Black hole chemistry: thermodynamics with Lambda

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The purpose of this article is to provide a convenient overview of content needed to fully understand derivation of the first law of black hole thermodynamics in the extended phase space embedded in the context of the usual first law derivation. Here by the extended phase space we mean including cosmological constant Λ as a thermodynamic variable equivalent to pressure which makes black hole mass no longer equivalent to internal energy, rather equivalent to enthalpy. We will also recover the actual result basing our analysis on derivation given in the article by David Kastor, Sourya Ray and Jennie Traschen[11] and recapitulated in the Appendix A of the article by David Kubizňák, Robert B. Mann and Mae Teo [12].

I. INTRODUCTION

Pondering upon the idea of relationship between gravitation, thermodynamics and quantum theory, the discipline of black hole thermodynamics arose. Inception of that idea may be traced to the early 1970s firstly with Hawking formulating the concept of a black hole's absolute horizon and proving that the surface areas of absolute horizons always increases. Such law, as Bekenstein noticed, closely resembled to the the second law of thermodynamics - the area theorem becomes the second law of thermodynamics if one merely replace the phrase 'horizon area' by the 'entropy'. Nevertheless, important fact was that Bekenstein didn't think of it as it was mere coincidence, he interpreted it as the equivalence between the notion of entropy and black hole horizon area. Lead by that idea Bekenstein calculated the proportionality of black hole entropy and black hole horizon area which lead him to the notion of so called characteristic temperature but noting: '...we emphasize that one should not regard T_{BH} as the temperature of the black hole; such an identification lead to all sorts of paradoxes, and is thus not useful.'[23] The idea that the laws of black hole mechanics are in more than analogous relation to the laws of thermodynamics (and that the notion of T_{BH} is not really paradoxical) wasn't established until Hawking proved that black holes radiate as though they actually had temperature proportional to their surface gravity, in a way similar to black body radiation. From then, black hole thermodynamics established itself as a valid discipline intending to investigate relationship between gravity and quantum physics - as thermal properties of ordinary system reflect the statistical mechanics of underlying microstates, it is to be questioned does black hole thermodynamics tells us something about the underlying quantum gravitational states.
Black hole chemistry extends on the idea of thermodynamics in the way that it actually extends the phase space to accommodate the cosmological constant as a thermodynamical variable. Needless to say, to explore black hole chemistry, we shall first explore some of the relations of black hole mechanics on which we will base our later formulation and argumentation. Also, in the appendix we provide recollection of crucial notions of general relativity and differential geometry. The reader is encouraged to refer to those, or to the references if any mentioned notion is unfamiliar.
Technical note For simplicity we will assume $\Lambda = 0$ in all derivations until chapter VII where non-vanishing Λ will be recovered by substituing $R \to R - 2\Lambda$.

II. BLACK HOLE THERMODYNAMICS

The black hole thormodynamics was officially established by Bardeen, Carter, and Hawking[25] under the assumption that the event horizon is a null hypersurface generated by a corresponding Killing vector field, thus associating the the four laws of black hole mechanics with the four laws of thermodynamics. Although in this paper we will focus our attention only on the first law, here we state all of them for completeness:
0th *The surface gravity κ is constant over the event horizon of a stationary black hole.*
This yields identification of surface gravity with the temperature of the black hole radiation. It can be shown[26] that the exact relation is:

$$T = \frac{\hbar}{2\pi}\kappa \tag{1}$$

1st *For a black hole with a mass M*

$$\delta M = \frac{\kappa}{8\pi G}\delta A$$

where κ is its surface gravity. As we said, surface gravity is to be identified with the temperature of a black hole radiation ($\kappa \sim T$), while area of the event horizon is now associated with the entropy ($A \sim S$) and the mass of the black hole with internal energy ($M \sim E_{int}$).
2nd *The area A of a black hole's event horizon can never decrease.*
This is Hawking's area theorem[24]:

$$\delta A \geqslant 0$$

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3rd *It is impossible to reduce the surface gravity κ to zero in a finite number of steps.*

III. LAGRANGIAN FORMULATION

Let us recall briefly Lagrangian formulation of a field theory. Let ψ denote some tensorial field and let ψ_λ be a smooth one-parameter family of field configurations starting from ψ_0 which satisfy appropriate boundary conditions. We will denote by $\delta\psi$ derivation of ψ_λ along parameter lambda:

$$\delta\psi \equiv \left.\frac{d\psi_\lambda}{d\lambda}\right|_{\lambda=0} \tag{2}$$

Suppose that there exists a smooth tensor field χ dual to ψ. We say that S is functionally differentiable at ψ_0 if for all one parameter families we have

$$\frac{dS}{d\lambda} = \int_M \chi\delta\psi \tag{3}$$

We call χ the functional derivative of S and denote it as

$$\chi = \left.\frac{\delta S}{\delta\psi}\right|_{\psi_0} \tag{4}$$

Now consider a functional S of the form

$$S[\psi] = \int_M \mathcal{L}[\psi] \tag{5}$$

where $\mathcal{L}$ is a local function of ψ and a finite number of its derivatives:

$$\mathcal{L}|_x = \mathcal{L}\left(\psi(x), \nabla\psi(x), \ldots, \nabla^k\psi(x)\right) \tag{6}$$

S is called an *action* and $\mathcal{L}$ is called a *Lagrangian density* if it is functionally differentiable and if the field configurations ψ which extremize S

$$\left.\frac{\delta S}{\delta\psi}\right|_{\psi} = 0 \tag{7}$$

are the ones which are solutions of the field equation for ψ.

