Local gauge invariance

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Abstract: We introduce and attempt to justify the local gauge invariance principle. We state its goal and with the main example being the electromagnetism, we give a short description of it. We then state the main geometrical aspects of the newly constructed theory which brings us to the formalism of principal fiber bundles, connections and curvature. We present the suitable mathematical language and give geometrical interpretations to physical quantities along the way.

1. Introduction

When trying to construct the relativistic quantum theory of electromagnetism, one usually builds a classical field theory of electromagnetism and then employs one of the quantization procedures. When approaching the former, there is a certain foundational freedom that is of no practical importance. We can start from Maxwell's equations and construct a Lagrangian whose Euler-Lagrange equations yield Maxwell's equations or derive the Lagrangian from some general principles.
Only when we try to describe the particles experiencing the strong and weak forces do we have a problem, since unlike in the electromagnetic case, we have no classical description of the forces. Because of this, it is of considerable interest to find a principle from which one builds a theory. Since the forces share many similarities, it is hoped that a general principle can be applied that constructs the theory of each force (or a unified one). It cannot be known, from a purely logical-philosophical point of view, whether the local gauge invariance is the only principle of such kind, but it is a very successful one.

To illustrate the principle, we will start with an elementary treatment of electromagnetism. Let $M = \mathbb{R}^4$ be the Minkowski spacetime with standard coordinates $x^\mu = (t, x, y, z) = (x^0, x^1, x^2, x^3)$. The smooth function $\psi : M \to \mathbb{C}^4$ describing the fermions is called the Dirac field. The *free* Dirac field satisfies the Dirac equation $i\gamma^\mu\partial_\mu\psi - m\psi = 0$ and the *free* Lagrangian density (shortly, Lagrangian) is

$$\mathcal{L}_{\text{free}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,. \tag{1}$$

But a free theory cannot describe nature, since it does not describe interactions. We therefore impose the local gauge invariance principle on our theory: it should be invariant under the transformation

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x)\,, \tag{2}$$

where $\alpha : M \to \mathbb{R}$ is a smooth function. Before we proceed, let us make the discussion more general so that the principle becomes suitable for geometric interpretation and applications. It is our hope that with certain changes we can obtain a theory for other fundamental forces.

Instead of $\mathbb{C}^4$ we will write V for a general vector space. Furthermore, the elements of the form $e^{i\phi}$, $\phi \in \mathbb{R}$ constitute a group $U(1)$. Instead, let G be a general Lie group and $\mathfrak{g}$ its Lie algebra (see Appendix for basic definitions). We therefore have the action of G on V. When G is compact and connected, a property enjoyed by $U(1)$ and other groups in particle physics, the exponential map $\exp : \mathfrak{g} \to G$ is surjective and every element $g \in G$ can be written as $g = \exp(A) = e^A$, $A \in \mathfrak{g}$. We will not use this, and write $g(x)$ instead of $e^{i\alpha(x)}$.

A quick look at the Lagrangian (1) tells us that in this form it cannot be invariant under gauge transformation.

The standard procedure here is to introduce the *covariant derivative*

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - A_\mu \tag{3}$$

$$\mathcal{L}_{\text{free}} \longrightarrow \mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{4}$$

In electromagnetism, where $G = U(1)$, A_μ is an imaginary valued function. In general, we don't know yet the character of these objects. From (4) we see that if $\psi'(x) = g(x)\psi(x)$ is the transformed field, then $\bar{\psi}' = \bar{\psi}g^{-1}$ and the appropriate transformation of the potential

$$A'_\mu = g^{-1}\partial_\mu g + g^{-1}A_\mu g \tag{5}$$

leaves the Lagrangian invariant.

As it stands, the Lagrangian (4) is still not yet complete. The reason is that it does not include a term that incorporates a change in the potential A_μ. Laws of nature are expressed in terms of differential equations. When adding such a term, we have to keep in mind that it must be invariant under gauge transformations (we say that it has to be *gauge invariant*). A way to achieve this is by introducing the commutator

$$[D_\mu, D_\nu] = -(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\,. \tag{6}$$

In electromagnetism, $[D_\mu, D_\nu] = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the celebrated electromagnetic field tensor. The full QED Lagrangian is then $\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. We will try to make sense of all of this.

In what follows we will work in a semiclassical approximation and speak of ψ as a *particle* field. When we introduced the transformation (2) we implicitly imposed additional structure on V. For every $x \in M$, in defining $\psi(x)$ along comes the determination of a "reference frame" at x. By this we mean, for example the zero phase angle, as in electromagnetism, where $G = U(1)$, or a basis in the isospin space, as in the Yang-Mills theory, where $G = SU(2)$. Let P_x denote the set of all such "reference frames" above x. A group transformation $p \mapsto pg$, for $p \in P_x$ and $g \in G$ maps one reference frame to another. The dependence of ψ on x can be "lifted" to dependence on p. Of course, $\psi(pg) = g^{-1}\psi(x)$, because when the axes are rotated in one direction, the vector is rotated in the other direction.

Denote by P the union of all P_x, $x \in M$. Our wish is to "glue" the P_x in a "smooth" way. What we are speaking of is a *principle fiber bundle*. It is a special kind of fiber bundle and can be visualised as a collection of "fibers" P_x tied up in a bundle, where tying up depends on M. M need not be $\mathbb{R}^4$; it can be a region determined by our laboratory. Also, P need not be equal to $M \times G$; it may be twisted. (Recall the difference between a cylinder and a Moebius strip. Some interesting physical results can be obtained when the topology of M or P isn't trivial, for example the Aharonov-Bohm effect. We will not go into it in this paper.)

When observing a sufficiently small region of space $U \subset M$, the bundle P looks "trivial", in the sense that it has the structure of $U \times G$. On U there is a function $s_U : U \to P$, such that $s_U(x) \in P_x$, called a *section* over U. Since it is also a smooth choice of reference frames, in physics jargon it is a choice of gauge. Once the gauge is chosen, we can pull down our field ψ to U: $\psi_U(x) = \psi \circ s_U(x)$. Let V be another region and s_V the corresponding section. Let $g_{UV} : U \cap V \to G$ be such that $s_V(x) = s_U(x)g_{UV}(x)$. We then have

$$\psi_V(x) = \psi(s_U(x)g_{UV}(x)) = g_{UV}(x)^{-1}\psi_U(x)\,, \tag{7}$$

which is exactly the transformation (2).

