is a subgroup of $C_r(M)$ called the (singular) **r-boundary group**.

Theorem 2.2 *If $Z_r(M)$ is the r-cycle group and $B_r(M)$ the boundary group then:*

$$B_r(M) \subseteq Z_r(M) \subseteq C_r(M).$$

2.3 Homology group

All the groups we have defined so far: $B_r(M)$, $Z_r(M)$ and $C_r(M)$ fail to be topological invariants. Therefore, we must construct, out of these, a new group that will, in fact, be a topological invariant. This turns out to be the homology group defined below.

Definition 2.16 *Let M be an m-dimensional manifold and $Z_r(M)$ and $B_r(M)$ the r-cycle group and the boundary group, respectively, then*

$$H_r(M) \equiv Z_r(M)/B_r(M), \quad 0 \le r \le m$$

is the **r-th homology group**[4]. *For $r > m$ and $r < 0$ we define the homology group to be $H_r(M) \equiv \{0\}$.*

[2] It is worth noting that this map may not be one-to-one in general and is thus not a triangularization of M, hence the term *singular*.

[3] The index of the boundary operator ∂_r and r-chain c_r will be omitted wherever possible as it is usually implicit from context, this notation choice is purely so the analogy with differential forms is more elegant since the exterior derivative operator is conventionally written without an index, as will be shown below.

As we have seen in definition 1.2 $H_r(M)$ is a set of equivalence classes:

$$H_r(M) = \{[z] \,|\, z \in Z_r(M)\},$$

defined by the equivalence relation:

$$z \sim z' \quad \Leftrightarrow \quad z - z' \in B_r(M).$$

The equivalence class $[z]$ is called a **homology class** and z and z' are said to be homologous to each other.

Theorem 2.3 *Homology groups are topological invariants.*

Definition 2.17 *The* ***r*-th Betti number** $b_r(M)$ *is the dimension of the r-th homology group,*

$$b_r(M) \equiv \dim H_r(M).$$

Theorem 2.4 (Euler-Poincaré) *If M is an m-dimensional topological space then:*

$$\chi(M) = \sum_{r=0}^{m} (-1)^r b_r(M),$$

where $\chi(M)$ is the Euler characteristic from definition 1.6.

2.4 Stokes theorem and the de Rham cohomology group

We now move our focus to the calculation of integrals. As we know, only integrals of volume forms $\omega(x)$ on an m-dimensional manifold (maximal forms[5], forms of degree m), are well defined. Instead of integrating over the entire manifold we can integrate over an r-chain $c \in C_r(M)$, however as before this means that we can only integrate r-forms. To define such an object our aim is to define it via integrals over simplices in $\mathbb{R}^r$.

Definition 2.18 *Let $c \in C_r(M)$, $\omega \in \Omega^r(M)$, σ_r be an oriented r-simplex in $\mathbb{R}^r$ and $\Theta : \sigma_r \mapsto \Theta(\sigma_r) \equiv s_r$ be the map defining singular r-simplices in M, then we define the* **integral of an *r*-form on *M* over an *r*-chain in *M***:

$$\begin{aligned}
\int_c \omega &= \int\limits_{\Sigma_i c_i s_{r,i}} \omega = \sum_i c_i \int\limits_{s_{r,i}} \omega = \sum_i c_i \int\limits_{\Theta(\sigma_{r,i})} \omega \\
&= \sum_i c_i \int\limits_{\sigma_{r,i}} \Theta^* \omega \\
&= \sum_i c_i \int\limits_{\sigma_{r,i}} f(x^1, \dots, x^r)\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^r \\
&= \sum_i c_i \int\limits_{\sigma_{r,i}} f(x)\, \mathrm{d}^r x,
\end{aligned}$$

as $\Theta^*\omega$ is an r-form in $\mathbb{R}^r$.

Definition 2.19 *If $c \in C_r(M)$ and $\omega \in \Omega^r(M)$, then we define an inner product $(\ ,\ ) : C_r(M) \times \Omega^r(M) \to \mathbb{R}$ as:*

$$c, \omega \mapsto (c, \omega) \equiv \int_c \omega.$$

Due to the additivity of an integral with respect to the domain of integration and the linearity of the operator, this inner product is bilinear,

$$(c_1 + c_2, \omega_1 + \omega_2) = (c_1, \omega_1) + (c_2, \omega_2) + (c_1, \omega_2) + (c_2, \omega_1),$$

and non-degenerate with respect to the chain.

Now that we have defined integration over chains in M, we can state one of the most important theorems of this paper.

Theorem 2.5 (Stokes) *Let $\omega \in \Omega^{r-1}(M)$ and $c \in C_r(M)$, then:*

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega.$$

Or, written using the inner product just defined:

$$(c,\, \mathrm{d}\omega) = (\partial c, \omega).$$

Therefore, in the sense of this inner product, the exterior derivative operator d and the boundary operator ∂ are mutually adjoint.

Definition 2.20 *Let M be an m-dimensional manifold. Defined are the:*

(a) ***r*-th cocycle group**:

$$Z^r(M) \equiv \ker \mathrm{d}_r = \{\omega \in \Omega^r(M) \,|\, \mathrm{d}\omega = 0\},$$

(b) ***r*-th coboundary group**:

$$\begin{aligned}
B^r(M) &\equiv \mathrm{im}\ \mathrm{d}_{r-1} \\
&= \{\omega \in \Omega^r(M) \,|\, \exists \psi \in \Omega^{r-1};\ \mathrm{d}\psi = \omega\}.
\end{aligned}$$

Since $\mathrm{d}^2 = 0$ it follows that $B^r(M) \subseteq Z^r(M)$. It can also be shown that:

(i) if $\omega \in Z^r(M)$ and $\psi \in Z^s(M)$, then $\omega \wedge \psi \in Z^{r+s}(M)$;

(ii) if $\omega \in Z^r(M)$ and $\psi \in B^s(M)$, then $\omega \wedge \psi \in B^{r+s}(M)$;

(iii) if $\omega \in B^r(M)$ and $\psi \in B^s(M)$, then $\omega \wedge \psi \in B^{r+s}(M)$.

[4] It is implicit here that the coefficients of the r-chain are real numbers since our homology group is over a manifold, that is $H_r(M) = H_r(M; \mathbb{R})$. However in general this need not be the case as the coefficients can be $\mathbb{Z}$ or even $\mathbb{Z}_2$ elements depending on the space in question.

[5] Volume forms other than being maximal, must also be globally defined and nonvanishing everywhere on M.

Definition 2.21 *The* ***r*-th de Rham cohomology group** *is the quotient group of the r-th cocycle group by the r-th coboundary group:*

$$H^r(M) \equiv Z^r(M)/B^r(M).$$

For $r > m$ and $r < 0$ we define the homology group to be trivial, $H^r(M) \equiv 0$.

Just as homology, cohomology too is a set of equivalence classes (**cohomology classes**):

$$H^r(M) = \{[\omega] \,|\, \omega \in Z^r(M)\},$$

with the equivalence relation:

$$\omega \sim \omega' \quad \Leftrightarrow \quad \omega - \omega' \in B^r(M).$$

In complete analogy, ω and ω' are called cohomologous. It is not difficult to see the meaning of the 0-th cohomology group $H^0(M)$. Since all forms of negative degree are defined as 0 this implies that $H^0(M) = Z^0(M)$, so we are looking for all 0-forms (scalar functions) on M whose gradient is 0. These are of course the constant functions of which there are $\mathbb{R}$ (every function is its own class since $B^0(M) = 0$), therefore for n connected components there are n times $\mathbb{R}$ classes and the 0-th de Rham cohomology group is:

$$H^0(M) \cong \underbrace{\mathbb{R} \oplus \cdots \oplus \mathbb{R}}_{n \text{ connected components}}.$$

Next is to show that the inner product from definition 2.19 induces an inner product Λ between elements of $H_r(M)$ and $H^r(M)$.

Definition 2.22 *Let $[c] \in H_r(M)$ and $[\omega] \in H^r(M)$. Define the inner product $\Lambda : H_r(M) \times H^r(M) \to \mathbb{R}$:*

$$\Lambda\left([c], [\omega]\right) \equiv (c, \omega) = \int_c \omega.$$

This product is well defined due to Stokes' theorem:

$$(c + \partial c', \omega) = (c, \omega) + (c', \mathrm{d}\omega) = (c, \omega),\ c' \in C_{r+1}(M),$$

$$(c, \omega + \mathrm{d}\psi) = (c, \omega) + (\partial c, \psi) = (c, \omega),\ \psi \in \Omega^{r-1}(M),$$

since $\mathrm{d}\omega = 0$ and $\partial c = 0$ as they are elements of a cohomology and homology class, respectively.

Theorem 2.6 (de Rham) *If M is a compact manifold, then $H_r(M)$ and $H^r(M)$ are finite dimensional. Inner product Λ is then bilinear and non-degenerate. Therefore, $H^r(M) \cong H_r(M)$ are dual vector spaces.*

Because of this isomorphism the dimensions are the same:

$$b^r(M) \equiv \dim H^r(M) = \dim H_r(M) \equiv b_r(M),$$

so the Betti number is the same with upper or lower indices and the Euler characteristic:

$$\chi(M) = \sum_{r=0}^{m} (-1)^r b^r(M).$$

The last point to make about the de Rham cohomology group is the relation between $H^r(M)$ and $H^{m-r}(M)$ given by the following.

Definition 2.23 (Poincaré duality) *Define an inner product $\langle \ \ , \ \ \rangle : H^r(M) \times H^{m-r}(M) \to \mathbb{R}$,*

$$\langle \omega, \eta \rangle \equiv \int_M \omega \wedge \eta.$$

This product is bilinear and non-singular and so defines the duality $H^r(M) \cong H^{m-r}(M)$.

This means that the Betti numbers are also the same

$$b_r = b_{m-r},$$

and that the Euler characteristic of odd-dimensional spaces vanishes.

Definition 2.24 *The* **cohomology ring** *of an m-dimensional manifold M, $H^*(M)$ is defined as:*

$$H^*(M) \equiv \bigoplus_{r=0}^{m} H^r(M),$$

with the exterior product as the multiplication:[6]

$$\wedge : H^*(M) \times H^*(M) \to H^*(M), \quad [\omega] \wedge [\eta] \equiv [\omega \wedge \eta],$$

and the direct sum as the addition.

If we have a smooth map between manifolds $f : M \to N$, then its pullback $f^* : \Omega^r(N) \to \Omega^r(M)$ naturally induces a linear map $H^r(N) \to H^r(M)$ since the pullback map commutes with the exterior derivative, $f^*\mathrm{d} = \mathrm{d}f^*$. This linear map also preserves the algebraic structure of the cohomology ring since: $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

3 Lie groups and Lie algebras

Groups are extremely important structures in almost all areas of physics and mathematics and are so also in the study of characteristic classes. The most important type of group relevant to this topic is the Lie group defined next.

[6] This product is well-defined because if we chose $\omega' = \omega + \mathrm{d}\psi$ as the representative,

$$[\omega'] \wedge [\eta] \equiv [(\omega + \mathrm{d}\psi) \wedge \eta] = [\omega \wedge \eta + \mathrm{d}(\psi \wedge \eta)] = [\omega \wedge \eta],$$

since η is closed (as is ω).

3.1 Lie groups

Definition 3.1 *A* **Lie group** *G is a group that is also a manifold, such that it is compatible with the smooth structure of the manifold; the following maps must be smooth:*

(a) composition:

$$\circ : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 \circ g_2,$$

(b) inverse:

$$^{-1} : G \to G, \quad g \mapsto g^{-1}.$$

It is useful to note that the set of all transformations of an arbitrary set X constitute a group with respect to the composition of transformations. If we add structure to this set (X, s) then all transformations that preserve s (automorphisms)[7] form a group G, a subgroup of the group of all transformations of set X. A few important examples of these are given below.

Example 3.1 *Let (X, s) be a (finite-dimensional) (complex) real vector space V. The group of automorphisms of V: $\mathrm{Aut}\, V \equiv \mathrm{GL}(V)$. It is the group of invertible linear operators on V and is (non-canonically)[8] isomorphic to ($\mathrm{GL}(n, \mathbb{C})$) $\mathrm{GL}(n, \mathbb{R})$, the group of non-singular (complex) real $n \times n$ matrices.*

Example 3.2 *Let $(X, s) = (V, g)$ be a vector space V with a defined bilinear map $g : V \times V \to \mathbb{R}$ that is symmetric[9] and non-degenerate[10] (a metric tensor in V) with signature (r, s). The group of automorphisms of this structure (invertible linear operators A that satisfy: $g(Au, Av) = g(u, v)$) $\mathrm{Aut}\, V \equiv \mathrm{O}(r, s)$ is the pseudo-orthogonal group isomorphic to the pseudo-orthogonal matrix group.*

Example 3.3 *Let $(X, s) = (V, \omega)$ be a vector space V with a defined volume form ω. The group of automorphisms of this structure (invertible linear operators A such that: $\omega(Av_1, \ldots, Av_n) = \omega(u_1, \ldots, u_n)$) is denoted $\mathrm{SL}(V)$ (isomorphic to $\mathrm{SL}(n, \mathbb{R})$) and called the special linear group of V.*

The following can be shown for $A \in \mathrm{O}(r, s)$:

(i) $\det A = \pm 1$;

(ii) there is a bijection between the two connected components of $\mathrm{O}(r, s)$; $\det A = 1$ and $\det A = -1$;

(iii) the $\det A = 1$ component is a subgroup of $\mathrm{O}(r, s)$ designated $\mathrm{SO}(r, s)$ and called the special (pseudo) orthogonal group;

(iv) $\mathrm{SO}(r, s) = \mathrm{O}(r, s) \cap \mathrm{SL}(r + s, \mathbb{R})$.

Example 3.4 *Let $(X, s) = (V, o)$ be a vector space V with a defined orientation. Then G, the group of automorphisms (those invertible linear operators A such that if $E = \{e_a\}$ is a right-handed basis, then $E' = \{Ae_a\}$ is also a right-handed basis), is denoted $\mathrm{GL}_+(V)$ (isomorphic to $\mathrm{GL}_+(n, \mathbb{R})$, the matrix group of $n \times n$ real non-singular matrices with $\det A > 0$, $A \in \mathrm{GL}(n, \mathbb{R})$).*

It is known that complex numbers can be written as elements of $\mathbb{R}^2$, the same can be done for elements of $\mathbb{C}^n$ in $\mathbb{R}^{2n}$. Define the map ρ:

$$\rho : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}_+(2n, \mathbb{R}),$$
$$A \equiv B + iC \mapsto \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

ρ is then an injective homomorphism with $\mathrm{im}\, \rho \subseteq \mathrm{GL}(2n, \mathbb{R})$. $\mathbb{C}^n$ can also be thought of as $\mathbb{R}^{2n}$ with a linear complex structure, this is because i, the imaginary unit, commutes with real numbers and is distributive with respect to vector addition. Therefore, we can write:[11]

$$\mathrm{GL}(n, \mathbb{C}) = \left\{ A \in \mathrm{GL}(2n, \mathbb{R}) \mid AJ = JA; J = \left(\begin{smallmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{smallmatrix} \right) \right\}.$$

Theorem 3.1 (Cartan) *Any topologically closed subgroup of a Lie group is a Lie group.*

Theorem 3.2 *The universal covering of a connected Lie group is a Lie group.*

We now move our attention to invariant fields of Lie groups and define the following.

Definition 3.2 *For any element $g \in G$ we define:*

(a) **left translation** *$L_g : G \to G$, $h \mapsto L_g h = gh$;*

(b) **right translation** *$R_g : G \to G$, $h \mapsto R_g h = hg$.*

[7] An automorphism is an isomorphism from an object onto itself. See section 3.3

[8] A map is canonical if it is uniquely determined by construction.
In linear algebra, the most important example of a non-canonical isomorphism is the isomorphism between a vector space V and its dual space V^*. This isomorphism depends on the choice of a basis in V.
In topology, non-canonical isomorphisms often arise between bundles because the fibres of bundles are homeomorphic but the homeomorphisms are not canonical; the canonicity of the homeomorphisms between fibres implies the triviality of the bundle. Therefore, the homotopy and homology groups of different fibres are isomorphic, but the isomorphisms are not always canonical because the products of the base and the fibres may be twisted. [3]

[9] If g is, instead, antisymmetric (a non-degenerate 2-form in V^*), then the group becomes the symplectic group $\mathrm{Sp}(m, \mathbb{R})$, where V must be even dimensional ($\dim V = 2m$).