Dynamical equations for ψ are obtained by introducing a variation that is arbitrary within V but vanishes everywhere on ∂V if the variation is about the actual path $q\left(x^\alpha\right)$:

$$\delta\psi|_{\partial V} = 0 \tag{8}$$

Upon variation we get:

$$\begin{aligned}
\delta S &= \int_V \left(\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\delta\partial_\alpha\psi\right)\sqrt{-g}\mathrm{d}^4x \\
&= \int_V \left((\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \nabla_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\delta\psi\right)\right. \\
&- \nabla_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\right)\delta\psi\sqrt{-g}\mathrm{d}^4x \\
&= \int_V \left((\frac{\partial\mathcal{L}}{\partial\psi} - \nabla_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\right)\right)\delta\psi\sqrt{-g}\mathrm{d}^4x \\
&+ \oint_{\partial V}\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)}\delta\psi\mathrm{d}\Sigma_\alpha
\end{aligned}$$

The surface integral will vanish upon using relation (8) giving:

$$\delta S = 0 \Rightarrow \nabla_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \tag{9}$$

We arrived at the Euler-Lagrange equation which determine the field equations. This can be generalized to higher order derivative dependence in $\mathcal{L}$.

In general relativity, the field variable is the spacetime metric g_{ab} defined on a four-dimensional manifold M. The integral of a continuous n -form field ϕ over an n -dimensional orientable manifold (with respect to the orientation ϵ) is given by:

$$I = \int \phi(x)\sqrt{|g|}d^nx \tag{10}$$

where

$$\epsilon = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n \equiv \sqrt{|g|}d^nx \tag{11}$$

is invariant volume element IX.5.
The volume element itself depends on the field variable, and hence its variation must be taken into account when calculating functional derivatives. One choice would be to incorporate the volume element into $\mathcal{L}$ but this would mean that $\mathcal{L}$ would become a totally antisymmetric four-index tensor rather than a scalar. Since the field variable in this case is scalar, we would like Lagrangian to keep its scalar property in order to ensure local Lorentz invariance (no preferred frame of reference). Furthermore, one should have in mind that first order partial derivative of metric tensor is not tensorial quantity and we cannot expect $\mathcal{L}$ depending only on zeroth and first derivative of metric to end up being a scalar object. In order to keep its scalar property, $\mathcal{L}$ has to depend on second derivative of metric tensor also. We define the action functional for general relativity by:

$$S[g^{ab}] = \int_{\mathcal{V}} \mathcal{L}(g^{ab}(x), \partial_c g^{ab}(x), \partial_c\partial_d g^{ab}(x))\sqrt{|g|}d^nx \tag{12}$$

where V is finite 4D volume and δV its 3D boundary. We will assume that the boundary is simply connected domain. One could argue that the simplest scalar object in which dependence on zeroth, first and second order partial derivative of metric tensor would be packed is Ricci scalar R, yielding Lagrangian density:

$$\mathcal{L} = \sqrt{-g}R \tag{13}$$

R is quantity which describes curvature and is defined by contractions of Riemann curvature tensor (for definition see Appendix, section IX.5):

$$\begin{aligned} R_{ab} &\equiv g^{cd}R_{acbd} \\ R &\equiv g^{ab}R_{ab} \end{aligned} \tag{14}$$

where R_{ab} represents Ricci tensor. The action of GR is:

$$S_G[g] = S_H[g] + S_B[g] - S_0 \tag{15}$$

where we have:

$$\begin{aligned} S_H[g] &= \frac{1}{16\pi}\int_{\mathcal{V}} R\sqrt{-g}\mathrm{d}^4x \\ S_B[g] &= \frac{1}{8\pi}\oint_{\partial V} \varepsilon K|h|^{1/2}\mathrm{d}^3y \\ S_0 &= \frac{1}{8\pi}\oint_{\partial V} \varepsilon K_0|h|^{1/2}\mathrm{d}^3y \end{aligned}$$

Here ε is equal to +1 where ∂V is timelike and -1 where ∂V is spacelike. Boundary is assumed to be nowhere null (or at least null on the set of measure zero). h denotes the determinant of the induced metric on the boundary ∂V and K is trace of the extrinsic curvature of the boundary (see (30)).

In the relation (15) we have $S_H[g]$ the Hilbert term (also known as Einstein-Hilbert action) which is the term associated with the bulk volume. It generates side of Einstein equation containing G_{ab} tensor. Then $S_B[g]$ the boundary term and S_0 will be the gravitational action of the flat spacetime. The difference $S_B - S_0$ is then well defined in the limit $r \to \infty$, where r is spatial radial coordinate, yielding a well defined gravitational action for asymptotically flat spacetimes ($S_G \to 0$). K_0 is extrinsic curvature of δV embedded in flat spacetime. Notice that the boundary term was vanishing in case of Lagrangian depending only on the zeroth and first order partial derivative of the field ψ while here it is necessary in order to make variation vanishing at the boundary, as equation (8) suggests.