Now that we have motivated the geometrical structure of gauge theories, we will present the mathematical theory of principal fiber bundles, connections and curvature. Along the way we will give definitions of quantities presented in the introduction, and also some new ones.

2. Principal fiber bundles

Definition 1. *Let P,M and F be smooth manifolds and $\pi : P \to M$ a smooth surjection. A* ***local trivialization*** *with fiber F for π is an open cover $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ together with a family $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F\}$ of diffeomorfisms such that the diagrams*

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times F \arrow[d, "\eta"] \\ & U_\alpha \end{tikzcd}$$

commute. Here η is a projection to the first factor. We say that ϕ_α are fiber-preserving. A ***fiber bundle*** *with fiber F is a smooth surjection $\pi : P \to M$ having a local trivialization with fiber F. Manifold P is called the* ***total space*** *and M is called the* ***base space****. A* ***fiber*** *of a fiber bundle $\pi : P \to M$ is the set $P_x := \pi^{-1}(x)$.*

Because π is a submersion, by the regular level set theorem, the each fiber P_x is a regular submanifold of P diffeomorfic to $\{x\} \times F$ via ϕ_α.
We have seen in the introduction that for our purposes the fiber must be a Lie group G. Moreover, there is map called a **smooth right action** of a Lie group G on a manifold P, $\mu : P \times G \to P$ denoted by $p \cdot g := \mu(p, g)$, such that

- $p \cdot e = p$, where e is the identity element of G,
- $(p \cdot g) \cdot h = p \cdot (gh)$, for all $g, h \in G$.

The action of G is **free** if the only element $g \in G$ such that $p \cdot g = p$, for all $p \in P$, is the identity element e. A manifold equipped with the right action of a Lie group G is called a **G-manifold**. Let $f : N \to M$ be a map between two G-manifolds. We say that f is **G-equivariant** if $f(p \cdot g) = f(p) \cdot g$. Now we are ready to give a definition of a principle fiber bundle.

Definition 2. *A smooth fiber bundle $\pi : P \to M$ with fiber G is a smooth* ***principal G-bundle*** *if G acts smoothly and freely on P on the right and the fiber-preserving local trivializations $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ are G-equivariant, where G acts on $U_\alpha \times G$ on the right by $(x, h) \cdot g = (x, hg)$.*

G-equivariance of the local trivializations in the previous definition is important: it preserves the structure. $U \times G$ are "local coordinates" of P, in a sense, and if there is a change in the "reference frame" $p \mapsto pg$, we want it to be visible in the coordinates.
A **section** of P over U is a function $s : U \to P$ such that $\pi \circ s = \mathrm{id}_U$. Consider now a right G-equivariant map $f : G \to G$. Then $f(g) = f(eg) = f(e)g = l_{f(e)}(g)$, where $l_h : G \to G$ is a left translation by h. This shows that f is necessarily a left translation. We conclude this section with a notion of transition functions.
Let $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \mid \alpha \in A\}$ be a local trivialization for a principle G-bundle $\pi : P \to M$ and $\alpha, \beta \in A$. Denote $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Then $\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ is a fiber preserving right G-equivariant diffeomorfism, and so by the preceding discussion it is a left translation on each fiber. It is therefore of the form $(x, h) \mapsto (x, g_{\alpha\beta}(x)h)$. Smooth functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ are called **transition functions**.

3. Connections

Throughout this section, G is a Lie group with Lie algebra $\mathfrak{g}$ and $\pi : P \to M$ is a principal G-bundle.

3.1. Vertical and Horizontal Distributions of the Tangent Bundle TP

On the total space P there is a natural notion of vertical tangent vectors. By the local triviality condition, since the trivializations are diffeomorphisms, the differential $\pi_{*,p} : T_pP :\to T_{\pi(p)}M$ is surjective. We define the **vertical subspace** $\mathcal{V}_p$ to be $\ker \pi_{*,p}$. To shed some light on this concept it is useful to define the fundamental vector field.
Let $A \in \mathfrak{g}$. Then the function $t \mapsto e^{tA}$, $t \in \mathbb{R}$ is an integral curve of a left invariant vector field generated by A starting at e. We define the **fundamental vector field $\underline{A}$ associated to A** by

$$\underline{A}_p = \frac{d}{dt}\bigg|_{t=0} p \cdot e^{tA} \in T_pP \, .$$

There is an alternative description of a fundamental vector field. For $p \in P$, define $j_p : G \to P$ by $j_p(g) = p \cdot g$. Using the curve $c(t) = e^{tA}$ we can calculate the action of a differential $j_{p*,e}$ on A:

$$j_{p*}(A) = j_{p*}(c'(0)) = \left.\frac{d}{dt}\right|_{t=0} j_p(e^{tA}) = \underline{A}_p \, .$$

The following proposition gives us a useful characterisation of vertical vectors.

Proposition 1. *For any $A \in \mathfrak{g}$, the fundamental field $\underline{A}$ is vertical at every point $p \in P$. For every $p \in P$, the map $j_{p*,e} : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_p$ is an isomorphism of vector spaces.*

Therefore, the vertical tangent vectors at a point are precisely the fundamental vectors. We can visualize this as follows: Let M be a plane. To each $x \in M$ we want to associate a group fiber. For example, if $G = U(1)$, which is diffeomorphic to the line segment with end points identified, we can stack the line segments (keeping in mind that the end points are identified) in such a way that the identity elements of all fibers coincide with the point in the plane. The fibers now look like straws stacked in a jar. For any p in the resulting bundle P, $p \cdot e^{tA}$ always stays in the same fiber, i.e., moves in the "up-down" direction. And so the vertical tangent vector at p will indeed be "vertical".

Now let $B_1, ..., B_l$ be a basis for a Lie algebra $\mathfrak{g}$. By the preceding proposition, the fundamental vector fields $\underline{B_1}, ..., \underline{B_l}$ form a basis of $\mathcal{V}_p$ for every $p \in P$. Therefore, by theorem 9 from the Appendix, the family $\mathcal{V} = \coprod_{p \in P} \mathcal{V}_p$ is a subbundle of a vector bundle TP. The subbundle of a tangent bundle of a manifold is also called a **distribution** on a manifold.

Now, for each $p \in P$ we can choose a subspace $\mathcal{H}_p$ of T_pP such that $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$,i.e.$T_pP = \mathcal{V}_p \cup \mathcal{H}_p$ and $\mathcal{V}_p \cap \mathcal{H}_p = \{0\}$. There is no canonical way to do this, unlike the vertical subspaces, which are always well defined on a principle bundle. If $\mathcal{H} = \coprod_{p \in P} \mathcal{H}_p$ forms a vector subbundle of the tangent bundle, it is called a **horizontal** distribution.