[10] Non-degenerate in the sense that if $g(u, v) = 0$, $\forall u \in V$, then $v = 0$.

[11] If A anticommutes with J then it corresponds to an antilinear operator on $\mathbb{C}^n$, so any matrix in $\mathrm{GL}(2n, \mathbb{R})$ corresponds uniquely to a sum of a linear and an antilinear operator on $\mathbb{C}^n$.

Translations are diffeomorphisms with the following properties:

$$L_{gh} = L_g \circ L_h \qquad R_{gh} = R_h \circ R_g;$$
$$L_g^{-1} = L_{g^{-1}} \qquad R_g^{-1} = R_{g^{-1}}.$$

Since L_g is a diffeomorphism (for each g), L_g^*, its pullback, can be applied to any tensor field on G with the result being in G again. Therefore of interest are fields that remain unchanged under this pullback.

Definition 3.3 *A tensor T on a Lie group G is said to be* **left-invariant** *if*

$$L_g^* T = T, \quad \forall g \in G.$$

Left-invariant fields are smooth and uniquely specified by their value at a single point in G.

Three points can be made for $T \in \mathcal{T}L^p_q(G)$, a left-invariant tensor field generated by (say) its value at the identity e:

(i) since T is completely determined by $T(e)$, there is an isomorphism between the space of all (p,q) tensors at e ($(T_e)^p_q$) and the space of all left-invariant (p,q) type tensor fields on G,

(ii) because of this isomorphism the dimension of the space of all left-invariant (p,q) type tensor fields on G is $(\dim G)^{p+q}$;

(iii) the space of all left-invariant scalar functions is the space of all constant functions on G.

It is useful to look at left-invariant vectors and 1-forms in more detail.

Example 3.5 *Let $\{E_a\}$ be a basis of $T_e(G)$ and denote by $\{e_a\}$ the set of left-invariant vector fields generated by $\{E_a\}$, $e_a(g) \equiv L_{g*}E_a$. Then:*

(i) $\{e_a\}$ is a global frame field on G,

(ii) therefore any Lie group must be parallelizable and orientable;

(iii) left-invariant vector fields $V = V^a e_a$ have constant components (V^a) with respect to left-invariant frame fields.

Example 3.6 *Let $\{E^a\}$ be a basis of $T_e^*(G)$ dual to $\{E_a\}$ and denote by $\{e^a\}$ the set of left-invariant 1-form fields generated by $\{E^a\}$, $e^a(g) \equiv (L_g^{-1})^* E^a$. Then:*

(i) $\{e_a\}$ is a global coframe field on G;

(ii) $\{e_a\}$ and $\{e^a\}$ are dual to each other;

(iii) left-invariant 1-form fields $\omega = \omega_a e^a$ have constant components (ω_a) with respect to left-invariant coframe fields;

(iv) if and only if $i_V\omega = const.$ for all left-invariant vector (1-form) fields V (ω) then ω (V) is left-invariant too.

Theorem 3.3 *Let $f : G \to H$ be a homomorphism of Lie groups and ω a left-invariant form field on H, then $f^*\omega$ is a left-invariant form field on G.*

Definition 3.4 *If G is a Lie group then a* **left-invariant volume form** *always exists and is defined uniquely up to a non-zero constant factor:*

$$\omega_L = \lambda\, e^1 \wedge \ldots \wedge e^n, \quad 0 \neq \lambda \in \mathbb{R},$$

where $\{e^a\}$ is a left-invariant coframe field and $n = \dim G$.

Then, the integral

$$\int_G f\omega_L = \int_G (f \circ L_g)\omega_L$$

is called *left-invariant.*[12]

3.2 Lie algebras

Definition 3.5 *A* **Lie algebra** $\mathfrak{g}$ *is a vector space over a field K together with the Lie bracket map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, that is antisymmetric, alternating, bilinear and satisfies the Jacobi identity:*

$$\big[[V,W],U\big] + \big[[W,U],V\big] + \big[[U,V],W\big] = 0.$$

Definition 3.6 *Let $X, Y \in T_e(G) \equiv \mathfrak{g}$ and $L_X, L_Y \in \mathfrak{X}_L(G)$ so $X = L_X(e)$ and $Y = L_Y(e)$.*[13] *We define the Lie bracket on $\mathfrak{g}$ as the value of the commutator of vector fields L_X and L_Y at e:*

$$[X,Y] \equiv [L_X, L_Y](e),$$

or alternatively as:

$$L_{[X,Y]} \equiv [L_X, L_Y].$$

$\mathfrak{g}$ together with this Lie bracket now defines a Lie algebra called **the Lie algebra of** *(associated to)* **Lie group G**.

[12] Two trivial examples are consequences of this:

(a) $G = \mathrm{U}(1)$: $\int_0^{2\pi} f(\alpha+\beta)d\alpha = \int_0^{2\pi} f(\alpha)d\alpha$;

(b) $G = \mathrm{GL}(1,\mathbb{R})$: $\int\limits_{-\infty}^{\infty} f(\alpha x)\frac{dx}{x} = \int\limits_{-\infty}^{\infty} f(x)\frac{dx}{x}$.

One can also define a right-invariant volume form and, consequently, a right-invariant integral over G. Which does not, in general, coincide with the left-invariant integral.

Lie algebras of Lie groups (usually called just Lie algebras for short) have the following property: if $\{E_a\}$ is the basis of a Lie algebra $\mathfrak{g}$ (also called the generators of $\mathfrak{g}$) then they satisfy:

$$[E_a, E_b] = C^c_{ab} E_c,$$

where C^c_{ab} are called the **structure constants** of $\mathfrak{g}$. Structure constants:

(i) constitute components of a left-invariant tensor field:

$$C = \frac{1}{2} C^c_{ab} \left(e^a \wedge e^b\right) \otimes e_c \in \Omega^2_L(G) \otimes \mathfrak{X}_L(G);$$

(ii) are anholonomy coefficients of left-invariant frame fields and since left-invariant fields are uniquely defined by one point (e) it follows that structure constants contain almost all information about the Lie group G;

(iii) satisfy the Maurer-Cartan structure equation:

$$\mathrm{d}e^a + \frac{1}{2} C^a_{bc}\, e^b \wedge e^c = 0;$$

(iv) satisfy:

$$C^f_{d[a} C^d_{bc]} = 0.$$

Definition 3.7 *If G is a Lie group and $\mathfrak{g}$ its Lie algebra then we define the $\mathfrak{g}$-valued* **canonical** *or* **Maurer-Cartan 1-form** $\theta \in \Omega^1(G, \mathfrak{g})$ *by:*

$$\theta : T_g G \to T_e G \equiv \mathfrak{g}; \quad X \mapsto (L_g^{-1})_* X, \quad X \in T_g G.$$

This Maurer-Cartan one-form has the following properties:

(i) it is left-invariant $L_{g*}\theta = \theta$;

(ii) if $\{E_a\}$ is the basis of $\mathfrak{g}$ and $\{e^a\}$ the left-invariant coframe field, then θ can be written as:

$$\theta = e^a E_a;$$

(iii) it satisfies the Maurer-Cartan equation:

$$\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta] = 0,$$

where we define for $\omega, \eta \in \Omega^1(G, \mathfrak{g})$:

$$[\omega \wedge \eta] \equiv (\omega^a \wedge \eta^b) \otimes [E_a, E_b].$$

Definition 3.8 One-parameter subgroups *are curves $\gamma(t)$ on a Lie group G defined by the property:*

$$\gamma(s+t) = \gamma(s)\gamma(t), \quad \gamma(0) = e, \quad s, t \in \mathbb{R}.$$

It can also be thought of as the image of the homomorphism $\gamma : (\mathbb{R}, +) \to G$.

Now, we have a one-to-one correspondence between the following concepts:

(i) element X of a Lie algebra $\mathfrak{g}$ of group G;

(ii) left-invariant vector field on G generated by X;

(iii) one-parameter subgroup with $\dot\gamma(0) = X$.

Definition 3.9 *An* **exponential map** *is defined by:*

$$\exp : \mathfrak{g} \to G, \quad X \mapsto \exp X \equiv \gamma^X(1),$$

where $\gamma^X(t)$ is the one-parameter subgroup generated by $X \in \mathfrak{g}$.

Because the one-parameter subgroup satisfies: $\gamma^X(kt) = \gamma^{kX}(t)$, for $k \in \mathbb{R}$, we can express the one-parameter subgroup via the exponential map:

$$\gamma^X(t) = \exp tX,$$

which means one-parameter curves correspond to straight lines in $\mathfrak{g}$. The exponential map has also the following properties:

$$\exp 0 = e,$$
$$\exp -X = (\exp X)^{-1},$$
$$\exp (s+t)X = \exp sX \exp tX.$$

Definition 3.10 *Let $f : G \to H$ be a homomorphism of Lie groups, then the map*

$$f' : \mathfrak{g} \to \mathfrak{h}, \quad X \mapsto f'(X) \equiv f_* X$$

is called a **derived homomorphism** *of Lie algebras.*

Definition 3.11 *An* **ideal** $\mathfrak{i}$ *of Lie algebra $\mathfrak{g}$ is a subalgebra such that the Lie bracket of any element in $\mathfrak{i}$ and $\mathfrak{g}$ is again an element of $\mathfrak{i}$, $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$.*[14]

Definition 3.12 *Define (inductively) two descending chains of subalgebras:*

(a) lower central series:

$$\mathscr{D}_1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], \quad \mathscr{D}_k\mathfrak{g} = [\mathfrak{g}, \mathscr{D}_{k-1}\mathfrak{g}];$$

(b) derived series:

$$\mathscr{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], \quad \mathscr{D}^k\mathfrak{g} = [\mathscr{D}^{k-1}\mathfrak{g}, \mathscr{D}^{k-1}\mathfrak{g}].$$

Before we move on to actions we give a rough classification of Lie algebras.

Definition 3.13 *Lie algebra $\mathfrak{g}$ is:*

[13] $\mathfrak{X}(M)$ denotes the space of all vector fields on a manifold M. The index L in $\mathfrak{X}_L(G)$ indicates the subset of all left-invariant vector fields on the Lie group G.

[14] The symbolic Lie bracket of two Lie algebras $\mathfrak{a}$ and $\mathfrak{b}$: $[\mathfrak{a}, \mathfrak{b}]$, designates also all linear combinations of such brackets.

(a) **nilpotent** *if* $\mathscr{D}_k\mathfrak{g} = \{0\}$ *for some* k*;*

(b) **solvable** *if* $\mathscr{D}^k\mathfrak{g} = \{0\}$ *for some* k*;*

(c) **perfect** *if the commutator subalgebra* $\mathscr{D}\mathfrak{g} \equiv \mathscr{D}_1\mathfrak{g} = \mathscr{D}^1\mathfrak{g} = \mathfrak{g}$*;*

(d) **simple** *if* $\mathfrak{g}$ *has no non-trivial ideals and* $\dim\mathfrak{g} > 1$*;*

(e) **semisimple** *if* $\mathfrak{g}$ *has no non-zero solvable ideals.*

Note here that no semisimple Lie algebra can ever be solvable.

3.3 Representations

Since representations fall into the much more general category of actions, we therefore define actions in general first. Using also [4], [5], [6], [7] and [8].

Definition 3.14 *Let G be a Lie group and M a manifold, then the* **(right) left action** *of G on M is a diffeomorphism (R_g) L_g such that $\forall g, h \in G$:*

$$L_g : M \to M \qquad L_{gh} = L_g \circ L_h \qquad L_e = \mathrm{id}_M;$$
$$R_g : M \to M \qquad R_{gh} = R_h \circ R_g \qquad R_e = \mathrm{id}_M\,.$$

Actions can also be thought of as maps $L, R : G \times M \to M$ that satisfy:

$$L(g,x) \equiv L_g x \qquad L(gh,x) = L(g, L(h,x));$$
$$R(g,x) \equiv R_g x \qquad R(x,gh) = R(R(x,g),h).$$

A manifold M with a (right) left action of group G is called a **(right) left G-space***.*

Definition 3.15 *If M is a G-space, then if any two points on M can be connected by an action of G, the action is called* **transitive***. A G-space with a transitive action is called a* **homogeneous space***.*

Definition 3.16 *Let M be a G-space and $x \in M$. The subgroup $G_x \subseteq G$ that contains only elements $g \in G$ that leave a point fixed is called the* **stabilizer** *or little group (or stationary subgroup) of x. In other words,*

$$G_x \equiv \{g \in G \,|\, L_g x = x\} \quad \text{or} \quad G_x \equiv \{g \in G \,|\, R_g x = x\},$$

depending on whether it is a left or right G-space.

Definition 3.17 *An action of a group on a G-space M is* **free** *if the stabilizer of every point is trivial,*

$$\forall x \in M, \quad G_x = \{e\};$$

and **effective** *if there exists at least one point in which the stabilizer is trivial,*

$$\exists x \in M, \quad G_x = \{e\}.$$

Before moving to representations we define one very important object of an action, the fundamental field.

Definition 3.18 *The* **fundamental field** *(or generator) of the right action R_g on a manifold M is a vector field defined for $X \in \mathfrak{g}$ and $p \in M$ as:*

$$\xi_X(p) \equiv \frac{d}{dt} R_{\exp tX}\, p\bigg|_{t=0} = \dot\gamma(0), \quad \gamma(t) \equiv p\exp tX.$$

Now, if we restrict ourselves to manifolds that are vector spaces V, actions on such manifolds are a very special class of actions called representations defined below. Before this, however, it is important to clarify the spaces associated to linear maps of spaces onto themselves. Endomorphisms are maps from an object onto itself, the set of all endomorphisms of an object V, $\mathrm{End}\,V$, is naturally endowed with an associative algebra structure. Endomorphisms that are invertible are called automorphisms, $\mathrm{Aut}\,V$, and have a group structure instead. Therefore, if we are interested in linear maps that are homomorphisms from a Lie group G, they cannot be to $\mathrm{End}\,V$ but to $\mathrm{Aut}\,V \subseteq \mathrm{End}\,V$ instead.

Definition 3.19 *A homomorphism*[15]

$$\rho : G \to \mathrm{Aut}\,V \equiv \mathrm{GL}(V) \subseteq \mathrm{End}\,V, \quad \rho(gg') = \rho(g)\rho(g')$$

is called a **representation of group G in vector space V***. The dimension of V is called the* **dimension of representation ρ***. Thus, a representation is a left linear action.*

As was needed for representations of groups to have group structure which lead to automorphisms, endomorphisms constitute an associative algebra and are thus used to represent Lie algebras.

Definition 3.20 *A* **representation of a Lie algebra $\mathfrak{g}$** *in a vector space V is a map*[16]

$$f : \mathfrak{g} \to \mathrm{End}\,V \equiv \mathfrak{gl}(V)$$

satisfying two linearities:

$$f(X + \lambda Y) = f(X) + \lambda f(Y);$$
$$f([X,Y]) = [f(X), f(Y)] \equiv f(X)f(Y) - f(Y)f(X).$$

As expected, linear operators, representations of basis vectors of $\mathfrak{g}$ (generators of the representation), $\rho'_i \equiv f(E_i)$, satisfy the same commutation relations:

$$[E_a, E_b] = C^c_{ab} E_c \quad \Rightarrow \quad [\rho'_a, \rho'_b] = C^c_{ab}\rho'_c.$$

Definition 3.21 *Let G be a Lie group and ρ its representation, then the derived homomorphism ρ':*

$$\rho\,(\exp X) = \exp\rho'(X), \quad \rho'(X) = \frac{d}{dt}\exp(tX)\bigg|_{t=0}$$

is a representation of $\mathfrak{g}$ and is called a **derived representation of $\mathfrak{g}$***.*

[15]If G is a Lie group, then, obviously, the representation map $G \times V \to V$, $(g,v) \mapsto \rho(g)v$ must be smooth.

[16]A Lie algebra homomorphism.

Definition 3.22 *Let $\rho : G \to \operatorname{Aut} V$ be a representation of Lie group G. Define $\tilde{\rho} : G \to \operatorname{Aut} V^*$:*[17]

$$\langle \tilde{\rho}(g)\omega, X\rangle \equiv \langle \omega, \rho(g^{-1})X\rangle, \quad \omega \in V^*,\ X \in V,$$

called the **dual representation**.