IV. HAMILTONIAN FORMULATION OF GENERAL RELATIVITY

Recall that for field theory in the flat spacetime we have Lagrangian density which depends on field and its derivatives $\mathcal{L}(\psi, \delta_\alpha\psi)$ and from that we calculate the associated canonical momentum $\pi = \frac{\delta\mathcal{L}}{\delta_t\psi}$. At the level

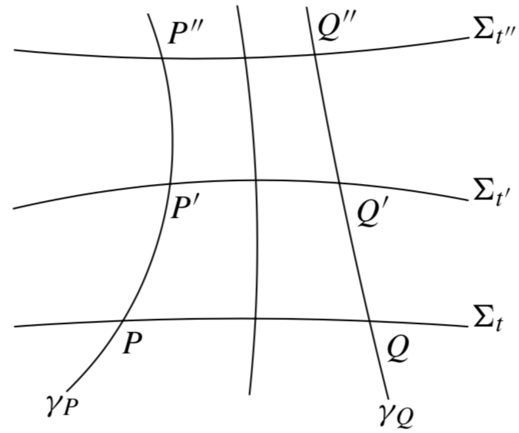

Figure 1. Foliation of spacetime by spacelike hypersurfaces [1]

of Lagrangian density, we still had manifest Lorentz invariance but introducing time derivative in canonical momentum forces us to pick a Lorentz frame. So at the level of Hamiltonian density $\mathcal{H} = \pi\delta_t\psi - \mathcal{L}$ we lost manifest Lorentz invariance. To generalise this to curved spacetime we have to consider more sophisticated approach, such that would not jeopardize general covariance.
Notice that in the Hamiltonian approach to field theory in flat spacetime we were really foliating Minkowski spacetime in terms of stacks of t=const flat surfaces. Considering time derivative really meant comparison of field configuration in successive slices i.e. considering how the field changes when we move from one hypersurface to another. In curved spacetime we want to do it in a similar manner but in a way which wouldn't tie us to a specific coordinate system.

IV.1. (3+1) decomposition

Let us consider foliation (refer to Figure 1 and Figure 7 for visual insight) of spacetime with arbitrary hypersurfaces - only condition constraining that arbitrariness is that hypersurfaces should not intersect, as this would produce ill defined coordinate system (for this reason we are restricting our analysis to a finite patch). Let coordinate system of our 4D space be some x^α. We introduce *time function* $t(x^\alpha)$ scalar field such that t=const. on each hypersurface Σ_t and such that it is unambiguously defined. Though it may not correspond to time in our coordinate system of course (as our choice of coordinate system is still arbitrary), this is gives us notion of time in a sense that it is monotonically increasing function tied to our foliation, constant as in Minkowski space on each hypersurface. The normal on the hypersuface $n_\alpha \sim \partial_\alpha t$ with appropriate norm $n_\alpha n^\alpha = -1$.

On each hypersurface we introduce a coordinate system y^a (see Figure1). In principle, we can choose on each hypersurface coordinate system independently but what we actually want is to propagate coordinate system picked on one hypersurface. For that purpose we introduce congruence of curves γ (family of non-intersectig curves) parameterized by t, intersecting our foliation so that each curve is curve of constant y^a. The curves are not assumed to be orthogonal to foliation, though they may chosen to be. Let t^α denote vector field tangent to the curves. If we look at the displacement along the curve

$$dx_\alpha = t^\alpha dt \tag{16}$$

and the change in t

$$dt = \frac{\partial t}{\partial x^\alpha} dx^\alpha = \left(\frac{\partial t}{\partial x^\alpha} t^\alpha\right) dt \tag{17}$$

$$\Rightarrow t^\alpha \partial_\alpha t = 1 \tag{18}$$

Note that this construction gives us an alternative choice of 4D coordinate system (t, y^α). In general there will be some relation $x^\alpha = x^\alpha((t, y^\alpha))$ yielding a system of parametric relations for curves γ. Tangent vector field to the congruence is given by:

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_{y_a} \tag{19}$$

and tangent to the hypersurface (coresponding to the displacement on each Σ_t)

$$e^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t \tag{20}$$

Since:

$$\frac{\partial^2 x^\alpha}{\partial y^a \partial t} = 0 \Rightarrow \mathfrak{L}_t e^\alpha_a = 0 \tag{21}$$

where is Lie derivative

$$\mathfrak{L}_t e^\alpha_a = t^\mu \nabla_\mu e^\alpha_a - e^\mu_a \nabla_\mu t^\alpha \tag{22}$$

So t^α is Lie transported along the tangent vectors of each hypersurface and vice versa, e^α_a will be Lie transported along the vector field tangent to the congruence of curves. Since congruence is not orthogonal to the hypersurface we will have:

$$n_\alpha = -N \partial_\alpha t \tag{23}$$

Notice that since $n_\alpha e^\alpha_a = 0$ we have good choice of coordinate system in terms of (n^α, e^α_a). N is a *lapse function* - it is normal component of the flow vector in the basis (n^α, e^α_a). In a sense N is measure of proper distance between two hypersurfaces. It will be useful to decompose t^α in terms of those as pictured on Figure 2:

$$t^\alpha = N n^\alpha + N^a e^\alpha_a \tag{24}$$

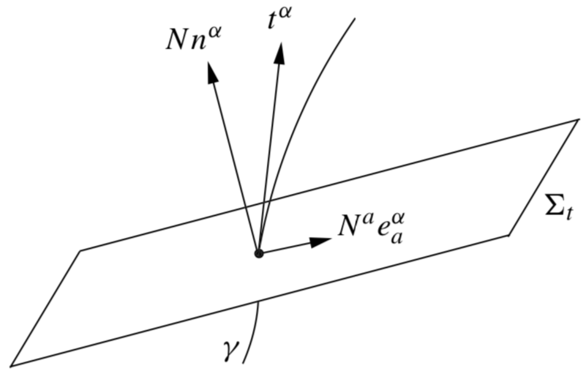

Figure 2. Decomposition of tangent congruence vectors t^α in terms of lapse and shift [1]