3.2. Ehresmann connection

When the horizontal distribution of a principle bundle $\pi : P \to M$ is chosen, there is a well defined notion of a vertical and horizontal component of a tangent vector. For each $p \in P$ let $v : T_pP \to \mathcal{V}_p$ and $h : T_pP \to \mathcal{H}_p$ be the natural projections. Every vector $Y_p \in T_pP$ can be written as a unique sum of a vertical and a horizontal vector. It can be shown that if Y is a smooth vector field on P, then $v(Y)$ and $h(Y)$ are also smooth. In this section we show (but do not prove) how the family of horizontal distributions is in a one to one correspondence with a family of $\mathfrak{g}$-valued 1-forms satisfying certain conditions.

Let $g \in G$ and $r_g : P \to P$ be the right action $r_g(p) = pg$. Since the action og G is free and transitive on each fiber, r_g is a diffeomorphism. We say that a distribution is **right-invariant** is $r_{g*}(\mathcal{H}_p) = \mathcal{H}_{pg}$. Proposition 1 allows the following theorem:

Theorem 1. *Let $\mathcal{H}$ be a smooth right-invariant horizontal distribution of a principal G.bundle $\pi : P \to M$. Define the $\mathfrak{g}$-valued 1-form ω by $\omega_p = j_{p*}^{-1} \circ v : T_pP \to \mathcal{V}_p \to \mathfrak{g}$, for each $p \in P$. Then ω satisfies the following properties:*

- *For any $A \in \mathfrak{g}$, we have $\omega(\underline{A}) = A$;*
- *(G-equivariance) For any $g \in G$, $r_g^* \omega = (\mathrm{Ad} g^{-1})\omega$*
- *ω is C^∞*

Definition 3. *An **Ehresmann connection** or simply a **connection** on a principal G-bundle $P \to M$ is a $\mathfrak{g}$-valued 1-form on P satisfying the three properties of the preceding theorem.*

Theorem 2. *If ω is a connection on a principal G-bundle $P \to M$, then $\mathcal{H}_p = \ker \omega_p$, $p \in P$ is a smooth right-invariant horizontal distribution.*

Lets fix a horizontal distribution $\mathcal{H}$ on $\pi : P \to M$ and let X be a vector field on M. For every $p \in P$, the differential $\pi_{*,p} : T_pP \to T_{\pi(p)}M$ induces an isomorphism between $T_{\pi(p)}M$ and $\mathcal{H}_p$:

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p \implies$$
$$\frac{T_pP}{\mathcal{V}_p} \xrightarrow{\sim} \mathcal{H}_p$$
$$\frac{T_pP}{\mathcal{V}_p} = \frac{T_pP}{\ker \pi_{*,p}} \xrightarrow{\sim} \mathrm{im}\pi_{*,p} = T_{\pi(p)}M$$

Consequently, for each $p \in P$ there is a unique horizontal vector $\tilde{X}_p \in \mathcal{H}_p$ such that $\pi_*(\tilde{X}_p) = X_{\pi(p)}$. The vector field $\tilde{X}$ is called the **horizontal lift** of X from M to P. The next proposition will be proven to illustrate the methods and concepts we have introduced thus far.

Proposition 2. *If $\mathcal{H}$ is a smooth right-invariant horizontal distribution on the total space P of a principal G-bundle $\pi : P \to M$, then the horizontal lift $\tilde{X}$ of a smooth vector field X on M is a smooth right-invariant vector field on P.*

Proof. Let $x \in M$ and $p \in \pi^{-1}(x) = P_x$. Let us first prove the right-invariance. From local triviality and G-equivariance of local trivializations, it follows that $\pi(pg) = \pi(p)$, i.e. $\pi_g = \pi$. Therefore, if $q \in P_x$ is another point in the fiber, by the transitivity of the action, $q = pg$, for some $g \in G$ and so we have

$$\pi_{*,pg}(r_{g*}\tilde{X}_p) = \pi_{*,p}(\tilde{X}_p) = X_x \,.$$

By the uniqueness of the horizontal lift, $r_{g*}\tilde{X}_p = \tilde{X}_{pg}$.
To prove the smoothnes of $\tilde{X}$, we prove it locally, over a trivializing open set $U \subset M$ with a trivialization $\phi : \pi^{-1}(U) \to U \times G$. Define $Z_{(x,g)} = (X_x, 0) \in T_{(x,g)}(U \times G)$ and let $\eta : U \times G \to U$ be the projection to the first factor. Then obviously Z is a smooth vector field on $U \times G$ such that $\eta_* Z_{(x,g)} = X_x$. Hence $Y := \phi_*^{-1}(Z)$ is a smooth vector field on $\pi^{-1}(U)$. By the triviality property, $\pi_*(Y_p) = (\eta \circ \phi)_*(\phi_*^{-1} Z_{\phi(p)}) = X_{\pi(p)}$. We have mentioned that if Y is smooth on P, then its horizontal component hY is smooth also. Since the vertical vectors are in the kernel of π_*, we have $\pi_*(Y_p) = \pi_*(h(Y_p) + v(Y_p)) = \pi_*(h(Y_p)) = X_{\pi(p)}$. Therefore, hY lifts X over U and by the uniqueness of the horizontal lift, $\tilde{X} = hY$ over U. This proves that $\tilde{X}$ is a smooth vector field. ☐

3.3. Gauge potentials

In the introduction we have seen how an introduction of a local gauge invariance principle implicitly introduces new "reference frames" resulting from the action of a group G. We can therefore "lift" our particle field from some region of space-time to the newly constructed principal bundle. Upon descending the field locally to some open set, via a *choice of gauge*, we get the desired properties of our particle fields. In this subsection we will try to make sense of A_μ from the introduction, know to us from electromagnetism as an electromagnetic field and show how it can be explained by descending a global object, a connection form, to an open set of a space-time.

Recall that a section over a fiber bundle $\pi : P \to M$ is a function $s : U \to P$ such that $\pi \circ s = \mathrm{id}_U$. Such section is also called a **local** section, as opposed to a **global** section whose domain is the entire base space M. It turns out that when the fiber bundle is a principal G-bundle, if there is a local section over U, then U is a trivializing open set. Conversely, on every trivializing open set there is a natural local section over it.