If we define a basis $\{v_a\}$ and $\{v^a\}$ of V and V^* respectively, and $\{E_i\}$ of $\mathfrak{g}$, then a representation of G and its dual, and the derived representation and its dual can be expressed as matrix elements:

$$\rho(g)v_a \equiv A^b_a v_b \quad \Rightarrow \quad \tilde{\rho}(g)v^a = (A^{-1})^a_b v^b;$$
$$\rho'(E_i)v_a \equiv \rho^b_{ai} v_b \quad \Rightarrow \quad \tilde{\rho}'(E_i)v^a = -\rho^a_{bi} v^b.$$

Definition 3.23 *A representation ρ of Lie group G (or ρ' of Lie algebra $\mathfrak{g}$) is said to be* **faithful** *if $\rho : G \to \operatorname{Aut} V$ (or $\rho' : \mathfrak{g} \to \operatorname{End} V$) is injective. If a representation is faithful then G is isomorphic to the subgroup $\operatorname{im}\rho \subseteq \operatorname{Aut} V$.*[18]

Definition 3.24 *Let G be a Lie group with representation ρ in an inner product space (V, h). If action ρ and the inner product h are compatible in the sense that:*

$$h(\rho(g)v, \rho(g)w) = h(v, w), \quad v, w \in V,\ g \in G,$$

then h is said to be a **ρ-invariant inner product**.

If we have a ρ-invariant inner product h of type (r, s) in V, then it can be shown that operators $\rho(g)$ must be (pseudo) orthogonal,

$$\rho : G \to \mathrm{O}(r, s) \equiv \operatorname{Aut}(V, h) \subseteq \operatorname{Aut} V, \quad r + s = \dim V.$$

Obviously operators of the derived representation of $\mathfrak{g}$ must satisfy:

$$h(\rho'(X)v, w) = -(v, \rho'(X)w), \quad X \in \mathfrak{g},$$

and are therefore (pseudo) antisymmetric operators,

$$\rho' : \mathfrak{g} \to \mathfrak{o}(r, s) \subseteq \operatorname{End} V.$$

Expressed in matrix notation (in a defined basis) this means:

$$\rho_{abi} = -\rho_{bai}, \quad \rho_{abi} \equiv h_{ac}\rho^c_{bi}.$$

In case of a complex inner product space, instead of the orthogonal representation, we get the unitary representation of G and derived antihermitian representation of $\mathfrak{g}$.

Definition 3.25 *Let (ρ, V) be a representation of group G. Define an* **invariant subspace** *$W \subseteq V$ as one that is closed with respect to all transformations of representation ρ,*

$$\rho(G)W \subseteq W \quad \Leftrightarrow \quad w \in W \Rightarrow \rho(g)w \in W, \forall g \in G.$$

If W can only be trivial ($W = V$ or $W = \{0\}$) then (ρ, V) is said to be an **irreducible representation** *otherwise it is called* **reducible**. *Furthermore, say there exists a non-trivial invariant subspace W, if there exists another invariant subspace $\bar{W}$ such that it is the complement of W: $V = W \oplus \bar{W}$, then (ρ, V) is called* **completely reducible** *or* **decomposable**.

A few properties on the reducibility of representations of Lie groups:

(i) reducibility or complete reducibility of ρ carries over to the derived representation ρ';

(ii) if there is a invariant inner product then reducibility necessarily implies decomposability;

(iii) for compact groups reducibility necessarily implies decomposability.

Definition 3.26 *Two representations (ρ_1, V_1) and (ρ_2, V_2) are said to be* **equivalent** *if there exists an isomorphism A such that:*

$$\rho_2(g) = A\rho_1(g)A^{-1}, \quad \forall g \in G.$$

This implies that if V has a ρ-invariant inner product, then the dual representation $\tilde{\rho}$ is equivalent to the representation ρ itself.

Theorem 3.4 (First Schur's lemma)[19] *Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible representations of group G and A a linear map such that:*

$$A : V_1 \to V_2, \quad A\rho_1(g) = \rho_2(g)A.$$

Then:

(i) either ρ_1 and ρ_2 are inequivalent and $A = 0$ or they are equivalent and A is an isomorphism;

(ii) if ρ_1 and ρ_2 are equivalent complex representations then A is unique up to a scalar factor $\in \mathbb{C}$.

[17] Here we have used the bracket notation for contraction $\langle \omega, X\rangle \equiv i_X\omega$ for $X \in V$ and $\omega \in V^*$.

[18] A well known, however, rarely emphasised, faithful representation is the (non-canonical see footnote on page 6) isomorphism between the abstract general linear group $\mathrm{GL}(V)$ (where V is an n-dimensional vector space above field K) and $\mathrm{GL}(n, K)$, the space of all $n \times n$ invertible matrices. Because this representation is faithful a distinction between these spaces is very rarely made and matrices, elements of $\mathrm{GL}(n, K)$, are understood as group elements of **matrix group** $\mathrm{GL}(n, K)$. The same is valid for the corresponding Lie algebra $\mathfrak{gl}(n, K)$ that consists of all $n \times n$ matrices. [12]

[19] It is important to note that Schur's lemma applies to any family of operators that act irreducibly and commute with the whole family, not just representations of groups.

Theorem 3.5 (Second Schur's lemma)[19] *Let* (ρ, V) *be an irreducible complex representation and* A *a linear operator in* V *that commutes with all operators of the representation:*

$$[A, \rho(g)] = 0, \quad \forall g \in G.$$

A *must then be of the form:*

$$A = \lambda \mathbb{1}, \quad \lambda \in \mathbb{C}.$$

Definition 3.27 *Define the conjugation map* $I_g : G \to G$ *(a bijective homomorphism) which maps* $h \mapsto ghg^{-1}$. *The derived homomorphism*

$$\mathrm{Ad}_g \equiv I_{g*} : G \to \mathrm{Aut}\,\mathfrak{g}$$

or,

$$\exp \mathrm{Ad}_g X \equiv g \exp X g^{-1}, \quad X \in \mathfrak{g};$$

is a representation called the **adjoint representation** *or inner automorphism.*

Since the adjoint representation uses $\mathfrak{g}$ as the carrier space it is not only linear but also preserves the Lie bracket,

$$\mathrm{Ad}_g[X, Y] = [\mathrm{Ad}_g X, \mathrm{Ad}_g Y], \quad X, Y \in \mathfrak{g}.$$

Adjoint representations arise when right-translating left invariant objects and when considering relations between right and left-invariant fields. Let L_X be a left-invariant vector field generated by X at $e \in G$, also let $\{E_i\}$ and $\{E^i\}$ be bases of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, and $\{e_i\}$ and $\{e^i\}$ the corresponding left-invariant frame and coframe fields. Then the following holds:

$$L_g^* L_X = L_X \quad R_g^* L_X = L_{\mathrm{Ad}_g X},$$

or for the frame and coframe fields:

$$\begin{aligned} L_g^* e_i &= e_i & R_g^* e_i &= (\mathrm{Ad}_g)^i_j e_j; \\ L_g^* e^i &= e_i & R_g^* e^i &= (\mathrm{Ad}_{g^{-1}})^j_i e_j. \end{aligned}$$

If θ is the Maurer-Cartan 1-form on G then we have:

$$\begin{aligned} L_g^* \theta &= \theta & (L_g^* \theta^i) E_i &= \theta^i E_i; \\ R_g^* \theta &= \mathrm{Ad}_{g^{-1}} \theta & (R_g^* \theta^i) E_i &= (\mathrm{Ad}_{g^{-1}})^i_j \theta^j E_i. \end{aligned}$$

Definition 3.28 *Let* $\rho = \mathrm{Ad}$ *be the adjoint representation of* G *in* $\mathfrak{g}$, *then the* **derived adjoint representation** ad *is:*

$$\rho' = \mathrm{Ad}' \equiv \mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}, \quad \mathrm{Ad}_{\exp X} = \exp \mathrm{ad}_X.$$

This is a representation of $\mathfrak{g}$ in $\mathfrak{g}$ with the following properties:

(i) $\mathrm{ad}_X Y = [X, Y]$;

(ii) $\mathrm{ad}_{X+\lambda Y} = \mathrm{ad}_X + \lambda\, \mathrm{ad}_Y$;

(iii) $\mathrm{ad}_{[X,Y]} = [\mathrm{ad}_X, \mathrm{ad}_Y]$;

(iv) if $\{E_i\}$ is a basis of $\mathfrak{g}$ then the representation matrix elements are given by the structure constants of $\mathfrak{g}$:
$$\mathrm{ad}_{E_i} E_j = (\mathrm{ad}_{E_i})^k_j E_k = C^k_{ij} E_k.$$

Definition 3.29 *If* G *is a Lie group and* $\mathfrak{g}$ *is its Lie algebra, then we define a bilinear symmetric form* $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

$$\begin{aligned} K : (X, Y) \mapsto K(X, Y) &\equiv \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y) \\ &\equiv \langle E^i, \mathrm{ad}_X \mathrm{ad}_Y E_i \rangle, \end{aligned}$$

where E_i *and* E^i *are basis vectors of* $\mathfrak{g}$ *and* $\mathfrak{g}^*$ *respectively.* K *is called the* **Killing-Cartan form**.

The Killing-Cartan form has very useful properties:

(i) the matrix components in basis $E_i \in \mathfrak{g}$
$$k_{ij} \equiv K(E_i, E_j) = C^k_{il} C^l_{jk}$$
satisfy:
$$k_{ij} = (\mathrm{Ad}_g)^i_r (\mathrm{Ad}_g)^j_s = k_{rs};$$

(ii) $K(X, Y)$ is invariant to all automorphisms of group G,
$$K(A(X), A(Y)) = K(X, Y), \quad A \in \mathrm{Aut}\,\mathfrak{g}$$
(including Ad-invariance: $A = \mathrm{Ad}_g$);

(iii) Ad-invariance can be stated in infinitesimal form as:
$$K(\mathrm{ad}_Z X, Y) = -K(X, \mathrm{ad}_Z Y),$$
or:
$$K([Z, X], Y) = -K(X, [Z, Y]),$$
or:
$$C_{ijk} + C_{jik} = 0, \quad C_{ijk} \equiv k_{il} C^l_{jk};$$

(iv) we can therefore construct a 3-form with components C_{ijk}:
$$C \equiv \frac{1}{3!} C_{ijk}\, E^i \wedge E^j \wedge E^k,$$
that is also Ad-invariant,
$$C(\mathrm{Ad}_g X, \mathrm{Ad}_g Y, \mathrm{Ad}_g Z) = C(X, Y, Z).$$

The Killing-Cartan form is not always non-degenerate, however, and does not always define an invariant scalar product in $\mathfrak{g}$.

Theorem 3.6 *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if:*

(i) **(Cartan's criterion)** *the Killing-Cartan form on* $\mathfrak{g}$ *is non-degenerate;*

(ii) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$, *where* $\mathfrak{g}_j$ *are simple Lie algebras.*

Theorem 3.7 *A Lie algebra $\mathfrak{g}$ is solvable if and only if its Killing-Cartan form K satisfies:*

(i) **(Cartan's criterion)**

$$K(X,Y)=0,\quad \forall X\in\mathfrak{g}, \forall Y\in[\mathfrak{g},\mathfrak{g}];$$

(ii) $K(\mathfrak{g},\mathscr{D}\mathfrak{g})=0$*;*

(iii) $K(\mathrm{ad}_X,\mathrm{ad}_X)=0,\quad \forall X\in\mathfrak{g}$.

We are now interested in the tensor field induced by the Killing-Cartan metric in $\mathfrak{g}$ (if $\mathfrak{g}$ is semisimple). Since K is non-degenerate so must the tensor field be, thus we have a metric tensor on G: $\mathcal{K}$. $\mathcal{K}$ is by construction left-invariant, however, it is also right-invariant:

$$\mathcal{K}\equiv k_{ij}\, e^i\otimes e^j,\quad k_{ij}\equiv K(E_i,E_j);$$

$$L_g^*\mathcal{K}=\mathcal{K},\quad R_g^*\mathcal{K}=\mathcal{K}.$$

Returning to volume forms, let $\omega_L\equiv e^1\wedge\cdots\wedge e^n$ and $\omega_R\equiv f^1\wedge\cdots\wedge f^n$ be left and right-invariant volume forms (we denote the right-invariant coframe field by $\{f^i\}$). It can be shown that they are related by:

$$\omega_R(g)=\det\mathrm{Ad}_g\,\omega_L(g).$$

Theorem 3.8 *Let G be a* **compact** *Lie group. All (right) left-invariant volume forms are also (left) right-invariant.*

Theorem 3.9 *There always exists a positive-definite Ad-invariant metric tensor on the Lie algebra of a* **compact** *Lie group.*

Definition 3.30 *A Lie algebra with a positive-definite Ad-invariant metric tensor is called a* **compact Lie algebra**. *Compact Lie algebras correspond to compact Lie groups.*

And lastly, we move our focus to the dual Lie algebra $\mathfrak{g}^*$.

Definition 3.31 *Let G be a Lie group and Ad_g its adjoint representation. The* **coadjoint representation** Ad_g^* *is then the dual representation to Ad_g:*[20]

$$\langle\mathrm{Ad}_g^*\,X^*,Y\rangle\equiv\langle X^*,\mathrm{Ad}_{g^{-1}}\,Y\rangle,\quad X^*\in\mathfrak{g}^*,\ Y\in\mathfrak{g}.$$

The coadjoint representation has the following properties:

(i) if there exists a non-degenerate Ad-invariant scalar product in $\mathfrak{g}$, then the coadjoint representation is equivalent to the adjoint;

(ii) $\mathrm{Ad}_g^*\,E^i=E^j(\mathrm{Ad}_{g^{-1}})^i_j$;

(iii) in the derived representation ad_X^* the generators are given by:

$$\langle\mathrm{ad}_X^*\,Z^*,Y\rangle=-\langle Z^*,[X,Y]\rangle,$$

$$\mathrm{ad}_{E_i}^*\,E^j=(\mathrm{ad}_{E_i}^*)^j_k E^k=-C^j_{ik}E^k.$$

To finish off this section we go back to actions of groups on manifolds. Let ξ_X be a fundamental field on manifold M, then it can be shown that ξ_X behaves with respect to the right action as:

$$R_g^*\xi_X=\xi_{\mathrm{Ad}_g X},\quad R_g^*\xi_{E_i}\equiv R_g^*\xi_i=(\mathrm{Ad}_g)^j_i\xi_j.$$

Two statements are valid:

(i) $[\xi_X,\xi_Y]=\xi_{[X,Y]},\quad [\xi_i,\xi_j]=C^k_{ij}\xi_k$;

(ii) the prescription $X\mapsto\rho'(X)\equiv\xi_X$ is a representation of the Lie algebra $\mathfrak{g}$ in $\mathcal{F}(M)\equiv\mathcal{T}^0_0(M)$.[21]

Definition 3.32 *Let $A\in\mathcal{T}^r_s(M)$ be a tensor field on a right G-space M, then the prescription*

$$A\mapsto\rho(g)A=R_g^*A,\quad A\mapsto\rho'(X)=\pounds_{\xi_X}A$$

defines a representation of G in $\mathcal{T}^r_s(M)$ and the derived representation of $\mathfrak{g}$.[22]

Say we have a set of N tensor fields $A^a\in\mathcal{T}^r_s(M)$, $a=1,\dots,N$ such that upon the action R_g^* of G they mix only amongst themselves and this mixing is defined by matrix r^a_b: $R_g^*A^a=r^a_b(g)A^b$, then:

(i) the prescription $g\mapsto r^a_b(g)$ is a right action (an antirepresentation);

(ii) the prescription $g\mapsto r^a_b(g^{-1})\equiv\hat\rho^a_b(g)$ is a representation;

(iii) if A^a is a component of a V-valued tensor field then $\hat\rho$ is a representation in space V with basis $\{E_a\}$: $\hat\rho(g)E_a=\hat\rho^b_a(g)E_b$ then the right action is:

$$(R_g^*A^a)E_a=A^a(\hat\rho(g^{-1})E_a)$$
$$\Updownarrow$$
$$R_g^*A^a=\hat\rho^a_b(g^{-1})A^b.$$

[20]It is important to note that, had we defined the coadjoint representation by:

$$\langle\mathrm{Ad}_g^*\,X^*,Y\rangle\equiv\langle X^*,\mathrm{Ad}_g\,Y\rangle,$$

we would have arrived at a right linear action and Ad_g^* would, therefore, have been an antirepresentation (one where the commutator has opposite sign).