Here N^a represents *shift function*, tangent to the hypersurface. To finalise our discussion on (3+1) decomposition we will find the metric in the (t, y^a) coordinate system following the decomposition of the displacement dx^α in terms of components along the congruence of curves and components along the hypersuface:

$$\begin{aligned} dx^\alpha &= t^\alpha dt + e^\alpha_a dy^a \\ &= (N n^\alpha + N^a e^\alpha_a) dt + e^\alpha_a dy^a \\ &= N dt n^\alpha + e^\alpha_a (dy^a + N^a dt) \end{aligned}$$

$$ds^2 = g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta$$
$$= g_{\alpha\beta}[N dt n^\alpha + e^\alpha_a (dy^a + N^a dt)][N dt n^\beta + e^\beta_b (dy^b + N^b dt)]$$
$$\Rightarrow$$

$$ds^2 = -N^2 \mathrm{d}t^2 + h_{ab} \left(\mathrm{d}y^a + N^a \mathrm{d}t\right)\left(\mathrm{d}y^b + N^b \mathrm{d}t\right) \tag{25}$$

with projection of the spacetime metric on each hypersurface:

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b \tag{26}$$

Relation (25) represents (3+1) decomposition. In terms of this discussion the 4D volume element $\sqrt{-g}$ takes the simple factorised form:

$$\sqrt{-g} = N\sqrt{h} \tag{27}$$

IV.2. Calculating the Hamiltonian

For purposes of our discussion we will restrict our attention on the gravitational part of the action, the part yielding the Einstein tensor $G_{\alpha\beta}$ in the Einstein equations. To work out the calculation we will first have to invoke some useful relations in *Technical note.*

Technical note The Einstein equation for the gravitational field of a matter-energy configuration described by the energy-momentum tensor T_{ab} is given by:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} \tag{28}$$

where R_{ab} represents Ricci tensor and R Ricci scalar. R_{ab} and R describe the curvature, as they are just contractions of the Riemannian curvature tensor as we will see bellow. From this it will be useful to express Ricci scalar:

$$\begin{aligned} -Rg_{ab}n^a n^b &= 2\left(G_{ab}n^a n^b - R_{ab}n^a n^b\right) \\ -Rn_b n^b &= 2\left(G_{ab}n^a n^b - R_{ab}n^a n^b\right) \\ R &= 2\left(G_{ab}n^a n^b - R_{ab}n^a n^b\right) \end{aligned} \tag{29}$$

We will be working in terms of foliation, meaning in terms of family of hypersurfaces Σ and associated normals. We will have the relations that are defined only on Σ and which are purely tangent to the hypersurface. To accomplish that we will pullback tensors defined on the whole spacetime (in other words, we will project it on the base space of Σ defined by the coordinate system e^α_a). We introduce the notation: $A_{a|b} = \nabla_\beta A_\alpha e^\alpha_a e^\beta_b$ for covariant derivative along the hypersurface.
The extrinsic curvature is defined as the gradient of the normal field of the hypersurfaces:

$$K_{ab} = \nabla_\alpha n_\beta e^\alpha_a e^\beta_b \tag{30}$$

We also introduce the Gauss-Codazzi equation:

$$R_{\alpha\beta\gamma\delta} e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d = R_{abcd} + \varepsilon\left(K_{ad}K_{bc} - K_{ac}K_{bd}\right) \tag{31}$$

to have a relation between R_{abcd} and the full Riemann tensor. Or alternatively:

$$R_{\mu\alpha\beta\gamma} n^\mu e^\alpha_a e^\beta_b e^\gamma_c = K_{ab|c} - K_{ac|b} \tag{32}$$

The spacetime Ricci tensor is given by

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu v} R_{\mu\alpha v\beta} \\ &= \left(\varepsilon n^\mu n^\nu + h^{mn} e^\mu_m e^v_n\right) R_{\mu\alpha v\beta} \\ &= \varepsilon R_{\mu\alpha v\beta} n^\mu n^v + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\nu_n \end{aligned} \tag{33}$$

and the Ricci scalar is

$$\begin{aligned} R &= g^{\alpha\beta} R_{\alpha\beta} \\ &= \left(\varepsilon n^\alpha n^\beta + h^{ab} e^\alpha_a e^\beta_b\right)\left(\varepsilon R_{\mu\alpha v\beta} n^\mu n^\nu + h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^v_n\right) \\ &= 2\varepsilon h^{ab} R_{\mu\alpha v\beta} n^\mu e^\alpha_a n^v e^\beta_b + h^{ab} h^{mn} R_{\mu\alpha\nu\beta} e^\mu_m e^\alpha_a e^v_n e^\beta_b \end{aligned} \tag{34}$$

where we used the Gauss-Codazzi equation. From this we have:

$$G_{ab} n^a n^b = \frac{1}{2}\left[{}^{(3)}R - K_{ab}K^{ab} + K^2\right] \tag{35}$$

where K_{ab} is the extrinsic curvature of Σ_t and K it's trace.