Proposition 3. *There is a natural correspondence between local trivializations $\phi_U : \pi^{-1}(U) \to U \times G$ and local sections on P.*

Proof. Let $s_U : U \to P$ be a local section and $x \in U$. s sends x to the fiber P_x over it. Since G acts transitively on each fiber, every element $q \in P_x$ can be written as $q = s_U(x)g$, for some $g \in G$. Therefore, the map $\phi_U : \pi^{-1}(U) \to U \times G \,, \phi_U(s_U(x)g) = (x, g)$ is well defined. Its inverse ϕ_U^{-1} is given by $\phi_U^{-1}(x, g) = s_U(x)g = \mu \circ (s_U \times \mathrm{id}_G)(x, g)$. $\psi_U := \phi_U^{-1}$ is obviously a smooth map, and both are G-equivariant. To show that ϕ_U is a local trivialization, it remains to show that it is smooth, i.e. a diffeomorphism.

Let $(x, g) \in U \times G$ be arbitrary. If we show that the differential of ψ_U is an isomorphism at (x, g), by the inverse function theorem it will follow that ψ_U has a smooth local inverse. Since the inverse is unique, this will show that ϕ_U is smooth in a neighbourhood of $\psi_U(x,g)$, i.e. that it is smooth.

We therefore need to calculate the differential of the action $\mu_{*,(x,g)} : T_pP \times T_gG \to T_{pg}P$. Let $l_g : G \to G$ be the left translation on a Lie group G by g. It is a diffeomorphism, so the tangent space of G at g can be identified with $l_{g*}\mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra of G. Since $\mu_*(X_p, l_{g*}A) = \mu_*(X_p, 0) + \mu_*(0, l_{g*}A)$, it suffices to calculate the each term separately.

Let $c(t)$ be a curve through p in P with initial vector X_p. Then $(c(t), g)$ is a curve through (p, g) with initial vector $(X_p, 0)$. So,

$$\begin{aligned} \mu_{*,(p,g)} &= \frac{d}{dt}\bigg|_{t=0} \mu(c(t), g) \\ &= \frac{d}{dt}\bigg|_{t=0} r_g(c(t)) \\ &= r_{g*}c'(0) = r_{g*}X_p\,. \end{aligned}$$

Since a curve through (p, g) with initial vector $(0, l_{g*}A)$ is (p, ge^{tA}), so

$$\begin{aligned} \mu_{*,(p,g)}(0, l_{g*}A) &= \frac{d}{dt}\bigg|_{t=0} \mu(p, ge^{tA}) \\ &= \frac{d}{dt}\bigg|_{t=0} pge^{tA} \\ &= \underline{A}_{pg}\,. \end{aligned}$$

We have thus found that $\mu_{*,(p,g)}(X_p, l_{g*}A) = r_{g*}X_p + \underline{A}_{pg}$. Let (x, g) be a point in $U \times G$, and let $Y_x \in T_xM$ and $l_{g*}A \in T_gG$. We thus have

$$\psi_{U*,(x,g)}(Y_x, l_{g*}A) = r_{g*}(s_{U*}Y_x)) + \underline{A}_{pg}.$$

Now, since $\pi \circ r_g \circ s_U = \pi \circ s_U = \mathrm{id}_U$, so the first term is never vertical, unless $Y_x = 0$. If $\psi_{U*,(x,g)}(Y_x, l_{g*}A) = 0$ it follows that $Y_x = A = 0$, so the differential is injective. Surjectivity is obvious, and so ϕ_U is a local trivialization associated to the local section s_U.

Conversely, if we are given a local trivialization $\phi_U : \pi^{-1}(U) \to U \times G$, then $s_U(x) = \phi_U^{-1}(x, e)$ defines a smooth local section over U. □

From hence forth, we will always work with the preferred local sections associated to local trivializations as above. The preceding proof also gave us a useful lemma.

Lemma 1. *Let $\mu : P \times G \to P$ be the action of a Lie group G on the principal bundle $P \to M$. The differential $\mu_{*,(p,g)} : T_pP \times T_gG \to T_{pg}P$ is given by*

$$\mu_{*,(p,g)}(X_p, l_{g*}A) = r_{g*}X_p + \underline{A}_{pg}\,.$$

Theorem 3. *Let $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \mid \alpha \in A\}$ be a local trivialization of a principal G-bundle $\pi : P \to M$ and ω a connection on P. Let s_α be the induced local sections and $\omega_\alpha = s_\alpha^*\omega$. If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, then on $U_{\alpha\beta}$ we have*

$$\omega_\beta(Y_x) = l_{g_{\alpha\beta}(x)^{-1}*}(g_{\alpha\beta*}(Y_x)) + \mathrm{Ad}g_{\alpha\beta}(x)^{-1}(\omega_\alpha(Y_x))\,, \tag{8}$$

for every $x \in U_{\alpha\beta}$ and $Y_x \in T_xM$

Proof. Let us write $\phi_\alpha(p) = (\pi(p), a_\alpha(p))$ and $\phi_\beta(p) = (\pi(p), a_\beta(p))$. Then, by the G-equivariance property, $\phi_\alpha = \phi_\beta \cdot (a_\alpha a_\beta^{-1})$ and therefore, if $x = \pi(p)$, the transition function is given by $g_{\alpha\beta}(x) = a_\alpha(p)a_\beta^{-1}(p)$. If q is another point in the fiber P_x, then $q = pg$, for some $g \in G$, and therefore, $a_\alpha(q)a_\beta^{-1}(q) = a_\alpha(p)g(a_\beta(p)g)^{-1} = a_\alpha(p)a_\beta^{-1}(p)$, so $g_{\alpha\beta}(x)$ is well defined.
Let us now find the relation between the induced local sections. By definition $\phi_\alpha(s_\alpha(x)a_\alpha(p)) = (x, a_\alpha(p)) =$