[21]It is actually just the derived representation of the representation of G in $\mathcal{F}(M)$ defined by the prescription: $\psi\mapsto\rho(g)\psi\equiv\psi\circ R_g$. This representation also has an interesting property, it preserves the product: $\rho(g)(\psi\phi)=(\rho(g)\psi)(\rho(g)\phi)$.

[22]The Lie derivative is denoted as $\pounds_X:\mathcal{T}^r_s(M)\to\mathcal{T}^r_s(M)$ for $X\in\mathfrak{X}(M)$.

Definition 3.33 *Let A be a $(V, \hat{\rho})$-valued tensor field. If A satisfies:*

$$R_g^* A = \hat{\rho}(g^{-1}) A,$$

then it is called a **tensor field of type $\hat{\rho}$**.

We can now explicitly state the representation ρ for $(V, \hat{\rho})$-valued tensors in $\mathcal{T}_s^r(M, V)$.

Definition 3.34 *The prescription:*

$$\rho(g) \equiv \hat{\rho}(g) \circ R_g^* = R_g^* \otimes \hat{\rho}(g)$$
$$\Updownarrow$$
$$\rho(g)A = \rho(g)(A^a E_a) \equiv (R_g^* A^a)(\hat{\rho}(g) E_a).$$

is the **representation of $(V, \hat{\rho})$-valued tensors** *in $\mathcal{T}_s^r(M, V)$.*

It then follows that tensor fields of type $\hat{\rho}$ are just fields that are invariant with respect to representation ρ,

$$\rho(g)A = A.$$

The derived representation of ρ is then:

$$\rho'(X) = \pounds_{\xi_X} + \hat{\rho}'(X)$$
$$\Updownarrow$$
$$\rho'(X)A = \rho'(X)(A^a E_a) = (\pounds_{\xi_X} A^a) E_a + A^a (\hat{\rho}'(X) E_a).$$

The requirement that a field be of type $\hat{\rho}$ can now be written as:

$$\pounds_{\xi_X} A = -\hat{\rho}'(X) A.$$

Now that we have introduced the very basics of Lie groups and algebras we move on to more complicated geometrical objects on manifolds.

4 Fibre bundles and connections

A basic differential manifold is a very rich structure, however, from it an even richer manifold can be constructed, a fibre bundle. Fibre bundles play an extremely important role in mathematical physics but can also be applied to almost all branches of physics. Therefore, this section is dedicated to the study of fibre bundles and a special additional structure that can be placed on them called a connection.

4.1 Fibre bundles

Definition 4.1 *A (differential)* **fibre bundle** *(E, π, M, F, G) or $\pi : E \to M$ for short, is a collection of :*[23]

(i) the **total space***, a differentiable manifold E;*[24]

(ii) the **base manifold***, a differentiable manifold M;*

(iii) the **fibre***, a differentiable manifold F;*

(iv) the **projection***, a surjection $\pi : E \to M$ with the inverse $\pi^{-1}(p) \equiv F_p \cong F$;*

(v) the **structure group***, a group*[25] *G which acts on the fibre from the left;*

(vi) the **local trivialization***, a diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$, $\forall f \in F$, where $\{U_i\}$ is an open covering of M;*

(vii) the **transition functions***, smooth maps $t_{ij} : U_i \cap U_j \to G$ if $U_i \cap U_j \neq \varnothing$ such that $\phi_j(p, f) = \phi_i(p, t_{ij}(p) f)$.*

So a fibre bundle is, essentially, an, in general,"twisted" product manifold $M \times F$. The transition functions $t_{ij} : U_i \cap U_j \to G$ must satisfy:

$$t_{ii}(p) = \mathbb{1},$$
$$t_{ij}(p) = t_{ji}(p)^{-1},$$
$$t_{ij}(p) t_{jk}(p) = t_{ik}(p).$$

These transition functions are not unique as they depend on the chosen local trivializations, $t_{ij} = \phi_i^{-1} \circ \phi_j$. If all transition functions can be chosen to be identity maps, then the bundle is globally a product bundle (in general bundles are only locally product structures) also called a **trivial bundle**. Fibre bundles are often called according to some additional structure the fibre manifold has, such as a vector bundle[26] if the fibre is a vector space or a principal bundle if the fibre is the structure Lie group.

Definition 4.2 *Let $\pi : E \to M$ be a fibre bundle. A map*

$$\sigma : M \supseteq U \to E, \quad \pi \circ \sigma = \mathrm{id}_U,$$

is called a **local section***, obviously, $\sigma(p) \in F_p \equiv \pi^{-1}(p)$. If $U = M$ then the section is said to be* **global***, though not all fibre bundles admit global sections. The set of all local sections on U is denoted by $\Gamma(U, F)$.*[27]

[23] Strictly mathematically, a fibre bundle cannot depend on the choice of open covering $\{U_i\}$ so this definition is more precisely a *coordinate bundle* $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$. Two coordinate bundles $(E, \pi, M, F, G, \{U_i\}, \{\phi_i\})$ and $(E, \pi, M, F, G, \{V_j\}, \{\psi_j\})$ are said to be equivalent if $(E, \pi, M, F, G, \{U_i\} \cup \{V_j\}, \{\phi_i\} \cup \{\psi_j\})$ is again a fibre bundle. Then, an equivalence class of coordinate bundles is a fibre bundle. In a physical context a covering is usually implicitly defined so this distinction is ignored.

[24] E itself is often referred to as the fibre bundle.

[25] Most often a Lie group.

[26] For example, the tangent bundle TM and cotangent bundle T^*M are vector bundles.

[27] For example, the set of all vector fields $\mathfrak{X}(M) \equiv \Gamma(M, TM)$ or 1-form fields $\Omega^1(M) \equiv \Gamma(M, T^*M)$.

Definition 4.3 *Let $\pi : E \to M$ and $\pi' : E' \to M'$ be fibre bundles. A pair of smooth maps $\bar{f} : E' \to E$ and $f : M' \to M$ is called a* **bundle map** *if $\pi' \circ \bar{f} = f \circ \pi$, that is, $\bar{f}$ maps each fibre F'_p of E' onto F_p of E. Two bundles are* **equivalent** *if there exists a bundle map such that $\bar{f}$ is a diffeomorphism and f is an identity map.*

Definition 4.4 *If $\pi : E \to M$ is a fibre bundle and $f : N \to M$, then we define a new bundle f^*E called a* **pullback bundle***:*

$$f^*E \equiv \{(p, u) \in N \times E \,|\, f(p) = \pi(u)\},$$

with the maps $\pi_1 : f^*E \to N$ and $\pi_2 : f^*E \to E$. The transition functions of a pullback bundle are then: $t^*_{ij}(p) = t_{ij}(f(p))$.

Theorem 4.1 *Fibre bundle $\pi : E \to M$ is trivial if M is contractible to a point.*

Definition 4.5 *Let $\pi : E \to M$ and $\pi : E' \to M'$ be vector bundles over M. The* **tensor product bundle** *$E \otimes E'$ is a bundle with fibre $F_p \otimes F'_p$, $\forall p \in M$. If $\bigotimes^r E$ is a tensor product bundle and $\{e_a\}$ is a basis of fibre F, then the fibre of $\bigotimes^r E$ is spanned by $\{e_{a_1} \otimes \cdots \otimes e_{a_r}\}$.*[28]

Definition 4.6 *A* **principal bundle** *is a fibre bundle $P(M, G) \equiv \pi : P \to M$ where the fibre is diffeomorphic to the structure group G. Other than the left action of the transition functions, there is also a defined free and transitive right action:*

$$R_g : P \to P, \quad R_{gh} = R_h \circ R_g, \quad \pi \circ R_g = \pi,$$

with the requirement that the local trivialization satisfy:

$$\phi_i : U_i \times G \to \pi^{-1}(U_i); \quad (p, hg) \mapsto ug \equiv R_g u, \quad \forall g \in G.$$

If a section $\sigma_i(p)$ is given, we can always construct a local trivialization $\phi_i(p, e) = \sigma_i(p)$ called the **canonical local trivialization***.*

For two principal bundles to be equivalent there is another requirement stemming from the additional right action defined on them: if $P(M, G)$ and $P'(M, G)$ are two G-bundles over the same base space M they are equivalent if there exists a diffeomorphism

$$f : P \to P', \quad R'_g \circ f = f \circ R_g, \quad \pi' \circ f = \pi.$$

Therefore if there exists such a diffeomorphism between a principal bundle P and $M \times G$, it is a global trivialization and so P is trivial.

Theorem 4.2 *A principal bundle is trivial if and only if it admits a global section.*

4.2 Vertical subspaces and lifts

Definition 4.7 *Let $\pi : E \to M$ be a fibre bundle. The existence of a projection mapping π means that there is always a special subspace of the tangent space $T_u E$:*

$$V_u E \subseteq T_u E, \quad V_u \equiv \ker \pi_{*u},$$

called a **vertical subspace***. Thus, a vector $W \in T_u E$ is said to be* **vertical** *if $\pi_* W = 0$ (written W^v or $\mathrm{ver}\, W$).*

Geometrically, if a vector is vertical this means that it is tangent to the fibre at that point. If our fibre bundle is a principal bundle then for the added structure of right action we have:

$$R_{g*} V_u P = V_{ug} P.$$

Definition 4.8 *If $\pi : E \to M$ is a fibre bundle then a* **lift** *is in general a procedure*[29] *which assigns to a geometrical object of the base manifold M a geometrical object on the total space E of the bundle.*

We introduce the concept of lifts through two examples of the tangent and cotangent bundles.

Example 4.1 (Curves in M) *Let*

$$\gamma : \mathbb{R} \to M, \quad t \mapsto \gamma(t)$$

be a curve in M, then:

$$\hat{\gamma} : \mathbb{R} \to TM, \quad t \mapsto \dot{\gamma}(t),$$

is a lift called the **natural lift** *of γ to TM. It can be shown that this lift is always exactly over γ: $\pi \circ \hat{\gamma} = \gamma$.*

Example 4.2 (Vectors on TM) *Let $u \in T_p M \equiv \pi^{-1}(p)$ be a vector on M (point in $T_p M$). Define a curve in fibre $\pi^{-1}(p)$:*

$$\Sigma(t) \equiv u + tw \in T_p M,$$

for point $w \in \pi^{-1}(p)$. Then, the tangent vector to curve Σ,

$$u^v \equiv \dot{\Sigma}(0) = \frac{d}{dt}\Big(u + tw\Big)\Big|_{t=0} \in V_w TM \subseteq T_w TM$$

is called a **vertical lift** *of vector u in p to $w \in TM$. If $u \in \mathfrak{X}(M)$ is a vector field, then we can simply define its vertical lift at every point by the previous construction and obtain the vertical lift of the vector field*[30] *u, $u^v \in \mathfrak{X}(TM)$.*[31]

[28] We can also define the $\Lambda^r E$ bundle with respect to the antisymmetrised tensor product $\wedge$, therefore, $\Omega^r(M) \equiv \Gamma(M, \Lambda^r T^* M)$.
[29] Most certainly not unique.

Let us take the canonical coordinates of a coordinate patch U of the tangent bundle TM ($\dim M = m$) given by its local trivialization:

$$\phi : \mathbb{R}^{2m}[x^1, \dots, x^m, v^1, \dots, v^m] \to \pi^{-1}(U).$$

This means that at a point $u \in TU$

$$\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^m} \right\}$$

span $T_u TU$. It trivially follows that $\left\{ \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^m} \right\}$ span the vertical subspace $V_u TU \subseteq T_u TU$. It is now natural to assume $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m} \right\}$ spans a "horizontal" subspace, however this is false in the sense that this subspace depends on the choice of coordinates. If we take a change of coordinates $x^i \mapsto x'^i(p)$ we see that $\left\{ \frac{\partial}{\partial v^i} \right\}$ transform amongst themselves and therefore still span the same space, whereas $\left\{ \frac{\partial}{\partial x'^i} \right\}$ contain $\frac{\partial}{\partial v^i}$ terms and thus no longer span the complementary space we called "horizontal". This implies that it is impossible to canonically decompose an arbitrary vector $w \in T_u TM$ into its "vertical" and "horizontal" components. This trivially generalises to arbitrary fibre bundles. The only way to define a horizontal space in a coordinate independent way is by adding additional structure to the bundle, this structure is called a connection and is defined next.

4.3 Connections and curvature on principal G-bundles

One way to define a connection on a vector bundle is to start by defining one on a principal bundle first. This is the construction that will be used here.

Definition 4.9 *A connection on a principle G-bundle is a unique separation of the tangent space $T_u P$ into the vertical $V_u P$ and horizontal $H_u P$ subspaces such that:*

(i) $T_u P = V_u P \oplus H_u P$;

(ii) a smooth vector field X on P is decomposed into smooth vector fields $X^v \in V_u P$ and $X^h \in H_u P$, $X = X^v + X^h$;

(iii) $H_{ug} P = R_{g*} H_u P$, *$\forall u \in P$ and $g \in G$.*

Definition 4.10 *Let $P(M, G)$ be a principle G-bundle and $\gamma : [0, 1] \to M$ be a curve in M, then its* **horizontal lift** *is a curve $\tilde{\gamma} : [0, 1] \to P$ such that $\pi \circ \tilde{\gamma} = \gamma$ and tangent vector*

$$\frac{d}{dt'} \tilde{\gamma}(t') \Big|_{t'=t} \in H_{\tilde{\gamma}(t)} P.$$

Theorem 4.3 *Let $\gamma : [0, 1] \to M$ be a curve in M, then:*

(i) if $u_0 \in \pi^{-1}(\gamma(0))$ there exists a unique horizontal lift $\tilde{\gamma}(t)$ in P such that $\tilde{\gamma}(0) = u_0$;

(ii) if there are two horizontal lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ such that $\tilde{\gamma}'(0) = \tilde{\gamma}(0)g$, it follows that $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$, $\forall t \in [0, 1]$ and $g \in G$.

Definition 4.11 *Define function Ψ_u:*

$$\Psi_u : \mathfrak{g} \to V_u P, \quad X \mapsto \xi_X(u),$$

where $X \in \mathfrak{g}$ and ξ_X is the fundamental field of action R_g on P.

One can show two properties:

(i) Ψ_p is a linear isomorphism;

(ii) any vertical vector at point $u \in P$ can always be written as a certain fundamental field in that point, that is, there is always an element $X \in \mathfrak{g}$ such that $V \equiv \operatorname{ver} V = \xi_X(u)$.

Definition 4.12 *Define at point $u \in P$ a Lie algebra valued 1-form ω_u by the property:*

$$\langle \omega_u, W_u \rangle \equiv \Psi_u^{-1}(\operatorname{ver} W),$$

or:

$$\omega_u \equiv \Psi^{-1} \circ \operatorname{ver} : T_u P \to \mathfrak{g}.$$

A **connection 1-form** *(Ehresmann connection) is a smooth 1-form field on P $\omega \in \Omega^1(P, \mathfrak{g})$ such that $\omega(u) \equiv \omega_u$, $\forall u \in P$.*

A connection 1-form has the following properties:

(i) if $W = W^h \equiv \operatorname{hor} W$, then $\langle \omega_u, W \rangle = 0$,

(ii) $H_u P = \ker \omega_u$,[32]

(iii) $R_g^* \omega = \operatorname{Ad}_{g^{-1}} \omega$, this means that ω is of type Ad: $\omega \in \Omega^1(P, \operatorname{Ad})$,

(iv) $\langle \omega, \xi_X \rangle = X$;

or infinitesimally:

(i) $\pounds_{\xi_X} \omega = -\operatorname{ad}_X \omega$,

(ii) $i_{\xi_X} \omega = X$,

(iii) $i_{\xi_X} \mathrm{d}\omega = -[X, \omega]$.

Therefore, the connection 1-form is just the projection operator onto the the vertical subspace $V_u P \cong \mathfrak{g}$.

Definition 4.13 *Define the* **horizontal part** *of a p-form $\alpha \in \Omega^p(P)$ as:*

$$(\operatorname{hor} \alpha)(U, V, \dots) \equiv \alpha(\operatorname{hor} U, \operatorname{hor} V, \dots).$$

[30] Notice that this construction of lifted fields is pointwise. In general, however, this need not be.
[31] Analogous to this construction is the vertical lift of a 1-form (field) in T^*M.
[32] This is sometimes regarded as the definition of a horizontal subspace.