The intention is to express everything in terms of extrinsic curvature K_{ab} because it will be easier to do the variation of Lagrangian with respect to it. So let us also rewrite the Ricci tensor R_{ab}:

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ R_{ab}n^a n^b &= R^c{}_{acb} n^b n^a \\ &= -\left(\nabla_a \nabla_c - \nabla_c \nabla_a\right) n^c n^a \\ &= \left(\nabla_a n^a\right)\left(\nabla_c n^c\right) - \nabla_a\left(n^a \nabla_c n^c\right) \\ &\quad - \left(\nabla_c n^a\right)\left(\nabla_a n^c\right) + \nabla_c\left(n^a \nabla_a n^c\right) \\ &= K^2 - K_{ac}K^{ac} - \nabla_a\left(n^a \nabla_c n^c\right) + \nabla_c\left(n^a \nabla_a n^c\right) \end{aligned}$$

Last two terms are divergences and can be neglected for the purposes of variation. This then yields:

$$R_{ab} n^a n^b = K^2 - K_{ac}K^{ac} \tag{36}$$

Now we are ready to go back to the Hamiltonian calculation. As we said, we will concentrate on the gravitational term of the Lagrangian:

$$\begin{aligned} \mathcal{L} &= \sqrt{-g} R \\ &= N\sqrt{h} R \\ &= 2N\sqrt{h}\left(G_{ab}n^a n^b - R_{ab}n^a n^a\right) \\ &= 2N\sqrt{h}\left(\frac{1}{2}\left[{}^{(3)}R - K_{ab}K^{ab} + K^2\right] - K^2 - K_{ab}K^{ab}\right) \\ \mathcal{L} &= N\sqrt{h}\left({}^{(3)}R + K_{ab}K^{ab} - K^2\right) \end{aligned} \tag{37}$$

where we used relation (27), (29), (35) and (36). Hamiltonian density is defined by:

$$\mathcal{H} = \pi^{ab} \dot{h}_{ab} - \sqrt{-g}\mathcal{L}_{bulk}\left(q_i, \dot{q}_i\right) \tag{38}$$

where π^{ab} is canonical momentum as canonically conjugate variable of $\dot{h}_{ab}$. Let us evaluate:

$$\begin{aligned} \pi^{ab} &= \frac{\partial\sqrt{-g}\mathcal{L}}{\partial \dot{h}_{ab}} \\ &= \frac{\partial K_{mn}}{\partial \dot{h}_{ab}} \frac{\partial}{\partial K_{mn}}\left(16\pi\sqrt{-g}\mathcal{L}\right) \\ &= \sqrt{h} N\left[\frac{\partial^{(3)}R}{\partial \dot{h}_{ab}} + \frac{\partial\left(K_{ab}K^{ab}\right)}{\partial \dot{h}_{ab}} - \frac{\partial K^2}{\partial \dot{h}_{ab}}\right] \\ &= \sqrt{h}\left(K^{ab} - h^{ab}K\right) \end{aligned} \tag{39}$$

Recall that the purpose of (3+1) decomposition was to keep Lorentz invariance at the level of Hamiltonian. We provide that by promoting $\partial_t\psi \to \mathfrak{L}_t\psi$ where L_t is Lie derivative along the time flow. This gives us:

$$\dot{h}_{ab} \equiv \mathfrak{L}_t h_{ab} \tag{40}$$

Substituting relation (26) we get:

$$\dot{h}_{ab} = \mathfrak{L}_t\left(g_{\alpha\beta} e^\alpha_a e^\beta_b\right) = \left(\mathfrak{L}_t g_{\alpha\beta}\right) e^\alpha_a e^\beta_b \tag{41}$$

We also have Lie derivative of the metric along t^α

$$
\begin{aligned}
\mathfrak{L}_t g_{\alpha\beta} &= \nabla_\beta t_\alpha + \nabla_\alpha t_\beta \\
&= \nabla_\beta \left(N n_\alpha + N_\alpha\right) + \nabla_\alpha \left(N n_\beta + N_\beta\right) \\
&= n_\alpha \partial_\beta N + n_\beta \partial_\alpha N \\
&+ N \left(\nabla_\beta n_\alpha + \nabla_\alpha n_\beta\right) + \nabla_\beta N_\alpha + \nabla_\alpha N_\beta
\end{aligned}
$$

where $N^\alpha = N^a e^\alpha_a$. Projecting this along $e^\alpha_a e^\beta_b$ gives

$$
\dot{h}_{ab} = 2NK_{ab} + N_{a|b} + N_{b|a} \tag{42}
$$

where we used the notation for the intrinsic curvature derivative of vector within the hypersurface:

$$
N_{a|b} = \nabla_\beta N_\alpha e^\alpha_a e^\beta_b \tag{43}
$$

This yields:

$$
K_{ab} = \frac{1}{2N}\left(\dot{h}_{ab} - N_{a|b} - N_{b|a}\right) \tag{44}
$$

From (39) we have:

$$
\sqrt{h} K^{ab} = \left(\pi^{ab} - \frac{1}{2}\pi h^{ab}\right) \tag{45}
$$

where $\pi \equiv h_{ab}\pi^{ab}$ is the trace.