$\phi_\alpha(p) \implies s_\alpha(x)a_\alpha(p) = p$ and similarly $s_\beta(x)a_\beta(p) = p$. Hence, $s_\beta(x) = s_\alpha(x)g_{\alpha\beta}(x) = \mu \circ (s_\alpha \times g_{\alpha\beta})(x)$. Put $p := s_\alpha(x)$ and $g := g_{\alpha\beta}(x)$. The previous lemma now gives us

$$\begin{aligned}
s_{\beta*}(Y_x) &= \mu_{*,(p,g)}(s_{\alpha*}(Y_x), g_{\alpha\beta*}(Y_x)) \\
&= \mu_{*,(p,g)}(s_{\alpha*}(Y_x), l_{g*}(l_{g^{-1}*}g_{\alpha\beta*}(Y_x))) \\
&= r_{g*}(s_{\alpha*}(Y_x)) + \underline{l_{g^{-1}*}g_{\alpha\beta*}(Y_x))}_{pg} \implies \\
\omega_\beta(Y_x) &= (s_\beta^*\omega)_x(Y_x) \\
&= \omega_{pg}(s_{\beta*}(Y_x)) \\
&= \omega_{pg}(r_{g*}(s_{\alpha*}(Y_x))) + \omega_{pg}(\underline{l_{g^{-1}*}g_{\alpha\beta*}(Y_x))}_{pg}) \\
&= (\mathrm{Ad}g^{-1})\omega_p(s_{\alpha*}(Y_x)) + l_{g^{-1}*}g_{\alpha\beta*}(Y_x)) \\
&= l_{g^{-1}*}g_{\alpha\beta*}(Y_x)) + (\mathrm{Ad}g^{-1})\omega_\alpha(Y_x)
\end{aligned}$$

□

When G is a matrix group, this formula can be put in a more simple form. Let $c(t)$ be a path through x with $c'(0) = Y_x$. Then

$$\begin{aligned}
l_{g^{-1}*}(g_{\alpha\beta*}(Y_x)) &= (l_{g^{-1}} \circ g_{\alpha\beta} \circ c)'(0) \\
&= \frac{d}{dt}\bigg|_{t=0} (g^{-1} \cdot g_{\alpha\beta}(c(t))) \\
&= g_{\alpha\beta}(x)^{-1} dg_{\alpha\beta}(Y_x)\,.
\end{aligned}$$

See the Appendix for the definition of the exterior derivative of a vector(matrix) valued form. Furthermore, since $\mathrm{Ad}g = c_{g*,e}$, where $c_g : G \to G$ is a conjugation by g, $h \mapsto ghg^{-1}$, for a matrix group we have

$$\mathrm{Ad}A(B) = c_{A*,I}(B) = \frac{d}{dt}\bigg|_{t=0} c_A(e^{tB}) = \frac{d}{dt}\bigg|_{t=0} Ae^{tB}A^{-1} = ABA^{-1}\,.$$

Therefore, equation (7) becomes

$$\omega_\beta = g_{\alpha\beta}^{-1}dg_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta}$$

Now, let us view this in terms of coordinate forms. We can shrink the domain, if necessary, so that $U_{\alpha\beta}$ becomes a coordinate open set, with coordinates x^μ. Set $\omega_\beta = A'_\mu dx^\mu$ and $\omega_\alpha = A_\mu dx^\mu$. When both ω_α and ω_β act on the coordinate vector field ∂_μ, we obtain (with a shortcut notation $g_{\alpha\beta}(x) = g(x)$)

$$A'_\mu = g^{-1}\partial_\mu g + g^{-1}A_\mu g\,,$$

which is *exactly* the equation (5). Thus, we have learned about the character of a gauge potential:

Definition 4. *Let $\pi : P \to M$ be a principal G-bundle over an open set of space-time, ω an Ehresmann connection and $s : U \to P$. The gauge potential A_μ on U is a coordinate component of a pullback of a connection $s^*\omega$, a Lie algebra-valued 1-form. A_μ is thus a Lie algebra-valued function. We sometimes refer to 1-forms $s^*\omega$ as gauge potentials.*

As we have announced in the introduction, a gauge transformation amounts to a change in a local section, $s(x) \mapsto s(x) \cdot g(x)$. We will give a formal definition of a gauge transformation in the last section.

A natural question arises here. Given a concrete physical system, how do we construct a connection and is it even possible? Many times it will occur that the global gauge potential is not available, i.e. the G-bundle is not trivial (magnetic monopole being a common example). This is answered by the following theorem.

Theorem 4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\pi : P \to M$ be a principal G-bundle, $\{\phi_\alpha(U_\alpha) : \pi^{-1}(U_\alpha) \mid \alpha \in A \to U_\alpha \times G\}$ its local trivialization and $s_\alpha : U_\alpha \to P$ the induced local sections. If $\{\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}) \mid \alpha \in A\}$ is a collections of $\mathfrak{g}$-valued 1-forms such that for $x \in U_\alpha \cap_\beta \neq \varnothing$ we have*

$$\omega_\beta(Y_x) = l_{g_{\alpha\beta}(x)^{-1}*}(g_{\alpha\beta*}(Y_x)) + \mathrm{Ad}g_{\alpha\beta}(x)^{-1}(\omega_\alpha(Y_x))\,, \tag{9}$$

then there is a unique connection ω on $P \to M$ such that $s_\alpha^*\omega = \omega_\alpha$, for every $\alpha \in A$.

To end this section, we will give meaning to the previously mentioned *covariant derivative.*

As we have seen, general philosophy has been either "lifting" to the total space P or "pulling down" to the base space M. In the introduction we have seen that the concept of a gauge potential and covariant derivative appear simultaneously in an attempt to replace the usual derivative ∂_μ, alongside with gauge invariance, which gave rise to the principal bundle formalism. We will therefore, try to lift the coordinate vector field to the total space.

We already have a notion of a horizontal distribution on P and a horizontal lift $\tilde{X}$ of a vector field X on M. Theorems 1 and 2 tell us that a horizontal distributions and connections are in a one-to-one correspondence. Therefore, let (U, x^μ) be a coordinate trivializing open set on a principal G-bundle $\pi : P \to M$ with connection ω, s_U the induced local section and $\omega_U = s_U^* \omega = A_\mu dx^\mu$. Let $\tilde{\partial_\mu}$ be a horizontal lift of a coordinate vector field ∂_μ. Let x be a point in U and $p = s_U(x)$. We have

$$\begin{aligned} \omega_{Ux}(\partial_\mu|_x) = \omega_p(s_{U*}(\partial_\mu|_x)) = A_\mu(x) = \omega_p(\underline{A_\mu(x)}_p) \implies \\ \omega_p(s_{U*}(\partial_\mu|_x) - \underline{A_\mu(x)}_p) = 0\,. \end{aligned}$$

Hence, $s_{U*}(\partial_\mu|_x) - \underline{A_\mu(x)}_p$ is horizontal and $\pi_{*,p}(s_{U*}(\partial_\mu|_x) - \underline{A_\mu(x)}_p) = \partial_\mu|_{\pi(p)}$. By the uniquiness of the horizontal lift, it follows that $\tilde{\partial_\mu} \circ s_U = s_{U*}(\partial_\mu) - \underline{A_\mu} \circ s_U$. Since s_U is a diffeomorphism into its own image, tangent vector Y_x at T_xM can be identified with $\underline{s_{U*}}(Y_x)$ at T_pP, and the fundamental vector fields can be identified with its Lie algebra generators. Thus, we have found the interpretation of a covariant derivative $D_\mu = \partial_\mu - A_\mu$ as a horizontal lift of a coordinate vector field ∂_μ.