This map, hor : $\Omega^p(P) \to \Omega^p(P)$, has the following properties:

(i) it is a projection:

$$\mathrm{hor} \circ \mathrm{hor} = \mathrm{hor}, \quad i_{\mathrm{ver}\, V} \,\mathrm{hor}\, \alpha = 0;$$

(ii) the connection form is completely vertical: $\mathrm{hor}\, \omega = 0$;

(iii) it is an endomorphism of the Cartan algebra $\Omega(P)$, its image defines a subalgebra

$$\bar{\Omega}(P) \equiv \mathrm{im}\,\mathrm{hor} \subseteq \Omega(P);$$

(iv) it preserves the type ρ of the form.

Definition 4.14 *The* **exterior covariant derivative** *is defined as:*

$$\mathrm{D}\alpha \equiv \mathrm{hor}\, \mathrm{d}\alpha.$$

It is a differential operator $\mathrm{D} : \Omega^p(P) \to \bar{\Omega}^p(P)$ *of degree* $+1$ *that preserves the type of forms.*

Definition 4.15 Curvature *is a* $\mathfrak{g}$*-valued 2-form defined as the exterior covariant derivative of the connection 1-form* ω*,*

$$\Omega \equiv \mathrm{D}\omega \equiv \mathrm{hor}\, \mathrm{d}\omega.$$

The curvature has the following properties:

(i) it can also be expressed as:

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega];$$

(ii) it is a horizontal 2-form of type Ad,

$$\mathrm{hor}\, \Omega = \Omega, \quad R_g^* \Omega = \mathrm{Ad}_{g^{-1}} \Omega;$$

(iii) it satisfies the **Bianchi identity**:

$$\mathrm{D}\Omega \equiv \mathrm{D}\,\mathrm{D}\omega = 0.$$

Definition 4.16 *Let* α *be a Lie algebra* $\mathfrak{g}$*-valued* p*-form and* β *a* q*-form with values in representation space* (W, ρ') *of* $\mathfrak{g}$*. Define the dot wedge product for* $\alpha = \alpha^i E_i$ *and* $\beta = \beta^a E_a$*:*

$$\rho'(\alpha) \dot{\wedge} \beta \equiv \alpha^i \wedge \beta^a \rho'(E_i) E_a.$$

If $\rho' = \mathrm{ad}$ *then* $\mathrm{ad}(\alpha) \wedge \beta = [\alpha \wedge \beta]$*.*

If $\omega = \omega^i E_i$ is the connection form, $\alpha = \alpha^a E_a$ a horizontal p-form of type ρ, and $\Phi = \Phi^a E_a$ a function of type ρ then their exterior covariant derivatives can be expressed as:

$$\begin{aligned} \mathrm{D}\alpha &= \mathrm{d}\alpha + \rho'(\omega) \dot{\wedge} \alpha & \qquad \mathrm{D}\alpha &= \rho'(\Omega) \dot{\wedge} \alpha; \\ \mathrm{D}\Phi &= \mathrm{d}\Phi + \rho'(\omega)\Phi & \qquad \mathrm{D}\Phi &= \rho'(\Omega)\Phi. \end{aligned}$$

This means that the Bianchi identity can be rewritten in the form:

$$\mathrm{d}\Omega + [\omega \wedge \Omega] = 0.$$

4.4 Local connections and local curvature

Definition 4.17 *Let* $\{U_i\}$ *be an open covering of the base manifold* M *and* σ_i *a local section for each* U_i*, then the* **local connection 1-form**[33] A_i *is a Lie algebra valued 1-form on* U_i*:*

$$A_i \equiv \sigma_i^* \omega \in \Omega^1(U_i, \mathfrak{g}).$$

Definition 4.18 *Let* Ω *be the curvature of the principle bundle* $P(M, G)$ *and* σ_i *a local section on a chart* U_i*, then we define the* **local curvature**[34] *on* $U_i \subseteq M$ *as the pullback of the curvature by* σ_i*,*

$$F_i \equiv \sigma_i^* \Omega \in \Omega^2(U_i, \mathfrak{g}).$$

Theorem 4.4 *Let* A_i *be a* $\mathfrak{g}$*-valued 1-form on* U_i *and* $\sigma_i : U_i \to \pi^{-1}(U_i)$ *a section on* U_i*, then there always exists a connection 1-form* ω *such that its pullback by* σ_i *is:* $A_i = \sigma_i^* \omega$*.*

Theorem 4.5 *Let* U_i *and* U_j *be overlapping charts on* M *and* (A_i) F_i *and* (A_j) F_j *local (connection forms) curvatures on these patches, then on* $U_i \cap U_j$ *they satisfy:*

$$\begin{aligned} A_j &= t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1}\, \mathrm{d}t_{ij}, \\ F_j &= \mathrm{Ad}_{t_{ij}^{-1}} F_i, \end{aligned}$$

where t_{ij} *are the transition functions between the charts.*

If v is a vector at point $p \in U \subseteq M$, and σ and σ' two sections on U related by the function $S : U \to G$, $\sigma'(p) = \sigma(p) S(p)$, then:

$$\sigma'_* v = R_{g*}(\sigma_* v) + \xi_X(u),$$

where $g = S(p)$, $X = \langle S^* \theta, v \rangle$ and $u = \sigma(p) S(p)$. The pullbacks by these sections of the connection, curvature, a horizontal p-form of type ρ: $\lambda = \sigma^* \Lambda$, and function of type ρ: $\phi = \sigma^* \Phi$, transform with respect to S in the following way:

$$\begin{aligned} A' &= \mathrm{Ad}_{g^{-1}} A + S^* \theta, \\ F' &= \mathrm{Ad}_{g^{-1}} F, \\ \lambda' &= \rho(S^{-1}) \lambda, \\ \phi' &= \rho(S^{-1}) \phi. \end{aligned}$$

[33] In a physical context more commonly called the gauge field or potential because of its significance in gauge theory.
[34] Also in physics commonly called the gauge field strength.

Let ρ be a representation of G in W (the representation space in which Λ, a type ρ horizontal p-form and Φ, a type ρ function have their values), then let E_a and E_i be bases in W and $\mathfrak{g}$ respectively and ρ' the derived representation of $\mathfrak{g}$. Then, obviously,

$$\begin{array}{lll} \omega = \omega^i E_i & A \equiv \sigma^*\omega = A^i E_i & \lambda \equiv \sigma^*\Lambda = \lambda^a E_a \\ \Omega = \Omega^i E_i & F \equiv \sigma^*\Omega = F^i E_i & \phi \equiv \sigma^*\Phi = \phi^a E_a \end{array}$$

are the components in these bases. In the derived representation we denote the local connection and curvature as:

$$\begin{array}{ll} \mathcal{A} \equiv \rho'(A) = A^i\rho'(E_i) & \mathcal{A}E_b = A^i\rho^c_{bi}E_c \equiv \mathcal{A}^c_b E_c; \\ \mathcal{F} \equiv \rho'(F) = F^i\rho'(E_i) & \mathcal{F}E_b = F^i\rho^c_{bi}E_c \equiv \mathcal{F}^c_b E_c. \end{array}$$

Definition 4.19 *Define the* **exterior covariant derivative on the base space** *(local exterior covariant derivative) by:*

$$\mathcal{D} \circ \sigma^* = \sigma^* \circ \mathrm{D}.$$

A local exterior derivative defined in this way satisfies:

(i) the Cartan structure equations,

$$\begin{aligned} \mathcal{F} &= \mathcal{D}\mathcal{A} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} \\ \mathcal{F}^a_b &= \mathcal{D}\mathcal{A}^a_b = \mathrm{d}\mathcal{A}^a_b + \mathcal{A}^a_c \wedge \mathcal{A}^c_b; \end{aligned}$$

(ii) the Bianchi identity,

$$\begin{aligned} \mathcal{D}\mathcal{F} &\equiv \mathrm{d}\mathcal{F} + \mathcal{A}\wedge\mathcal{F} - \mathcal{F}\wedge\mathcal{A} = 0 \\ \mathcal{D}\mathcal{F}^a_b &\equiv \mathrm{d}\mathcal{F}^a_b + \mathcal{A}^a_c\wedge\mathcal{F}^c_b - \mathcal{F}^a_c\wedge\mathcal{A}^c_b = 0; \end{aligned}$$

(iii) the action on horizontal p-forms and functions of type ρ:

$$\begin{array}{ll} \mathcal{D}\lambda = \mathrm{d}\lambda + \mathcal{A}\wedge\lambda & \mathcal{D}\lambda^a = \mathrm{d}\lambda^a + \mathcal{A}^a_b\wedge\lambda^b, \\ \mathcal{D}\phi = \mathrm{d}\phi + \mathcal{A}\phi & \mathcal{D}\phi^a = \mathrm{d}\phi^a + \mathcal{A}^a_b\lambda^b; \end{array}$$

(iv) the Ricci identity:

$$\begin{array}{ll} \mathcal{D}\mathcal{D}\lambda = \mathcal{F}\wedge\lambda & \mathcal{D}\mathcal{D}\lambda^a = \mathcal{F}^a_b\wedge\lambda^b, \\ \mathcal{D}\mathcal{D}\phi = \mathcal{F}\phi & \mathcal{D}\mathcal{D}\phi^a = \mathcal{F}^a_b\phi^b. \end{array}$$

This concludes our study of connections and curvatures on principal bundles. Next we will see how to move on from principal bundles and its connections to vector bundles.

4.5 Associated bundles

Definition 4.20 *Let $P(M,G)$ be a principal bundle. An* **associated fibre bundle** *(E,π,M,G,F,P) is the quotient space,*

$$P\times_\rho F \equiv (P\times F)/G,$$

in which a point is an equivalence class,

$$[(u,f)] \sim [(ug, \rho(g^{-1})f)],$$

for $u\in P$, $f\in F$, $g\in G$ and ρ a left action of G on F. The projection is given by:

$$\pi_E : E\to M, \quad [u,f]\mapsto \pi(u)\equiv p,$$

the transition functions by $\rho(t_{ij}(p))$ if $t_{ij}(p)$ are those of P, and the local trivialization by $\psi_i : U_i\times F\to \pi_E^{-1}(U_i)$.

The most important associated bundle is the associated vector bundle in which the fibre $F=V$ is a vector space and (V,ρ) a representation of G. Notice, also, that sections of associated vector bundles can be canonically identified with quantities of type ρ.[35]

Example 4.3 *Let M be a manifold, then TM and T^*M are associated vector bundles to the frame bundle LM. More generally, all tangent tensor bundles T^r_sM are associated to LM.*

Obviously we can reverse the "direction" of association; a vector bundle E, also, induces an associated principal bundle $P(E)\equiv P(M,G)$ where G is the structure group of E.[36] Since the transition functions of the associated bundles are the same as those of the principal bundle this implies the global topological properties of these bundles must also coincide. In particular we have the following.

Theorem 4.6 *A vector bundle E is trivial if and only if its associated principal bundle admits a global section (is trivial).*

Moving on now to connections on associated bundles. A connection on $P(M,G)$ implies a connection on E as well defined through the use of the covariant derivative of a section of E.

Definition 4.21 *Let $P(M,G)$ be a principal bundle and E an vector bundle associated to it, and $s: M\to E$, $s(p)=[\sigma_i(p),\eta(p)]$ a section. Along a curve $\gamma:[0,1]\to M$ we choose $s(t)=[\tilde\gamma(t),\eta(t)]$ with respect to a horizontal lift of γ. Then, we define a* **covariant derivative of section s on associated vector bundle** *E along curve γ at $p_0=\gamma(0)$ by:*

$$\nabla_X s \equiv \left[\left(\tilde\gamma(0), \frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right)\right].$$

[35] Quantities of type ρ are a type of mapping known as *equivariant* maps on P, in this case they are maps that commute with the action of G:

$$\Phi : P\to (V,\rho), \quad \Phi\circ R_g = \rho(g^{-1})\circ\Phi.$$

For section $s: M\to E$ we can identify $s(p)=[u,v]=[u,\Phi(u)]$ and since this does not depend on the choice of u over p: $[ug,\Phi(ug)]=[ug,\rho(g^{-1})\Phi(u)]=[u,\Phi(u)]=s(p)$, we have a one-to-one correspondence.

[36] Therefore, we can also say LM is the associated principal bundle of T^r_sM.

It is very important to notice that this derivative does not depend on the choice of horizontal lift $\tilde{\gamma}$. Say we picked another lift $\tilde{\gamma}'(t) = \tilde{\gamma}(t)g$, $g \in G$:

$$s(t) = [\tilde{\gamma}(t), \eta(t)] = [\tilde{\gamma}'(t), \rho(g^{-1})\eta(t)],$$

$$\begin{aligned} &\left[\left(\tilde{\gamma}'(0), \frac{d}{dt}\left(\rho(g^{-1})\eta(\gamma(t))\right)\Big|_{t=0}\right)\right] \\ &= \left[\left(\tilde{\gamma}'(0)g^{-1}, \frac{d}{dt}\eta(\gamma(t))\Big|_{t=0}\right)\right] = \nabla_X s. \end{aligned}$$

Therefore, the covariant derivative can be understood as a map:

$$\nabla : \Gamma(M,E) \to \Gamma(M,E) \otimes \Omega^1(M), \quad \nabla s(X) \equiv \nabla_X s,$$

for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(M,E)$. Moving on to the local case, a horizontal lift of a curve can always be written as $\tilde{\gamma}(t) = \sigma_i g_i(t)$, where $g_i : [0,1] \to G$ is a unique function. Say the basis of V is $\{E_a\}$, then a frame field of E along γ $e_a(t)$, can be written as:

$$e_a(t) \equiv [\sigma_i(t), E_a] = [\tilde{\gamma}(t), \rho(g_i(t)^{-1})E_a].$$

The covariant derivative in terms of the local connection form is then:

$$\nabla_X e_a = [\sigma_i, \mathcal{A}_i(X)E_a] = \frac{dx^\mu}{dt}\mathcal{A}^b_{i\mu a}e_b,$$

that is:

$$\nabla e_a = \mathcal{A}^b_{ia}e_b.$$

For a general section of E, $s(p) = [\sigma_i(p), \zeta_i(p)] = \zeta^a_i(p)e_a$, we can write:

$$\begin{aligned} \nabla_X s &= \left[\left(\sigma_i, \frac{d\zeta_i}{dt} + \mathcal{A}_i(X)\zeta_i\Big|_{t=0}\right)\right] \\ &= \frac{dx^\mu}{dt}\left(\frac{\partial \zeta^a_i}{\partial x^\mu} + \mathcal{A}^a_{i\mu b}\zeta^b_i\right)e_a. \end{aligned}$$

We can generalise the map ∇ to vector valued p-forms:

$$\nabla(s \otimes \eta) = (\nabla s) \wedge \eta + s \otimes \mathrm{d}\eta, \quad \eta \in \Omega^p(M).$$

The following can then be shown:

$$\nabla\nabla e_a = e_b \otimes \mathcal{F}^b_{ia},$$

for $e_a = [(\sigma_i, E_a)] \in \Gamma(U_i, E)$. The Ricci identity for a section $s(p) = \zeta^a(p)e_a(p)$ is then:

$$\nabla\nabla s = e_a \otimes \mathcal{F}^a_{ib}\zeta^b.$$

Definition 4.22 *If $\pi : E \to M$ is a vector bundle with a symmetric inner product defined at every point $p \in M$ by:*

$$g_p : \pi^{-1}(p) \otimes \pi^{-1}(p) \to \mathbb{R},$$

then a connection that preserves the inner product:

$$\mathrm{d}g(s_1, s_2) = g(\nabla s_1, s_2) + g(s_1, \nabla s_2),$$

or:

$$\mathrm{d}g_{ab} = \mathcal{A}^c_{ia}g_{cb} + \mathcal{A}^c_{ib}g_{ac},$$

is called a **metric connection**.

Now we make a brief detour to the concepts of torsion and the Levi-Civita connection with the help of [14].

Definition 4.23 *Let M be a differential manifold, $P(M,G)$ a principal bundle on it with a connection, (F,ρ) a representation space of G, $P \times_\rho F$ an associated vector bundle to P, and let there exist an isomorphism $\kappa : TM \to TT_pM$, $p \in M$. A* **canonical** *(or solder)* **form** *is a horizontal equivariant 1-form field on P, that is, pointwise, a map:*

$$\theta_u : T_uP \to TF; \quad U \mapsto u_*^{-1}(\kappa^{-1}\pi_* U),$$

where $u \in P$ is understood as a map $u : F \to T_{\pi(u)}M$.