Hamiltonian density is then given by:

$$
\begin{aligned}
\mathcal{H} &= \pi^{ab}\dot{h}_{ab} - \mathcal{L}\left(q_i, \dot{q}_i\right) \\
&= -\sqrt{h} N {}^{(3)}R + \frac{N}{\sqrt{h}}\left[\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right] + 2\pi^{ab} D_a N^a \\
&= \sqrt{h}\left[N\left[-{}^{(3)}R + h^{-1}\pi^{ab}\pi_{ab} - \frac{1}{2}h^{-1}\pi^2\right]\right. \\
&\left. - 2N^a\left[D_a\left(h^{-1/2}\pi^{ab}\right)\right] + 2D_a\left(h^{-1/2}N^a\pi^{ab}\right)\right. \\
&= \sqrt{h}\left[N\left[-{}^{(3)}R + h^{-1}\pi^{ab}\pi_{ab} - \frac{1}{2}h^{-1}\pi^2\right]\right. \\
&\left. - 2N^a\left[D_a\left(h^{-1/2}\pi^{ab}\right)\right]\right]
\end{aligned} \tag{46}
$$

where we introduced D_a the covariant derivative operator with respect to h_{ab} on Σ in stead of vertical bar notation to get expression as in[11,12]. We write gravitational Hamiltonian more compactly: $\mathcal{H} = NH + N^a H_a$, where

$$
\begin{aligned}
H &= -R^{(d-1)} + \frac{1}{|h|}\left(\frac{\pi^2}{d-2} - \pi^{ab}\pi_{ab}\right) \\
H_b &= -2D_a\left(|h|^{-\frac{1}{2}}\pi^{ab}\right)
\end{aligned} \tag{47}
$$

Notice that hamiltonian depends on the laps and shift meaning it depends on the choice of flow (t^αs of our congruence of curves). It depends on how we choose to slice our spacetime (choice of foliation Σ_t) and on how we choose to pick our boundary (we tend to push the boundary all the way to infinity where we constraint hypersurfaces to asymptotically approach hypersurfaces of some space e.g. flat space or AdS etc though we may choose to have two component boundary as we will in our discussion (cf.[22])).

Note on picking the choice of foliation

Let us considere asymptotically flat spacetime and let us constraint our hypersurfaces to approach surface of constant time in Minkowski spacetime we would. Notice that we still have freedom when it comes to flow choice, meaning we still have the freedom to pick any N and N^a in $t^\alpha = Nn^\alpha + N^a e^\alpha_a$. But let's make a chocie, let's pick $N = 1$ and $N^a = 0$. Notice that this will give us $t^\alpha = n^\alpha$ which will point in the direction of timelike constant vector in asymptotic Minkowski space. This means that we have picked the flow to be *asymptotic time translation*. In this way evaluated hamiltonian is associated with the notion of mass for the total spacetime. If we pick a flow to be constraint with $N = 0$ and $N^a = 1$ we will get *asymptotic spatial translation*. Hamiltonian evaluated in this case would be total linear momentum. If we pick $N = 0$ and $N_\alpha = \phi_a = \frac{y_a}{\bar{\phi}}$ where $\bar{\phi}$ is *rotational coordinate in asymptotically flat spacetime*, we would get angular momentum of our spacetime.

Note on the initial value problem

To solve equations of motion we need to impose the initial values for the metric tensor and its derivative. First step is making an arbitrary choice of a spacelike hypersurface and choose one of them as the surface on which we will specify initial data. We define some coordinate system on hypesurface and decompose the specatime metric $g_{\alpha\beta}$ in terms of components along the hypersurface and components that characterize displacements away from the hypersurface. Initial values for the spacetime metric is given by intrinsic metric h_{ab}. Given that, it is to be noticed that this implements the fact that the choice of coordinates is arbitrary as the initial values for g_{ab} can only be the six components of the induced metric h_{ab} leaving the remaining four components arbitrary.

Next we need to fix the initial values for the time derivative of the metric. It is described by extrinsic curvature tensor K_{ab} which can be seen from the fact that it K_{ab} carries information about the derivative of the metric in the direction normal (i.e. time derivative) to the hypersurface.

Therefore the initial-value problem consist in specifying two symmetric tensor fields, h_{ab} and K_{ab}, on a spacelike hypersurface Σ. But h_{ab} and K_{ab} are not arbitrary, they are related to G_{ab} and curvature tensor in way we have seen in a technical note above, i.e. there exist constraint equations fixing our initial value tensor fields. Presence of such constraints in the Hamiltonian formulation is a characteristic feature of gauge theories or generally covariant theories.

IV.3. Stationary and axially symmetric spacetimes (Komar formulae and pertubation)

As we said in *Note on picking the choice of foliation*, Hamiltonian evaluated for the flow picked to be associated with asymptotic time translation gives us the notion of mass for the total spacetime and a flow associated with asymptotic rotational translation gives us angular momentum of the total spacetime. Specifically, for stationary and axially symmetric spacetimes we have formulae know as Komar formulae:

$$M = -\frac{1}{8\pi}\lim_{S_t\to\infty}\oint_{S_t}\nabla^\alpha \xi^\beta_{(t)}\mathrm{d}S_{\alpha\beta} \tag{48}$$

and

$$J = \frac{1}{16\pi}\lim_{S_t\to\infty}\oint_{S_t}\nabla^\alpha \xi^\beta_{(\phi)}\mathrm{d}S_{\alpha\beta} \tag{49}$$

where we denoted Killing vectors with $\xi_{(t)}$ and $\xi_{(\phi)}$ and the surface element is given by

$$\mathrm{d}S_{\alpha\beta} = -2n_{[\alpha}r_{\beta]}\sqrt{\sigma}\mathrm{d}^2\theta \tag{50}$$

where n_α and r_α are the timelike and spacelike normals to S_t respectively. Surface S_t is $2D$ surface, boundary of hypersurface Σ. Our intrinsic coordinate system defined on a S_t is $y^a = (\lambda, \Theta^A)$, in accordance with the discussion in IX.1. Note that Hamiltonian definitions for mass and angular momentum do not involve a specific choice of coordinates. We can use Stokes' theorem to get hypersurface integrals:

$$M = 2\int_\Sigma\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right)n^\alpha\xi^\beta_{(t)}\sqrt{h}\mathrm{d}^3y \tag{51}$$

$$J = -\int_\Sigma\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right)n^\alpha\xi^\beta_{(\phi)}\sqrt{h}\mathrm{d}^3y \tag{52}$$

Those relations allows us to define mass density and angular momentum density for spacetime via energy momentum tensor (remember that $\sqrt{h}\mathrm{d}^3y$ is invariant volume element so densities are the rest of the integrand).