4. Curvature

In Appendix you can find a short discussion of a connection and curvature on a *vector bundle* and their connection and curvature matrix forms. There is also a short treatment of vector valued forms and their products. This will help to motivate the definition of a curvature of a connection on a *principal bundle.*

If ∇ is a connection on a vector bundle $\pi : E \to M$, then its connection and curvature matrices ω_e and Ω_e on a framed open set (U, e) are related by the equation $\Omega_e = d\omega_e + \omega_e \wedge \omega_e = d\omega_e + \frac{1}{2}[\omega_e, \omega_e]$. On a general Lie group, wedge product is not defined, but the Lie bracket is always defined. This brings us at the definition of curvature.

Definition 5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and ω an Ehresmann connection on a principal G-bundle $\pi : P \to M$. Then the* ***curvature*** *of the connection ω is a $\mathfrak{g}$-valued 2-form*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

The next theorem gives us some essential properties of a curvature

Theorem 5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\pi : P \to M$ is a principal G-bundle, ω connection on P, and Ω curvature form of ω. Then*

- (Horizontality) *For every $p \in P$ and $X_p, Y_p \in T_pP$,*

$$\Omega_p(X_p, Y_p) = (d\omega)_p(hX_p, hY_p)\,.$$

- (G-equivariance) *For every $g \in G$, we have $r_g^* \Omega = (\mathrm{Ad} g^{-1})\Omega$.*
- (Second Bianchi identity) $d\Omega = [\Omega, \omega]$.

The first property also states that Ω is horizontal. A differential form on P is said to be **horizontal** if it vanishes on vertical vectors.

Of course, just as a connection pulls back by local sections to gauge potentials, the curvature form can also be pulled back to something identified with *field strength.* Let $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \mid \alpha \in A\}$ be a local trivialization for a principal G-bundle $\pi : P \to M$ and $s_\alpha : U_\alpha \to P$ the induced local sections. Let Ω_α be the pullback $s_\alpha^*\Omega$. Since the pullback commutes with the exterior derivative and with Lie bracket, we have from definition 5

$$\Omega_\alpha = d\omega_\alpha + \frac{1}{2}[\omega_\alpha, \omega_\alpha]\,.$$

Let U_α be a coordinate open set and write $\omega_\alpha = A_\mu dx^\mu$. If $B_1, ..., B_l$ is a basis for $\mathfrak{g}$, we can write $A_\mu = A^i_\mu B_i$ and $\omega_\alpha = (A^i_\mu dx^\mu)B_i$, for some real valued functions A^i_μ , $i = 1, ..., l$. Then, by proposition 5 from the Appendix, we have

$$\begin{aligned}
[\omega_\alpha, \omega_\alpha] &= (A^i_\mu dx^\mu) \wedge (A^j_\nu dx^\nu)[B_i, B_j] \\
&= [A^i_\mu B_i, A^j_\nu B_j] dx^\mu \wedge dx^\nu \\
&= [A_\mu, A_\nu] dx^\mu \wedge dx^\nu \,; \\
d\omega_\alpha &= (dA^i_\nu \wedge dx^\nu)B_i \\
&= (\partial_\mu A^i_\nu) dx^\mu \wedge dx^\nu B_i \\
&= (\partial_\mu A_\nu) dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}(\partial_\mu A_\nu)(dx^\mu \wedge dx^\nu - dx^\nu \wedge dx^\mu) \\
&= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \implies \\
\Omega_\alpha :&= \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \\
&= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) dx^\mu \wedge dx^\nu \,.
\end{aligned}$$

This has *exactly the same* form as a commutator (6) introduced in the first section. Thus we see that the curvature is responsible for the dynamics of the gauge field.

It is very useful to know how the curvature responds to a change of gauge. This is answered by the following theorem.

Theorem 6. *Let $\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \mid \alpha \in A\}$ be a local trivialization of a principal G-bundle $\pi : P \to M$ and Ω the curvature form of a connection ω on P. Let s_α be the induced local sections and $\Omega_\alpha = s_\alpha^*\Omega$. If $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$, then on $U_{\alpha\beta}$ we have*

$$\Omega_\beta = \mathrm{Ad} g^{-1}_{\alpha\beta} \Omega_\alpha$$

Proof. Let x be a point in $U_{\alpha\beta}$, $X_x, Y_x \in T_xM$ and put $p = s_\alpha(x)$. In the proof of theorem 3 we have learned that $s_\beta(x) = s_\alpha(x)g_{\alpha\beta}(x) := pg$ and $s_{\beta*}(Y_x) = r_{g*}(s_{\alpha*}(Y_x)) + \underline{l_{g^{-1}*}g_{\alpha\beta*}(Y_x))}_{pg}$. Because fundamental vectors are vertical, we have

$$\begin{aligned}
\Omega_{\beta x}(X_y, Y_x) &= \Omega_{pg}(s_{\beta*}(X_x), s_{\beta*}(Y_x)) \\
&= \Omega_{pg}(r_{g*}(s_{\alpha*}(X_x)), r_{g*}(s_{\alpha*}(Y_x))) \quad \text{Horizontality} \\
&= (r_g^*\Omega)_p(s_{\alpha*}(X_x), s_{\alpha*}(Y_x)) \\
&= (\mathrm{Ad} g^{-1})\Omega_p(s_{\alpha*}(X_x), s_{\alpha*}(Y_x)) \quad G\text{-equivariance} \\
&= (\mathrm{Ad} g^{-1})\Omega_{\alpha x}(X_y, Y_x)\,.
\end{aligned}$$

□

When G is a matrix group, then $\Omega_\beta = g^{-1}_{\alpha\beta}\Omega_\alpha g_{\alpha\beta}$.

5. Particle fields and gauge transformations

In this section we will formalize the discussion of a particle field and gauge transformation from the end of Introduction.