Definition 4.24 *If a vector bundle admits a canonical 1-form it is said to be* **soldered** *to M.*[37]

Example 4.4 *A frame bundle LM and its associated tangent bundle TM are soldered to M. In this case, the canonical 1-form is nothing more then the form of which the pullback by a section $\sigma : U \subseteq M \to P$ is the coframe field:*

$$\sigma^*\theta^a = e^a.$$

So θ for LM is a $\mathbb{R}^{\dim M}$-valued 1-form of type id. *Since canonical 1-forms are linearly independent they can be used as a global basis for horizontal forms on LM. Therefore we can decompose the curvature with respect to this basis as:*

$$\Omega^a_b = \frac{1}{2}\theta^c \wedge \theta^d,$$

the components Ω^a_{bcd} constitute a global function of type $\mathrm{id} \otimes \tilde{\mathrm{id}} \otimes \tilde{\mathrm{id}} \otimes \tilde{\mathrm{id}}$. *The traditional curvature tensor is the local curvature of connection ω:*

$$\mathcal{R}^a_{bcd} = R^a_{bcd} \equiv \sigma^*\Omega^a_{bcd}.$$

Definition 4.25 *Let M be a differential manifold, LM its frame bundle with a connection form ω, and θ the canonical form, then:*

$$\Theta \equiv \mathrm{D}\theta \equiv \left(\mathrm{d}\theta^a + \omega^a_b \wedge \theta^b\right)E_a,$$

is called the **torsion 2-form** *of connection ω. Components Θ^a_{bc} are global functions on LM of type* $\mathrm{id} \otimes \tilde{\mathrm{id}} \otimes \tilde{\mathrm{id}}$. *Local torsion (usually just called torsion) is defined by section $\sigma : U \subseteq M \to P$:*

$$\mathcal{T}^a = T^a \equiv \sigma^*\Theta^a = \frac{1}{2}T^a_{bc}\, e^b \wedge e^c.$$

[37]Notice that $\dim V = \dim M$ must hold.

Definition 4.26 *Let $E = TM$ and ∇ be a metric connection, then if torsion vanishes ∇ is called a* **Levi-Civita connection***.*

The final note to make in this section is on the reduction of structure groups. The structure group of a general (real) vector bundle is $\mathrm{GL}(n, \mathbb{R})$. However, if this bundle has additional structure the structure group is reduced to a subgroup. Therefore, the restriction of a structure group to one of its subgroups corresponds to the introduction of additional structure and may not always be possible.[38]

Example 4.5 *Say we have a manifold M, some of the possible restrictions to LM are:*

(a) $\mathrm{O}(p, q) \subset \mathrm{GL}(n, \mathbb{R})$*, corresponds to the introduction of a metric tensor of signature (p, q) on M;*

(b) $\mathrm{GL}_+(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$*, corresponds to the introduction of an orientation on M;*

(c) $\mathrm{SL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$*, corresponds to the introduction of a volume form on M;*

(d) $\{e\} \subset \mathrm{GL}(n, \mathbb{R})$*, corresponds to complete parallelism on M.*

If we have a connection ω on a principal bundle $P(M, G)$, and assume the Lie algebra is decomposable $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}$. A connection on a subbundle $Q(M, H)$ (H is a subgroup of G) $\omega_{\mathfrak{h}}$ can be produced by the decomposition:

$$\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{l}},$$

if $\mathfrak{l}$ happens to be Ad_H-invariant ($\mathrm{Ad}_H \mathfrak{l} \subseteq \mathfrak{l}$). In the opposite direction, if we have a connection on Q we can always extend it to P by relation: $R_g^* \omega = \mathrm{Ad}_g \omega$, from which it follows that:

$$\omega_{qg} = \mathrm{Ad}_g R_{g*} \omega_q, \quad qg \in P.$$

This is obviously not the most general connection on P however it is important because it is compatible with, that is, it preserves, the additional structure that has arisen from the reduction of the structure group.

This ends our introduction to fibre bundles and their connections. We now, finally, have all the basic tools needed to start the construction of characteristic classes in the next section.

5 Characteristic classes

We are finally ready to give an exposition of characteristic classes, a bridge between algebraic topology and differential geometry. What we are interested in is characterising the topology of fibre bundles, that is how "much" they differ from trivial bundles. This will (in most cases) be expressed as de Rham cohomology classes associated to the curvature of the bundle. It will be seen that the geometry of a bundle is a way to compute completely topological quantities. (Based also on [9].)

5.1 Invariant polynomial and Chern-Weil theory

Definition 5.1 *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Define the space $S^r(\mathfrak{g})$ as the vector space of all completely symmetric, multilinear $\mathbb{C}$-valued functions on $\mathfrak{g}$:*

$$\tilde{P} : \bigotimes^r \mathfrak{g} \to \mathbb{C},$$

and the ring $S^*(\mathfrak{g})$ as the formal sum

$$S^*(\mathfrak{g}) = \bigoplus_{r=0}^{\infty} S^r(\mathfrak{g})$$

with the product:

$$\begin{aligned}\tilde{P}\tilde{Q}(X_1, \ldots, X_{p+q}) \\ = \frac{1}{(p+q)!} \sum_\sigma \tilde{P}(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \\ \cdot \tilde{Q}(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}),\end{aligned}$$

where σ denote permutations of $\{1, \ldots, p+q\}$.

Definition 5.2 *Let $\tilde{P} \in S^r(\mathfrak{g})$ and $g \in G$, then if:*

$$\tilde{P}(\mathrm{Ad}_g X_1, \ldots, \mathrm{Ad}_g X_r) = \tilde{P}(X_1, \ldots, X_r),$$

*$\tilde{P}$ is called an Ad-***invariant polynomial***. The set of all invariant polynomials of degree r is denoted $I^r(G)$.*

Definition 5.3 *A* **homogeneous invariant polynomial** *of degree r is a map $P : \mathfrak{g} \to \mathbb{C}$ such that $\exists \tilde{P} \in I^r(G)$ for which:*

$$P(X) \equiv \tilde{P}(X, \ldots, X).$$

Definition 5.4 *Any invariant polynomial P defines also a symmetric, invariant and r-linear form by expanding $P(t_1 X_1 + \cdots t_r X_r)$ as a polynomial in t_i. The coefficient of the highest order term multiplied by $1/r!$ is invariant and symmetric and is called the* **polarization** *$\hat{P}$.*

Definition 5.5 *If $P(M, G)$ is a principal bundle, obviously we can extend the domain of invariant polynomials to $\mathfrak{g}$-valued p-forms on M by the definition:*

$$\tilde{P}(\lambda_1, \ldots, \lambda_r) \equiv \lambda_1^{i_1} \wedge \ldots \wedge \lambda_r^{i_r} \tilde{P}(E_{i_1}, \ldots, E_{i_r}),$$

where $\lambda \in \Omega^p(M, \mathfrak{g})$ and $\{E_i\}$ is a basis in $\mathfrak{g}$.

There are two points important to make:

[38] For example, if a manifold is non-orientable then the structure group cannot be restricted to $\mathrm{GL}_+(n, \mathbb{R})$.

(i) curvature on a principal bundle is a global form, whereas the local curvature F_i is only defined with respect to some **local** section. However, since the transition between two coordinate patches is given by the adjoint action $F_j = \mathrm{Ad}_{t_{ij}^{-1}} F_i$, the Ad-invariant polynomial of F is **globally** defined because of this invariance;

(ii) local curvature is defined with respect to some local section (gauge). The field strength 2-form gauge transforms adjointly, $F' = \mathrm{Ad}_{g^{-1}} F$, therefore the invariant polynomial is gauge invariant (does not depend on the choice of defining section).

Theorem 5.1 (Chern-Weil) *If $P \in I^r(G)$, then:*

(i) $P(F) \in Z^{2r}(M)$;

(ii) if we have two local curvatures F and F' corresponding to two connections A and A':

$$P(F) - P(F') \in B^{2r}(M).$$

The first part of the theorem is obvious since $P(\Omega)$ is Ad-invariant and horizontal:

$$\begin{aligned} R_g^* P(\Omega) &= P(\Omega) \;\Rightarrow\; P(\Omega) \text{ is of type id} \\ \mathrm{d}P(\Omega) &= \mathrm{D}P(\Omega) \\ &= \tilde{P}(\mathrm{D}\Omega, \dots, \Omega) + \cdots + \tilde{P}(\Omega, \dots, \mathrm{D}\Omega) = 0 \\ \mathrm{d}P(F) &= \mathrm{d}P(\sigma^*\Omega) = \sigma^* \mathrm{d}P(\Omega) = 0. \end{aligned}$$

The second part is more complicated, however, it can be shown that:

$$P(F') - P(F) = \mathrm{d}\left[r \int_0^1 dt \hat{P}(A' - A, F_t, \dots, F_t) \right],$$

where $F_t = F + t\,\mathrm{D}(A' - A) + t^2(A' - A)^2$ is the interpolating field strength of $A_t = A + t(A' - A)$.[39] This means that an invariant polynomial defines a cohomology class of M. From this it follows that the integral of an invariant polynomial over a $2m$-dimensional manifold without a boundary (a $2m$-cycle) M does not depend on the connection (and then also curvature) chosen,

$$\begin{aligned} \int_M P_m(F') - \int_M P_m(F) &= \int_M \mathrm{d}TP_m(A', A) \\ &= \int_{\partial M} P_m(A', A) = 0. \end{aligned}$$

An invariant polynomial does also not depend on the cycle it is integrated over within the same homology class (it is invariant to transformations of the kind $M_{2m} \mapsto M_{2m} + \partial M_{2m+1}$). Then, by construction we have a topological invariant:

$$\int_M P(F).$$

Definition 5.6 *A* **characteristic class** *is a mapping χ associating to each fibre bundle over a manifold M an element of the de Rham cohomology ring $H^*(M)$ such that if two bundles (E_1 and E_2) are homeomorphic to each other $\chi_{E_1} = \chi_{E_2}$.*

Theorem 5.2 *Let $P \in I^*(G)$ and E be a fibre bundle over M with structure group G. The mapping*

$$\chi_E : I^*(G) \to H^*(M)$$

is a homomorphism called a **Weil homomorphism**. *A Weil homomorphism has the following naturality property:*

$$\chi_{f^*E} = f^* \chi_E,$$

for a differentiable mapping between manifolds f.

It is obvious now that an invariant polynomial defines a characteristic class, some examples of which will be given in the following sections. A characteristic class measures how much a bundle differs from a product bundle, that is how "twisted" it is. Therefore, the following theorem is no surprise.

Theorem 5.3 *All characteristic classes of a trivial (product) bundle are trivial.*

5.2 Pontryagin classes

Observe that the following determinant can be written as a polynomial in $\lambda \in \mathbb{R}$:

$$\det(\lambda \mathbb{1} + A) = \sum_{k=0}^{n} \lambda^{n-k} P_k(A),$$

where $A \in M_n(\mathbb{R})$ is a real $n \times n$ matrix. The coefficients $P_k(A)$ of λ^{n-k} are polynomials in matrices A. A simple example is for $n = 2$:

$$\det(\lambda \mathbb{1} + A) = \lambda^2 + \lambda \operatorname{tr} A + \det A.$$

If we understand A as an element of $\mathfrak{gl}(n, \mathbb{R})$ (or any of its subalgebras), it is trivial to show that this determinant is Ad-invariant and, therefore, so must the polynomials $P_k(A)$ be:

$$\begin{aligned} \det(\mathrm{Ad}_B(\lambda \mathbb{1} + A)) &= \det(\lambda B \mathbb{1} B^{-1} + BAB^{-1}) \\ &= \det(B\,(\lambda \mathbb{1} + A)\,B^{-1}) \\ &= \det(\lambda \mathbb{1} + A). \end{aligned}$$

[39] The expression in square brackets is called the transgression of P_r, TP_r:

$$TP_r(A', A) \equiv r \int_0^1 dt \hat{P}(A' - A, F_t, \dots, F_t).$$

Lets examine the case of $A \in \mathfrak{o}(n) \subset \mathfrak{gl}(n, \mathbb{R})$ further, A must be antisymmetric, $A^T = -A$ so

$$\begin{aligned} \det(\lambda \mathbb{1} + A) &= \det(\lambda \mathbb{1} + A)^T \\ &= \det(\lambda \mathbb{1} + (-A)) \\ \Rightarrow \qquad P_k(A) &= (-1)^k P_k(-A) \\ \Rightarrow \qquad P_{2k+1}(A) &= 0. \end{aligned}$$

Returning to bundles; say we have a real n-dimensional vector bundle, in general its structure group is $\mathrm{GL}(n, \mathbb{R})$ so it has a principal $\mathrm{GL}(n, \mathbb{R})$-bundle associated to it. However, if we construct a fibre metric on this vector bundle then it can be reduced to $\mathrm{O}(p, q)$ (where $p + q = n$). A curvature on a principal $\mathrm{O}(p, q)$-bundle is antisymmetric. Then, we can extend this connection (and then the curvature) to the full bundle and, since it is compatible with $\mathfrak{o}(p, q)$ it must too be antisymmetric ($\Omega^T = -\Omega$). All this consideration also descends to the associated vector bundle (although only through sections with values in the reduced principal bundle)[40] and its curvature $\mathcal{F}$.[41]

Definition 5.7 *Let $\pi : E \to M$ be a vector bundle with curvature F, then:*

$$p(E) \equiv \det\left(\mathbb{1} + \frac{\mathcal{F}}{2\pi}\right) = \sum_{k=0}^{n} P_k(\mathcal{F}/2\pi) \equiv \sum_{k=0}^{n} \mathcal{P}_k(\mathcal{F}),$$

is the **total Pontryagin class** *where the $4j$-forms, $p_j(E) \equiv \mathcal{P}_{2j}(\mathcal{F})$, are called the* **j-th Pontryagin classes**.[42] *The integral of a Pontryagin class over a $4j$-cycle,*

$$p_j \equiv \int_{M_{4j}} p_j(E),$$

is called the **j-th Pontryagin number** *and is a topological invariant (and always an integer).*

Obviously we need at least a 4-dimensional manifold to benefit from Pontryagin classes. The first three Pontryagin classes are given below.

$$\begin{aligned} p_0(E) &= \quad 1 \\ p_1(E) &= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \operatorname{tr}\mathcal{F} \\ p_2(E) &= \quad \frac{1}{8}\left(\frac{1}{2\pi}\right)^4 \left((\operatorname{tr}\mathcal{F}^2)^2 - 2\operatorname{tr}\mathcal{F}^4\right) \\ &\;\;\vdots \\ p_{\lfloor j/2 \rfloor} &= \left(\frac{1}{2\pi}\right)^j \det\mathcal{F} \end{aligned}$$

It can be shown that:

$$p(E \oplus F) = p(E) \wedge p(F).$$

5.3 Chern classes

Lets construct a new determinant for matrix A, an antihermitian square $n \times n$ matrix ($A^\dagger = -A$), as:

$$\det(\lambda \mathbb{1} + iA) = \sum_{k=0}^{n} \lambda^{n-k} P_k(iA).$$

This determinant is real:

$$\begin{aligned} \overline{\det(\lambda \mathbb{1} + iA)} &= \det(\lambda \mathbb{1} + iA)^\dagger \\ &= \det(\lambda \mathbb{1} - iA^\dagger) = \det(\lambda \mathbb{1} + iA), \end{aligned}$$

and, therefore, so is the polynomial $P_k(iA)$. Moving on to a complex vector bundle with fibre $\mathbb{C}^n$ now, it has a principal $\mathrm{GL}(n, \mathbb{C})$-bundle $\tilde{P}(M, \mathrm{GL}(n, \mathbb{C}))$ associated to it. Say we have a connection $\tilde{\omega}$ on it and say $\tilde{\Omega}$ is its curvature 2-form. The curvature is a $\mathfrak{gl}(n, \mathbb{C})$-valued 2-form, in matrix representation over $\mathbb{C}^n$ it is a square $n \times n$ matrix with $\mathbb{C}$-valued 2-form elements ($\tilde{\Omega}^a_b = \alpha^a_b + i\beta^a_b$, $\alpha^a_{b\mu\nu}(\tilde{u}), \beta^a_{b\mu\nu}(\tilde{u}) \in \mathbb{R}$, $u \in \tilde{P}$). Repeating the procedure of Pontryagin classes, we introduce a Hermitian fibre metric.[43] The introduction of this additional structure induces a reduction in the structure group $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{U}(n)$ to which, in matrix representation, correspond unitary matrices. Therefore we now have a smaller $\mathrm{U}(n)$ principal bundle $P(M, \mathrm{U}(n))$ with

[40] It was shown, however, that Ad-invariant polynomials do not depend on the choice of section.