Let us consider test energy momentum tensor representing a flow of matter (along the same lines with classical procedure where we invoke test mass or charge) meaning $T^{\alpha\beta}$ very small such that it does not participate in building up curvature in our spacetime. From this we are interested in calculating transfer of mass and angular momentum across a hypersurface in a stationary, axially symmetric spacetime where we have $\xi_{(t)}$ and $\xi_{(\phi)}$ as Killing vectors.

We will not provide here entire derivation of expressions but let us try to motivate them a bit in purpose of better understanding. Consider a perfect fluid with stressenergy tensor $T^{\alpha\beta} = \rho u^\alpha u^\beta$ (i.e. dust model). Here ρ is the rest mass density and u^α the 4-velocity. Energy-momentum conservation we can get:

$$0 = \nabla_\beta T^{\alpha\beta} = \rho((\nabla_\beta u^\alpha)u^\beta + u^\alpha(\nabla_\beta u^\beta)) \tag{53}$$

Notice that here we have $a^\alpha = u^\beta\nabla_\beta u^\alpha$ acceleration in the first term and velocity u^α in the second. Since velocity and acceleration are always orthogonal to each other, both terms in relation (53) must be zero in order to yield zero:

$$a^\alpha = 0 \implies u^\alpha \text{ satisfies the geodesic equation}$$

and that

$$j^\alpha = \rho u^\alpha \text{ is a conserved vector:}$$

$$\nabla_\alpha j^\alpha = 0 \tag{54}$$

This vector can be interpreted as the flux of rest mass for the fluid. To get the energy carried by the fluid element we have to consider a conserved quantity that is going to be carried by the geodesic motion of the fluid element. That will involve Killing vectors: $\tilde{E} \equiv -u_\alpha\xi^\alpha_{(t)}$ conserved energy per unit rest mass and $\tilde{L} \equiv u_\alpha\xi^\alpha_{(\phi)}$ the conserved angular momentum per unit rest mass (both $\tilde{E}$ and $\tilde{L}$ are constants of the motion). Then $\varepsilon^\alpha = \tilde{E}j^\alpha$ represents a flux of energy density, while $\ell^\alpha = \tilde{L}j^\alpha$ is a flux of angular-momentum density. Notice that

$$\varepsilon^\alpha = \tilde{E}j^\beta = -u_\alpha\xi^\beta_{(t)}j^\beta = -T^\alpha_\beta\xi^\beta_{(t)}$$

similarly

$$\ell^\alpha = \tilde{L}j^\beta = u_\alpha\xi^\beta_{(\phi)}j^\beta = T^\alpha_\beta\xi^\beta_{(\phi)}$$

Those vectors are divergence-free. For example

$$\nabla_\alpha\varepsilon^\alpha = -\nabla_\alpha T^\alpha_\beta\xi^\beta_{(t)} - T^\alpha_\beta\nabla_\alpha\xi^\beta_{(t)} = 0$$

the first term vanishes by virtue of energy-momentum conservation, and the second vanishes because $\nabla_\alpha\xi^\beta_{(t)}$ is an antisymmetric tensor field while T^α_β is symmetric. This implies that the integral of ε^α or ℓ^α over a hypersurface ∂V is identically zero:

$$\oint_{\partial V}\varepsilon^\alpha\mathrm{d}\Sigma_\alpha = 0 \tag{55}$$

meaning that the total transfer of energy across a closed hypersurface ∂V is zero. This is clearly a statement of conservation of total energy (i.e. total mass). We can now claim that the transfer of mass energy across hypersurface Σ_α is:

$$\delta M = \int_\Sigma\varepsilon^\alpha\mathrm{d}\Sigma_\alpha = \int_\Sigma -T^\alpha_\beta\xi^\beta_{(t)}\mathrm{d}\Sigma_\alpha \tag{56}$$

where we integrate across a piece of ∂V denoted by Σ.

$$\delta J = \int_\Sigma\ell^\alpha\mathrm{d}\Sigma_\alpha = \int_\Sigma T^\alpha_\beta\xi^\beta_{(\phi)}\mathrm{d}\Sigma_\alpha \tag{57}$$

The generalisation of Komar formulae to the D dimensions is given by:

$$\frac{D-2}{8\pi G}\int_{\partial\Sigma} \mathrm{d}S_{ab}\nabla^a\xi^b = 0 \tag{58}$$

This statement is proved by rewriting the volume integral using Gauss' law for $A^c = \nabla_b B^{bc}$ as

$$\int_\Sigma \mathrm{d}v n_c A^c = \int_{\partial\Sigma_\infty} \mathrm{d}a r_b n_c B^{bc} - \int_{\partial\Sigma_h} \mathrm{d}a r_b n_c B^{bc} \tag{59}$$

where n_a is the unit normal to Σ (future pointing) and r_b is the unit normal to $\partial\Sigma$ within Σ taken to point toward infinity. We have for the surface volume element $dS_{bc} = 2dar_{[b}n_{c]}$. Invoking identity for Killing vector:

$$\nabla_a\nabla^a\xi^b = -R^b_c\xi^c$$

and noting that the resulting volume integrand then vanishes by the vacuum Einstein equations $R_{ab} = 0$ gives relation (59). Note that we here picked the boundary of Σ to have two components, an inner boundary at the horizon and an outer boundary at infinity. We could have picked it in different manner but we are choosing here what is useful for the context of this discussion[11] (cf.[22]).