Definition 6. *Let $\pi : P \to M$ be a principal G bundle and $\rho : G \to \mathrm{GL}(V)$ a representation of G on a finite dimensional vector space V. The notation will be $\rho(g)v = g \cdot v = gv$. The **associated bundle** $E := P \times_\rho V$ is the quotient of $P \times V$ by the equivalence relation*

$$(p, v) \sim (pg, g^{-1} \cdot v)\,,$$

for $g \in G$ and $(p, v) \in P \times V$. The equivalence class of (p, v) is denoted by $[p, v]$. Since $\pi(pg) = \pi(p)$, the natural projection $\beta : E \to M$ given by $\beta([p, v]) = \pi(p)$ is well defined.

Proposition 4. *If $\rho : G \to \mathrm{GL}(V)$ is a finite dimensional representation of a Lie group G and U is a manifold, then there is a fiber preserving diffeomorphism*

$$\begin{aligned} \phi : (U \times G) \times_\rho V &\xrightarrow{\sim} U \times V\,, \\ [(x, g), v] &\mapsto (x, g \cdot v)\,. \end{aligned}$$

Since the principal bundle $P \to M$ is locally $U \times G$, this proposition shows that the associated bundle $P \times_\rho V \to M$ is locally trivial with fiber V. This makes the associated bundle into a vector bundle, with the well defined operations on fibers given by

$$\begin{aligned} [p, v_1] + [p, v_2] &= [p, v_1 + v_2] \\ \lambda[p, v] &= [p, \lambda v]\,. \end{aligned}$$

Definition 7. *A V-valued k-form ϕ on P is said to be **right-equivariant of type** ρ if for every $g \in G$ we have $r_g^*\phi = \rho(g)^{-1} \cdot \phi$. A form that is both horizontal and right-equivariant of type ρ is called **tensorial of type** ρ. The set of all tensorial V-valued k-forms on P are denoted by $\Omega^k_\rho(P, V)$.*

Theorem 7. *$\Omega^k(M, E)$ and $\Omega^k_\rho(P, V)$ are isomorphic vector spaces.*

When $k = 0$ in the above theorem, $\Omega^0(M, E)$ consists of maps from M to the associated bundle E, i.e. of sections of E. $\Omega^0_\rho(P, V)$ on the other hand consists of maps $f : P \to V$ that are right-equivariant with respect to ρ, i.e. $f(pg) = g^{-1} \cdot f(p)$, for all $p \in P$ and $g \in G$. The previous theorem for the special case is stated in the following corollary.

Corollary 1. *Let G be a Lie group, $P \to M$ a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ a representation of G. Then there is a ono-to-one correspondence*

$$\{\text{right-equivariant maps of type}\, \rho\,,\; f : P \to V\} \longleftrightarrow \Gamma(P \times_\rho V, M)\,.$$

Definition 8. *Let G be a Lie group, $\rho : G \to \mathrm{GL}(V)$ its finite dimensional representation and $\pi : P \to M$ a principal G-bundle, where M is a region of space-time. A **particle field** is a section of the associated vector bundle $P \times_\rho V \to M$.*

Lifting the dependence from M to P allows us to make the discussion global and to observe the geometric properties. We know that there are passive transformations (i.e. a change of reference frame or, equivalently, action of $g(x)$ to the right) and active transformations (i.e. action on the field in the opposite direction). In principle, these are indistinguishable and hence the associated bundle formalism. That's why we are identifying the points (p, v) and $(pg, g^{-1}v)$. All such descriptions are equivalent, and therefore to each $x \in M$ we are associating an equivalence class.

By corollary 1, a particle field ψ can also be viewed as a right-equivariant map $\psi : P \to V$. Descending on a trivializing open set U_α, i.e. working in a particular gauge, means pulling the field back by a local section s_α associated to the given trivialization, $\psi_\alpha = s_\alpha^*\psi$. When a gauge is changed, $s_\beta = s_\alpha g_{\alpha\beta}$, the field is changed according to

$$\psi_\beta(x) = \psi(s_\alpha(x) g_{\alpha\beta}(x)) = g_{\alpha\beta}(x)^{-1} \cdot \psi_\alpha(x)\,,$$

which we have seen before. We conclude with the following definition.

Definition 9. *An* ***automorphism*** *of a principal G-bundle $\pi : P \to M$ is a G-equivariant diffeomorphism $f : P \to P$. This means that for all $p \in P$ and $g \in G$, $\pi(f(pg)) = \pi(f(p)g) = \pi(f(p))$, which means that each fiber is diffeomorphically transformed to another. A* ***gauge transformation*** *of $P \to M$ is a fiber preserving automorphism, i.e. a G-equivariant diffeomorphism $f : P \to P$ such that the diagram*

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\mathrm{id}_M} & M \end{array}$$

commutes.

6. Concluding remarks

In this paper, our philosophy was not to present theorems, but to motive definitions. We tried to describe a natural mathematical language appropriate for discussion of gauge theories and the local gauge invariance principle. The word *local* is somewhat misterious. A *global* transformation is of the kind $p \mapsto pg$. A local transformation, in the physicists sense, is a transformation of the kind $p \mapsto ph(x)$, where the group elements are dependent on the base space points. These transformations exist only on product bundles, i.e. *locally*, in a mathematicians sense (on an open neighbourhood). This is one of the many places where mathematics and physics struggle to communicate. Our intention was to try to make this communication easier.

We have only attempted to lay the foundations of the formalism. We have not discussed the vast applications it provides. We haven't discussed the dynamical aspects of gauge theory, the Yang-Mills theory, the Standard Model or the inclusion of gravity.

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Appendix A. Lie groups and Lie algebras

Let G be a smooth manifold that is also a group. If the group operations $\mu : G \times G \to G$ and $i : G \to G$, $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth, then G is called a *Lie group.* A *Lie algebra* V is a vector space together with the alternating bilinear map called the *Lie bracket*, $(x,y) \mapsto [x,y]$, such that it satisfies the *Jacobi identity*: For every $x,y,z \in V$, we have $[[x,y],z] + [[z,x],y] + [[y,z],x] = 0$.

The tangent space of a Lie group G at the identity can be given a Lie algebra structure in the following way: Let $l_g : G \to G$ be the left translation by an element $g \in G$. A vector field $X \in \mathcal{X}(G)$ is said to be left invariant if $l_{g*}X = X$. It can be shown that the left translation commutes with the Lie bracket of vector fields, so $L(G)$, a space of all left invariant vector fields is a Lie subalgebra of the Lie algebra of smooth vector fields on G, with the Lie bracket being the usual bracket of vector fields.