[41] In matrix representation over the fibre vector space $\mathcal{F} \equiv \rho'(F)$.

[42] More often written as $p(F)$ and $p_j(F)$ in literature, however this is misleading as the class does not depend on the choice of connection.

[43] For details see [11], [10] and [1].

Definition 5.8 *A complex (0,2)-type tensor field $g \in \Gamma(E \otimes \overline{E})^*$ on a vector bundle E over a(n almost) complex manifold M is called a* **Hermitian metric** *if it satisfies for $Z, W \in \Gamma(E)$:*

(i) $g(Z, \overline{W}) = \overline{g(W, \overline{Z})}$;

(ii) $g(Z, \overline{Z}) > 0$, *for any non-zero vector field.*

Theorem 5.4 *A complex manifold always admits a Hermitian metric.*

connection ω and curvature Ω both $\mathfrak{u}(n)$-valued. In matrix representation curvature Ω is an $n\times n$ antihermitian matrix of 2-forms. Motivated by the determinant from the beginning we define the following.

Definition 5.9 *Let $\pi : E \to M$ be a complex vector bundle with fibre $\mathbb{C}^n$. Then, the structure group G is a subgroup of $\mathrm{GL}(n,\mathbb{C})$, the connection and curvature, A and F, are $\mathfrak{g}$-valued. The* **total Chern class** *$c(E)$ is given by:*

$$c(E) \equiv \det\left(\mathbb{1} + \frac{i\mathcal{F}}{2\pi}\right) = \sum_{k=0}^{n} P_k(i\mathcal{F}/2\pi) \equiv \sum_{k=0}^{n} c_k(E),$$

where $c_k(E)$ are the ***k*-th Chern classes**. *The integral of the k-th Chern class is a topological invariant,*

$$c_k \equiv \int_{M_{2k}} c_k(E),$$

called the **Chern number**.

On an m-dimensional manifold Chern classes with $2k > m$ or $k > n$ vanish identically. Again, since $c_k(E)$ is closed, it defines an equivalence class $[c_k(E)] \in H^{2k}(M)$. The first three Chern classes are given below.

$$\begin{aligned}
c_0(E) &= 1\\
c_1(E) &= \frac{i}{2\pi}\operatorname{tr}\mathcal{F}\\
c_2(E) &= \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2\left((\operatorname{tr}\mathcal{F})^2 - \operatorname{tr}\mathcal{F}^2\right)\\
&\vdots\\
c_k(E) &= \left(\frac{i}{2\pi}\right)^k \det\mathcal{F}
\end{aligned}$$

The total Chern class of a Whitney sum bundle is:

$$c(E\oplus F) = c(E)\wedge c(F).$$

Since the total Chern class of a trivial bundle is 1, the following can be shown: if $E = E_1 \oplus E_2$ and E_2 is trivial, then:

$$c_i(E) = 0, \quad \dim E_1 + 1 \le i \le \dim E_1 + \dim E_2.$$

Theorem 5.5 (Splitting principle) *Let $\pi : E \to M$ be a n-dimensional complex vector bundle. The total Chern class of E is identical to that of bundle $L_1 \oplus L_2 \oplus \cdots \oplus L_n$:*

$$c(E) = c(L_1)\wedge \ldots \wedge c(L_n).$$

5.4 Chern characters and the Chern-Simons form

Another characteristic class are the Chern characters, important because of their appearance in the Atiyah-Singer index theorem.

Definition 5.10 *Let $\pi : E \to M$ be a complex vector bundle with curvature F and fibre dimension n, then the* **total Chern character** *is:*

$$\mathrm{ch}(E) \equiv \operatorname{tr}\exp\left(\frac{i\mathcal{F}}{2\pi}\right) = \sum_{j=0}^{n}\frac{1}{j!}\operatorname{tr}\left(\frac{i\mathcal{F}}{2\pi}\right)^j \equiv \sum_{j=0}^{n}\mathrm{ch}_j(E),$$

where we have defined the ***j*-th Chern character** *$\mathrm{ch}_j(E)$.*

The two following properties of Chern characters can be shown:

$$\begin{aligned}
\mathrm{ch}(E\otimes F) &= \mathrm{ch}(E)\wedge\mathrm{ch}(F),\\
\mathrm{ch}(E\oplus F) &= \mathrm{ch}(E)+\mathrm{ch}(F),
\end{aligned}$$

and from the splitting principle:

$$\mathrm{ch}(E) = \mathrm{ch}(L_1) + \cdots + \mathrm{ch}(L_n),$$

where L_i are complex line bundles. Finally we define the integral of the j-th Chern character:

$$\mathrm{Ch}_j(E) \equiv \int_M \mathrm{ch}_j(E).$$

The Chern character can also be expressed in terms of Chern classes:

$$\begin{aligned}
\mathrm{ch}_0(E) &= n,\\
\mathrm{ch}_1(E) &= c_1(E),\\
\mathrm{ch}_2(E) &= \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right),\\
&\vdots\quad .
\end{aligned}$$

Definition 5.11 *Let $P_j(E)$ be an arbitrary $2j$-form characteristic class. By Poincaré's lemma, since it is closed we can write $P_j(E)$ locally as:*

$$P_j(\mathcal{F}) = \mathrm{d}Q_{2j-1}(\mathcal{A},\mathcal{F}), \quad Q_{2j-1}(\mathcal{A},\mathcal{F}) \in \Omega^{2j-1}(M,\mathfrak{g}).$$

$Q_{2j-1}(\mathcal{A},\mathcal{F})$ is called the **Chern-Simons form** *of $P_j(E)$.*

From section 5.1 it is obvious that the Chern-Simons form can be expressed as:

$$Q_{2j-1}(\mathcal{A},\mathcal{F}) = TP_j(\mathcal{A},0) = j\int_0^1 dt\hat{P}_j(\mathcal{A},\mathcal{F}_t,\ldots,\mathcal{F}_t),$$

(see footnote on page 20) where $\mathcal{F}_t = t\mathcal{F} + (t^2 - t)\mathcal{A}^2$. It is important to note that this is valid only for a local chart on which A' can be set to zero; however, since a

Chern-Simons form is only defined locally this is consistent. Observe the following:

$$\int_M P_l(\mathcal{F}) = \int_M \mathrm{d}Q_{2l-1}(\mathcal{A},\mathcal{F}) = \int_{\partial M} Q_{2l-1}(\mathcal{A},\mathcal{F}),$$

for $\dim M = 2l$, this means that a Chern-Simons form is to be understood as a characteristic class of the boundary of M. We calculate the first three Chern-Simons forms of the Chern characters. Since the polarization is the symmetrised trace str, from the definition of the j-th Chern character we arrive at:

$$Q_{2j-1}(\mathcal{A},\mathcal{F}) = \frac{1}{(j-1)!}\left(\frac{i}{2\pi}\right)^j \int_0^1 dt\, \mathrm{str}(\mathcal{A},\mathcal{F}_t,\ldots,\mathcal{F}_t).$$

Then it follows that the first three Chern-Simons forms of the Chern character are:

$$Q_1(\mathcal{A},\mathcal{F}) = \frac{i}{2\pi}\,\mathrm{tr}\,\mathcal{A},$$
$$Q_3(\mathcal{A},\mathcal{F}) = \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}\left(\mathcal{A}\,\mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right),$$
$$Q_5(\mathcal{A},\mathcal{F}) = \frac{1}{3!}\left(\frac{i}{2\pi}\right)^3 \mathrm{tr}\left(\mathcal{A}(\mathrm{d}\mathcal{A})^2 + \frac{3}{2}\mathcal{A}^3\,\mathrm{d}\mathcal{A} + \frac{3}{5}\mathcal{A}^5\right).$$

It is important to note that a Chern-Simons form is a polynomial of both connection $\mathcal{A}$ and field strength $\mathcal{F}$ and is therefore not necessarily gauge invariant since a connection does not transform adjointly (remember that $A' = \mathrm{Ad}_{g^{-1}} A + S^*\theta$).[44] It can be shown that the Chern-Simons form of the Chern character gauge transforms by a locally exact form.

5.5 Euler classes and the Gauss-Bonnet theorem

So far we have constructed characteristic classes from determinants and traces. There is also a third option.

Definition 5.12 *Let V be an even-dimensional vector space ($\dim V = 2m$) and V^* its dual space with bases $\{E_a\}$ and $\{E^a\}$ respectively. Also, let A be an antisymmetric $2m \times 2m$ matrix ($A^T = -A$), then we define 2-form:*

$$\alpha_A^E \equiv \frac{1}{2}A_{ab}E^a \wedge E^b \in \Lambda^2 V^*,$$

with respect to basis $\{E^a\}$. The **Pfaffian** *of matrix A is defined by the relation:*

$$\alpha_A^E \wedge \ldots \wedge \alpha_A^E \equiv m!\,\mathrm{Pf}(A)\; E^1 \wedge \ldots \wedge E^{2m}.$$

The Pfaffian has the following properties:

(i) explicitly, it is given by

$$\mathrm{Pf}\,A = \frac{1}{2^m m!}\epsilon^{a_1a_2\cdots a_{2m-1}a_{2m}} A_{a_1a_2}\cdots A_{a_{2m-1}a_{2m}};$$

(ii) if $B \in \mathrm{GL}(2m,\mathbb{R})$ and $\mathcal{B}$ is its representation in V, then:

$$\alpha^E_{B^TA\,B} = \alpha_A^{E\mathcal{B}},$$

or:

$$\mathrm{Pf}(B^TA\,B) = \det B\;\mathrm{Pf}\,A;$$

(iii) $(\mathrm{Pf}\,A)^2 = \det A$.

Because of these properties the Pfaffian can be used to construct a characteristic class. From property (ii) we see that if $B \in \mathrm{SO}(2m)$:

$$\mathrm{Pf}(B^TA\,B) = \mathrm{Pf}(B^{-1}A\,B) = \mathrm{Pf}(\mathrm{Ad}_B\,A) = \mathrm{Pf}\,A.$$

Therefore, if we can reduce our structure group to $\mathrm{SO}(2m)$, we can use the Pfaffian to construct a characteristic class for even-dimensional manifolds. Such a reduction is made by, in addition to defining a fibre metric, choosing an orientation in each fibre. This is not always possible.[45]

Definition 5.13 *Let $\pi : E \to M$ be a real orientable vector bundle with $2m$-dimensional fibre. The* **Euler class** *is:*

$$e(E) \equiv \mathrm{Pf}\left(\frac{\mathcal{F}}{2\pi}\right)$$
$$= \frac{1}{(4\pi)^m m!}\epsilon^{a_1a_2\cdots a_{2m-1}a_{2m}}\mathcal{F}_{a_1a_2}\wedge\ldots\wedge\mathcal{F}_{a_{2m-1}a_{2m}},$$

where $\mathcal{F}_{ab} \equiv \delta_{ac}\mathcal{F}^c_b$.

The two and four-dimensional Euler classes are:

$$e(E) = \frac{1}{2\pi}\mathcal{F}^1_2,$$
$$e(E) = \left(\frac{1}{2\pi}\right)^2(\mathcal{F}^1_2\mathcal{F}^3_4 - \mathcal{F}^1_3\mathcal{F}^2_4 + \mathcal{F}^1_4\mathcal{F}^2_3).$$

For the case of $E = TM$ since the tangent bundle is canonically defined for a manifold M, the Euler class is a topological characteristic over M itself, so it is often written $e(M) \equiv e(TM)$. From the expression for the two-dimensional Euler class we get:

$$e(M_2) = \frac{1}{2\pi}K\omega,$$

where K is the Gaussian curvature and ω the volume element. The integral of this quantity over a closed 2-manifold is the Euler characteristic. This is known as the **Gauss-Bonnet theorem**. Its generalization to higher $2m$ dimensions is the following theorem.

Theorem 5.6 (Chern-Gauss-Bonnet *or generalised Gauss-Bonnet***)** *For any closed and oriented $2m$-dimensional manifold M the following holds:*

$$\int_M e(M) = \chi(M).$$

[44] The precise transformation is given by Cartan's homotopy operator which will not be given here.

[45] An example of this is the Möbius strip, understood as a vector bundle over a circle.

5.6 Stiefel-Whitney classes

The final characteristic class is slightly different in that it cannot be expressed in terms of the curvature of the bundle. Its main importance is in the fact that the second Stiefel-Whitney class determines whether a manifold admits a spin or not. Therefore, we begin with the definition of a spin group.

Definition 5.14 *A* **spin group** $\mathrm{Spin}(p,q)$ *is the two-sheeted covering of the* $\mathrm{SO}(p,q)$ *group. For* $n = p+q > 2$ *it coincides with the universal covering.*

Some accidental isomorphisms with the spin groups are:

$$\begin{aligned}
\mathrm{Spin}(1,1) &\cong \mathrm{GL}(1,\mathbb{R}),\\
\mathrm{Spin}(2,1) &\cong \mathrm{SL}(2,\mathbb{R}),\\
\mathrm{Spin}(3,1) &\cong \mathrm{SL}(2,\mathbb{C}),\\
\mathrm{Spin}(2,2) &\cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}),\\
\mathrm{Spin}(1) &\cong \mathrm{O}(1),\\
\mathrm{Spin}(2) &\cong \mathrm{U}(1),\\
\mathrm{Spin}(3) &\cong \mathrm{SU}(2),\\
\mathrm{Spin}(4) &\cong \mathrm{SU}(2) \times \mathrm{SU}(2).
\end{aligned}$$

Before defining Stiefel-Whitney classes we need to define spin bundles and Čech cohomology groups.

Definition 5.15 *Let* (M,g) *be an* m*-dimensional orientable (pseudo) Riemannian manifold. The* **spin bundle** SM *is a principal* $\mathrm{Spin}(p,q)$*-bundle with an equivariant function of type* ρ*,* ρ *being the spinor representation of group* $\mathrm{Spin}(p,q)$ *such that* SM *is a two-sheeted covering of* OM*, the total space of the bundle of (right-handed) orthonormal frames* $\pi : OM \to M$*.*

Let t_{ij} be transition functions of LM, then a spin structure on M must have the following properties:

$$\begin{aligned}
\varphi(\tilde{t}_{ij}) &= t_{ij},\\
\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} &= \mathbb{1},\\
\tilde{t}_{ii} &= \mathbb{1};
\end{aligned}$$

where homomorphism φ is the two-sheeted covering $\mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$.

Definition 5.16 *Let* $\mathbb{Z}_2 \equiv \mathbb{Z}/2\mathbb{Z} = \{-1,+1\}$ *be the multiplicative cyclic group, and let* M *be a manifold. A* **Čech** $\boldsymbol{r}$**-cochain** *is a totally symmetric function* $f(i_0,\ldots,i_r) \in \mathbb{Z}_2$ *on* $U_{i_0} \cap \cdots \cap U_{i_r} \neq \varnothing \subseteq M$*:*

$$f(i_{\sigma(0)},\ldots,i_{\sigma(r)}) = f(i_0,\ldots,i_r),$$

where σ *is a permutation of* $\{0,\ldots,r\}$*. The group of all Čech* r*-cochains is denoted* $C^r(M,\mathbb{Z}_2)$*.*

Definition 5.17 *Define the* **coboundary operator** $\delta : C^r(M,\mathbb{Z}_2) \to C^{r+1}(M,\mathbb{Z}_2)$ *by:*

$$(\delta f)(i_0,\ldots,i_{r+1}) = \prod_{j=0}^{r+1} f(i_0,\ldots,i_{j-1},i_{j+1},\ldots,i_r).$$

The coboundary operator is nilpotent:

$$\delta^2 f = 1.$$

Definition 5.18 *Define the:*

(a) **cocycle group** $Z^r(M,\mathbb{Z}_2)$ *by:*

$$Z^r(M,\mathbb{Z}_2) = \{f \in C^r(M,\mathbb{Z}_2) \,|\, \delta f = 1\},$$

(b) **coboundary group** $B^r(M,\mathbb{Z}_2)$ *by:*

$$\begin{aligned}
&B^r(M,\mathbb{Z}_2)\\
&\quad = \{f \in C^r(M,\mathbb{Z}_2) \,|\, f = \delta f',\ f' \in C^{r-1}(M,\mathbb{Z}_2)\}.
\end{aligned}$$

Definition 5.19 *The* **Čech cohomology group** $H^r(M,\mathbb{Z}_2)$ *is defined by:*

$$\begin{aligned}
H^r(M,\mathbb{Z}_2) &= \ker\delta_r / \operatorname{im}\delta_{r-1}\\
&= Z^r(M,\mathbb{Z}_2)/B^r(M,\mathbb{Z}_2).
\end{aligned}$$

Finally, a Stiefel-Whitney class is a characteristic class that takes its values in $H^r(M,\mathbb{Z}_2)$. We will define the first two.

Definition 5.20 *Let* (M,g) *be a (pseudo) Riemannian manifold of dimension* m*, then the structure group is* $\mathrm{O}(p,q)$*. If* $\{U_i\}$ *is a simple open covering of* M *then define function:*

$$f(i,j) \equiv \det t_{ij} = \pm 1,$$

where $t_{ij} : U_i \cap U_j \to \mathrm{O}(p,q)$ *is the transition function of the local frames,* $e_{i\alpha} = t_{ij} e_{\alpha j}$*. The* **first Stiefel-Whitney class** *is then:*

$$w_1(M) \equiv [f] \in H^1(M,\mathbb{Z}_2).$$

It is trivial to show for $f \in Z^r(M,\mathbb{Z}_2)$:

$$\delta f(i,j,k) = \det t_{ij} \det t_{jk} \det t_{ki} = 1.$$

Theorem 5.7 *If* TM *is the tangent bundle with fibre metric,* M *is orientable if and only if* $w_1(M)$ *is trivial.*

Definition 5.21 *Let* M *be an orientable* m*-dimensional manifold and* TM *its tangent bundle with fibre metric. If* $t_{ij} \in \mathrm{SO}(p,q)$ *are transition functions, define* $\tilde{t}_{ij} \in \mathrm{Spin}(p,q)$ *such that:*

$$\varphi(\tilde{t}_{ij}) = t_{ij}, \quad \tilde{t}_{ji} = \tilde{t}_{ij}^{-1},$$

where $\varphi : \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q)$ *is the 2:1 covering homomorphism. Define a Čech 2-cochain* $f : U_i \cap U_j \cap U_k \to \mathbb{Z}_2$ *by:*

$$\tilde{t}_{ij}\tilde{t}jk\tilde{t}_{ki} = f(i,j,k)\mathbb{1} \in \ker\varphi = \{\pm\mathbb{1}\}.$$

The equivalence class defined by f *is the* **second Stiefel-Whitney class** $w_2(M) \equiv [f] \in H^2(M,\mathbb{Z}_2)$*.*

Since a spin bundle must have $\tilde{t}_{ij}\tilde{t}_{jk}\tilde{t}_{ki} = \mathbb{1}$ satisfied, this is only possible if $f(i,j,k) = 1$. This leads us to the following theorem.

Theorem 5.8 *If* TM *is the tangent bundle with fibre metric of an orientable manifold* M*, then there exists a spin bundle over* M *if and only if* $w_2(M)$ *is trivial.*

5.7 Physical examples

In addition to [1] in example 5.3 [13] was also used.

Example 5.1 (2D Gravity) *The standard Einstein-Hilbert action for gravity is:*

$$S[g] \sim \int R\sqrt{|g|}\,\mathrm{d}^n x,$$

where R is the scalar curvature (Ricci scalar). For two-dimensional gravity this would read:

$$S[g] \sim \int R\sqrt{|g|}\,\mathrm{d}^2 x \sim \int K\omega,$$

where $\omega = \sqrt{|g|}\,\mathrm{d}^2 x$ is the volume element, however, this is the integral of the two-dimensional Euler class. Therefore, since a characteristic class transforms up to an exact form upon change of connection, this results in a surface term not affecting the equations of motion. This means the standard Einstein-Hilbert action does not give dynamical results in 2D.

Example 5.2 (Dirac monopole) *The Dirac monopole is defined on Euclidean space with the origin removed. $\mathbb{R}^3 - \{0\}$ (for simplicity everything is assumed to be time-independent) is homotopic to S^2, therefore, the appropriate principal bundle is the $\mathrm{U}(1)$-bundle: $P(S^2, \mathrm{U}(1))$. S^2 is covered by two charts:*

$$\begin{aligned} U_N &\equiv \left\{(\theta,\phi)\,\middle|\, 0 \le \phi \le 2\pi,\, 0 \le \theta \le \frac{\pi}{2} + \epsilon\right\}, \\ U_S &\equiv \left\{(\theta,\phi)\,\middle|\, 0 \le \phi \le 2\pi,\, \frac{\pi}{2} - \epsilon \le \theta \le \pi\right\}. \end{aligned}$$

Let ω be a connection form on P, and σ_N and σ_S be sections on U_N and U_S respectively, then we have the gauge potentials (local connection forms) given by:

$$A_N = \sigma_N^*\omega, \quad A_S = \sigma_S^*\omega.$$

Gauge potentials A_N and A_S can be chosen to be of the form:

$$A_N = ig(1-\cos\theta)\,\mathrm{d}\phi, \quad A_S = -ig(1+\cos\theta)\,\mathrm{d}\phi,$$

with g the strength of the monopole. The transition function defined on an infinitesimally thick ribbon around the equator is:

$$t_{NS}(\phi) = \exp -i\alpha(\phi), \quad \alpha : S^1 \to \mathbb{R}.$$

Therefore, the gauge potentials are related by:

$$A_N = t_{NS}^{-1} A_S t_{NS} + t_{NS}^{-1}\,\mathrm{d}t_{NS} = A_S - i\,\mathrm{d}\alpha,$$

it follows then that:

$$\mathrm{d}\alpha = i(A_N - A_S) = -2g\,\mathrm{d}\phi.$$

For the transition functions to be well defined the integral of function α over 2π must be a multiple of 2π:

$$\Delta\alpha = \int_0^{2\pi} \mathrm{d}\alpha = -2g\int_0^{2\pi}\mathrm{d}\phi = -4\pi g \in 2\pi\,\mathbb{Z},$$

or $-2g \in \mathbb{Z}$. On the other hand we have the topological invariant from the Chern character:

$$\begin{aligned} \mathrm{ch}\,P &= 1 + \frac{iF}{2\pi}; \\ \mathbb{Z} \ni \mathrm{Ch}_1\,P &= \int_{S^2} \mathrm{ch}_1\,P \\ &= \int_{S^2} -\frac{g}{2\pi}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi = -2g. \end{aligned}$$

Thus, the quantization of the magnetic monopole g is a topological consequence, and its magnitude characterizes the twisting of the principal P bundle.

Example 5.3 (Chiral anomaly) *Let M be an even-dimensional manifold ($\dim M = m = 2l$) (for simplicity in this case $\mathbb{R}^{2l}$) and ψ be the field of a massless Dirac particle. The partition functional is:*

$$Z = \int \mathfrak{D}A\,\mathfrak{D}\psi\mathfrak{D}\bar{\psi}\,\exp\int_M \bar{\psi} i\not{D}\psi\,\mathrm{d}^m x,$$

where $i\not{D} =$ is the Dirac operator. A chiral transformation is:

$$\begin{aligned} \psi(x) &\to e^{i\gamma^{m+1}\alpha(x)}\psi(x) \\ \bar{\psi}(x) &\to \bar{\psi}(x)e^{i\gamma^{m+1}\alpha(x)} \end{aligned}$$

or infinitesimally,

$$\begin{aligned} \psi &\to \psi + i\alpha\,\gamma^{m+1}\psi \\ \bar{\psi} &\to \bar{\psi} + i\alpha\,\bar{\psi}\gamma^{m+1}. \end{aligned}$$

Classically, the action is invariant with respect to this chiral symmetry because of the conservation of the axial current $j^\mu_{m+1}(x) = \bar{\psi}(x)\gamma^{m+1}\gamma^\mu\psi(x)$:

$$\begin{aligned} S[A,\psi,\bar{\psi}] &= \int_M \bar{\psi} i\not{D}\psi\,\mathrm{d}^m x \\ &\to \int_M \bar{\psi} i\not{D}\psi\,\mathrm{d}^m x - i\int_M \alpha(x)\partial_\mu j^\mu_{m+1}(x)\,\mathrm{d}^m x \\ \partial_\mu j^\mu_{m+1}(x) &= 0. \end{aligned}$$

In the quantum case, however, in the partition functional we must also transform the path integral measures $\mathfrak{D}\psi$ and $\mathfrak{D}\bar{\psi}$. The Jacobian of this transformation can be shown to be:

$$\mathfrak{D}\psi\mathfrak{D}\bar{\psi} \to \mathfrak{D}\psi\mathfrak{D}\bar{\psi}\exp -2i\int_M \alpha(x)\sum_i \psi_i^\dagger(x)\gamma^{m+1}\psi_i(x),$$

where ψ_i are the eigenstates of the Dirac operator, $i\not{D}\psi_i = \lambda_i\psi_i$. The full transformation of the path integral is then:

$$Z \to \int \mathfrak{D}A\,\mathfrak{D}\psi\mathfrak{D}\bar{\psi}\,\exp\int_M \Big(\bar{\psi}i\not{D}\psi - i\alpha\partial_\mu j^\mu_{m+1} - 2i\alpha\sum_i \psi_i^\dagger\gamma^{m+1}\psi_i\Big)\,\mathrm{d}^m x,$$

therefore, the anomalous axial Ward identity is:[46]

$$\langle\partial_\mu j^\mu_{m+1}\rangle = -2\sum_i\langle\psi_i|\gamma^{m+1}|\psi_i\rangle.$$

Calculation of the right hand side for $m = 4$ produces:

$$\langle\partial_\mu j^\mu_5\rangle = -\frac{1}{16\pi^2}\operatorname{tr}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

but this is nothing other then the second Chern character $\mathrm{ch}_2\,F$*:*

$$\begin{aligned}\int_M \mathrm{ch}_2\,F &= \int_M \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2\left(\frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu\wedge\,\mathrm{d}x^\nu\right)^2\\ &= \int_M -\frac{1}{16\pi^2}\operatorname{tr}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\,\mathrm{d}^4x\\ &= \int_M\langle\partial_\mu j^\mu_5\rangle\,\mathrm{d}^4x.\end{aligned}$$

Generalising this to an arbitrary dimension $m = 2l$ it follows that,

$$\mathrm{Ch}_l\,F \equiv \int_M \mathrm{ch}_l\,F = \int_M\langle\partial_\mu j^\mu_{m+1}\rangle\,\mathrm{d}^m x,$$

the chiral anomaly (also called the axial or Abelian anomaly) is a topological effect of the twisting of the spin complex and is the consequence of the Atiyah-Singer index theorem.[47]

Example 5.4 (Euler characteristic of S^2) *Let $M = S^2$ be endowed with a Levi-Civita connection of curvature $\mathcal{R}$. Consider TS^2, if we choose the coordinates on S^2 as (θ,ϕ) in the standard manner, the coframe fields, $\mathrm{e}^a = \sigma^*\theta^a$, can be chosen to be:*

$$e^1 = \mathrm{d}\theta, \qquad e^2 = \sin\theta\,\mathrm{d}\phi.$$

From this we can calculate the components of the curvature:

$$\mathcal{R}^1_2 = e^1\wedge e^2, \qquad \mathcal{R}^2_1 = -e^1\wedge e^2,$$

and then the trace:

$$\operatorname{tr}\mathcal{R}^2 = -2(e^1\wedge e^2)^2.$$

Symbolically, the first Pontryagin class of TS^2 is:[48]

$$\begin{aligned}p_1(TS^2) &= -\frac{1}{8\pi^2}\operatorname{tr}\mathcal{R}^2 = \frac{1}{4\pi^2}(e^1\wedge e^2)^2\\ &= \left(\frac{1}{2\pi}\sin\theta\,\mathrm{d}\theta\wedge\,\mathrm{d}\phi\right)^2.\end{aligned}$$

As the Euler class is just the square of the Pontryagin class it is then given by:

$$e(S^2) = \frac{1}{2\pi}\sin\theta\,\mathrm{d}\theta\wedge\,\mathrm{d}\phi,$$

in agreement with results from section 5.5 for two-dimensional manifolds: $\mathrm{e}(E) = \frac{1}{2\pi}\mathcal{R}^1_2$. Integrating the Euler class we obtain:

$$\int_{S^2} e(S^2) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\int_0^\pi \sin\theta\,d\theta = 2,$$

the Euler characteristic of a sphere as per the generalised Gauss-Bonnet theorem.

Example 5.5 (Massive vector field) *Let M be a three-dimensional manifold and A a connection form on a principal* $\mathrm{U}(1)$*-bundle on it. The second Chern character is $\mathrm{ch}_2\,F = -\frac{1}{8\pi^2}F\wedge F$, since the gauge group is* $\mathrm{U}(1)$.[49] *Lets find the Chern-Simons form of this character, we guess the form to be:*

$$\begin{aligned}&Q_3 \sim F\wedge A,\\ &\mathrm{d}Q_3 \sim \mathrm{d}F\wedge A + F\wedge\,\mathrm{d}A = F\wedge F;\\ &\quad\Downarrow\\ &Q_3 = -\frac{1}{8\pi^2}F\wedge A,\end{aligned}$$

where the Bianchi identity was used ($\mathrm{d}F = 0$ for $\mathrm{U}(1)$). This is in agreement with the explicit result obtained in section 5.4 for the third Chern-Simons form of the Chern character:

$$\begin{aligned}Q_3(A,F) &= \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2\operatorname{tr}\left(A\wedge\,\mathrm{d}A + \frac{2}{3}A^3\right)\\ &= -\frac{1}{8\pi^2}F\wedge A,\end{aligned}$$

[46] Since the vacuum expectation value of an arbitrary operator H in the path integral formalism is:

$$\langle H\rangle = \frac{1}{Z}\int\prod_i\mathfrak{D}A\,\mathfrak{D}\phi_i H(A,\phi_i)\exp S[A,\phi_i]; \qquad Z = \int\prod_i\mathfrak{D}A\,\mathfrak{D}\phi_i\exp S[A,\phi_i],$$

Z being the partition functional of the given theory.

[47] The complex of two eigenspaces of sections of the spin bundle (those with positive and those with negative chirality) generated by the Dirac operator D.

[48] It is actually zero because one cannot have a 4-form on a two-dimensional manifold, however this is unimportant since it is only a tool to compute the "square root" i.e. the Euler class.

[49] Again, it is actually zero, but for the purposes of this calculation we pretend it is not.

in the case of U(1)*. Since the Chern-Simons form of a Chern character gauge transforms by an exact form, we can build a gauge invariant action from it (up to constant factors):*

$$\begin{aligned}\mathcal{L}\,\mathrm{d}^3x &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}^3x + \frac{1}{4}m\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\nu\,\mathrm{d}^3x \\ &= -\frac{1}{2}F\wedge *F + \frac{1}{4}m\,F\wedge A.\end{aligned}$$

The gauge invariance trivially follows as for U(1) *F does not transform and A transforms as $A \to A + \mathrm{d}f$ for some function f:*

$$\begin{aligned}\mathcal{L} &\to -\frac{1}{2}F\wedge *F + \frac{1}{4}m\,F\wedge A + \frac{1}{4}mF\wedge\,\mathrm{d}f \\ &= -\frac{1}{2}F\wedge *F + \frac{1}{4}m\,F\wedge A + \frac{1}{4}m\,(\,\mathrm{d}(fF) - f\,\mathrm{d}F) \\ &= -\frac{1}{2}F\wedge *F + \frac{1}{4}m\,F\wedge A + \frac{1}{4}m\,\mathrm{d}(fF).\end{aligned}$$

An exact form only contributes on the boundary which makes this action gauge invariant according to the usual assumptions that fields die off sufficiently quickly towards the boundary. The Euler-Lagrange equations produce the following field equation:

$$\partial_\mu F^{\mu\nu} - \frac{1}{2}m\,\epsilon^{\mu\nu\rho}F_{\nu\rho} = 0.$$

However, it pays off to view it as an equation of forms as the following transformations are much more compact in coordinate-free notation:

$$*\,\mathrm{d}{*F} + m * F = 0, \qquad \mathrm{d}{*F} + mF = 0.$$

And consequently,

$$\begin{gathered}*\,\mathrm{d}{*}\,\mathrm{d}{*F} - m^2 * F = 0, \\ \Updownarrow \\ (\Box + m^2)(*F)_\mu = 0.\end{gathered}$$

This is nothing else but the field equation of a massive vector field. Therefore, the Chern-Simons form of the second Chern character describes the mass term of the action of a massive vector field in 3D.

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