If $A \in T_eG$ then $\tilde{A}_g = l_{g*}A$ is a smooth left invariant vector field. Hence there is a one-to-one correspondence $A \mapsto \tilde{A}$ and $X \mapsto X_e$ between T_eG and $L(G)$. With the Lie bracket $[A,B] := [\tilde{A},\tilde{B}]_e$, the tangent space at the identity $\mathfrak{g} := T_eG$ becomes a Lie algebra.

Now, let $c_X(t)$ be an integral curve of a left invariant vector field X on G passing through e. It can be shown that it is define for all $t \in \mathbb{R}$.

Definition 10. *The* ***exponential map*** *for a Lie group G with Lie algebra* $\mathfrak{g}$ *is the map* $\exp : \mathfrak{g} \to G$ *defined by* $\exp(A) := e^A = c_{\tilde{A}}(1)$.

Theorem 8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

- *For $A \in \mathfrak{g}$, the integral curve starting at e of a left invariant vector field $\tilde{A}$ is* $\exp(tA)$.
- *For $A \in \mathfrak{g}$ and $g \in G$, the integral curve starting at g of a left invariant vector field $\tilde{A}$ is* $g\exp(tA)$.
- *The exponential map is smooth.*

For $g \in G$, define $c_g : G \to G$, the *conjugation* by g: $c_g(h) = ghg^{-1}$. The differential at the identity is denoted by $\mathrm{Ad}(g) := \mathrm{Ad}g = c_{g*,e} : \mathfrak{g} \to \mathfrak{g}$. Then the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is easily seen to be a group homomorphism and is called the *adjoint representation* of a Lie group G.

Appendix B. Vector bundles and connections

Let $\pi : E \to M$ be a fiber bundle whose fiber is a vector space V of dimension r. If the local trivializations restrict to a linear isomorphism on each fiber, then $\pi : E \to M$ is a C^∞ *vector bundle* of rank r.

Definition 11. *A C^∞* subbundle *of a C^∞ vector bundle $\pi : E \to M$ is a C^∞ vector bundle $\rho : F \to M$ such that*

- *F is a regular submanifold of E,*
- *the inclusion $i : F \to E$ is a bundle homomorphism, i.e. is fiber preserving.*

Definition 12. *A k*-frame *of a C^∞ vector bundle $\pi : E \to M$ over an open set $U \subset M$ is a collection of k sections $s_1, ...s_k$ of E over U such that at every $p \in U$, the vectors $s_1(p), ..., s_k(p)$ are linearly independent. An r-frame for a bundle of rank r is simply called a* frame.

Theorem 9. *Let $\pi : E \to M$ be a smooth vector bundle of rank r and $F := \coprod_{p \in M} F_p$ a subset of E such that for every $p \in M$, F_p is a k-dimensional vector subspace of E_p. If for every $p \in M$ there exists an open neighbourhood U of p and $m \geq k$ smooth sections $s_1, ..., s_m$ of E over U that span F_q at every point $q \in U$, then F is a smooth subbundle of E.*

Denote the vector space of smooth sections of E over U with $\Gamma(E,U)$. When $U = M$, we write $\Gamma(E)$. A *connection* on a vector bundle $\pi : E \to M$ is an $\mathbb{R}$-bilinear map $\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$, $(X,s) \mapsto \nabla_X s$, that is $C^\infty(M)$-linear in X and satisfies the *Leibniz rule*: if $f \in C^\infty(M)$, then $\nabla_X(fs) = (Xf)s + f\nabla_X s$.
This can be restricted on an open set. Let (U,e) be a framed open set, i.e. on open set together with a frame $e = (e_1, ..., e_r)$ for E over U. Then there is a collection of 1-forms ω^i_j on U called *connection forms* such that

$\nabla_X e_j = \omega^i_j(X) e_i$.

The curvature of a connection is a map $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E)$ given by $R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$. On a framed open set it can be represented by a collection of 2-forms called *curvature forms* Ω^i_j given by $R(X,Y)e_j = \Omega^i_j(X,Y)e_i$.

Appendix C. Vector valued forms and their products

Let V be a vector space and M a smooth manifold. A V-valued k-form on M is defined to be $\Omega^k(M,V) := \Gamma\left(\left(\bigwedge^k T^*M\right) \otimes V\right)$. This definition is very convenient because it can easily generalize to a k-forms with values in a vector bundle $E \to M$ as a section of a vector bundle $\left(\bigwedge^k T^*M\right) \otimes E$. This is not needed for our purposes and so if $v_1, ...v_n$ is a basis for V, we define a V-valued form α to be $\alpha = \alpha^j v_j$, for some real valued k-forms α^j. The exterior derivative $d\alpha$ is defined simply to be $d\alpha^j v_j$.

If V is a vector space of matrices, then ω^i_j and Ω^i_j from the previous section form a matrix valued 1-form ω_e and 2-form Ω_e called a connection and curvature matrix with respect to the frame e. A wedge product of two matrix valued forms A and B is a matrix valued form $C = A \wedge B$ with entries $C_{ij} = A_{ik} \wedge B_{kj}$.

Given a vector bundle $E \to M$, one can form a special kind of a principal $\mathrm{GL}(n,\mathbb{R})$-bundle called a *frame bundle* of E, whose local sections are frames for E. There is a preferred horizontal distribution on a frame bundle. When one forms an Ehresmann connection ω out of that distribution and pulls it back by e, one gets precisely a connection form ω_e. The curvature Ω of ω then pulls back under e to Ω_e. So, connection and curvature forms are a local manifestation of a singe global object.

Now, if V, W, Z are finite dimensional vector spaces and $\mu : V \times W \to Z$ is a bilinear map, a product of a V-valued and W-valued form α and β is defined by

$$\alpha \cdot \beta(X_1, ...X_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\mathrm{sgn}\sigma) \mu(\alpha(X_{\sigma(1)}, ..., X_{\sigma(k)}), \beta(X_{\sigma(k+1)}, ..., X_{\sigma(k+l)}))\,.$$

When $V = W = Z = \mathfrak{g}$ is a Lie algebra, we will always take μ to be the Lie bracket. We have the following important result:

Proposition 5. *Let $\{A_1, ...A_n\}$ be a set of vectors in a Lie algebra $\mathfrak{g}$ and let $\alpha \in \Omega^k(M,\mathfrak{g})$ and $\beta \in \Omega^l(M,\mathfrak{g})$ be written as $\alpha = \alpha^i A_i$ and $\beta = \beta^j A_j$. Then*

$$[\alpha, \beta] = (\alpha^i \wedge \beta^j)[A_i, A_j]\,. \tag{C.1}$$