V. STATIONARY BLACK HOLE

The case of stationary balck holes arises when we have time independent metric (in a suitable coordinates) and time independent stress energy tensor. More generally, the spacetime is said to be stationary if there exists a timelike Killing vector field. Static black holes are more restrictive category where we demand that the metric is not only time independent but that there exist time reversal invariance which yields a restriction on Killing vector to be a hypersurface orthogonal. By the theorem of Stephen Hawking[24], stationary black hole is either static (non rotating) or axially symmetric (rotating). In both cases we will have two Killing vectors: $\xi_{(t)}$ and $\xi_{(\phi)}$ (though $\xi_{(\phi)}$ is not that important in static case of course). Hawking also showed there is going to be a linear combination of a Killing vectors such that it is null on the event horizon:

$$\xi^\alpha_{(t)} + \xi^\alpha_{(\phi)}\Omega_H \equiv \xi^\alpha_{(\Theta)} \tag{60}$$

A linear combination of two Killing vectors with a constant coefficients is again a Killing vector. Here Ω_H is the black hole's angular velocity (in the static case $\Omega_H = 0$ but in the stationary case it nonzero).
Since $\xi^\alpha_{(\Theta)}$ is null recall that it will be both tangent and orthogonal. As a consequence $\xi^\alpha_{(\Theta)}$ will satisfy geodesic equation on event horizon:

$$\xi^\beta_{(\Theta)}\nabla_\beta\xi^\alpha_{(\Theta)} = \kappa\xi^\alpha_{(\Theta)} \tag{61}$$

where we parameterized by some non affine parameter v. Here κ measures the failure of v to be an affine parameter on the horizon and it can be proved that $\kappa = const$ (cf. 1). Physically κ is *surface gravity* . The surface gravity is basically the force required of an observer at infinity to hold a particle (of unit mass) stationary at the event horizon.

VI. FIRST LAW OF THERMODYNAMICS

During a quasi-static process a stationary black hole of mass M, angular momentum J, and surface area A under the infinitesimal change of its parameters is taken to another stationary state. A statement of the first law is that changes in mass, angular momentum, and surface area are related by:

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega_H \delta J \tag{62}$$

Let us prove that. Suppose that a black hole is perturbed by a small quantity of matter described by the infinitesimal stress-energy tensor $T_{\alpha\beta}$. Recall that the changes of mass and momentum generated by transfer of matter across the hypersufrace, in this case horizon, is given by relations (56) and (57):

$$\delta M = -\int_H T^\alpha_\beta \xi^\beta_{(t)}\mathrm{d}\Sigma_\alpha$$

and

$$\delta J = \int_H T^\alpha_\beta \xi^\beta_{(\phi)}\mathrm{d}\Sigma_\alpha$$

where the integrations are over the entire event horizon. To the first order we can use $d\Sigma^\alpha = \xi^\alpha_{(\Theta)}\mathrm{d}S\mathrm{d}v$ the directed surface element where $\xi^\alpha_{(\Theta)}$, as already discussed, denotes direction of an element, $\mathrm{d}v$ integration along the generators and $\mathrm{d}S = \sqrt{\sigma}d^2\theta$ cross sectional area on the event horizon. Further, we will denote by $\mathcal{H}(v)$ cross-sections by which horizon is foliated. Substituting this into

$$\begin{aligned}\delta M - \Omega_H\delta J &= \int_H T_{\alpha\beta}\left(\xi^\beta_{(t)} + \Omega_H\xi^\beta_{(\phi)}\right)\xi^\alpha_{(\Theta)}\mathrm{d}S\mathrm{d}v \\ &= \int \mathrm{d}v \oint_{H(v)} T_{\alpha\beta}\xi^\alpha_{(\Theta)}\xi^\beta_{(\Theta)}\mathrm{d}S\end{aligned} \tag{63}$$

Let us consider a small bundle of generators as depicted on Figure 3. By looking at the rate of change of small area δS as we move along the generators, we define quantity θ as fractional rate of change of the congruence's cross-sectional area:

$$\theta = \frac{1}{\delta S}\frac{\mathrm{d}}{\mathrm{d}v}\delta S \tag{64}$$

We need θ in this discussion sice there is one convenient result regarding it which we will invoke to work out the integral; it is called Raychaudhuri's equation:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta}k^\alpha k^\beta \tag{65}$$

where $B_{\alpha\beta} = \nabla_\beta u_\alpha$. It describes divergence or convergence of geodesics within the congruence in terms of quantities introduced: the expansion scalar $\theta = B^\alpha_\alpha = \nabla_\alpha u^\alpha$, the shear tensor $\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta}$ and $\omega_{\alpha\beta} = B_{[\alpha\beta]}$ the rotation tensor. Because θ and $\sigma_{\alpha\beta}$ are quantities of the first order in $T_{\alpha\beta}$, it is appropriate to neglect the quadratic terms and Raychaudhuri's equation simplifies to

$$\frac{\mathrm{d}\theta}{\mathrm{d}v} = \kappa\theta - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta \tag{66}$$

We now have:

$$\begin{aligned}\delta M - \Omega_H \delta J &= -\frac{1}{8\pi}\int \mathrm{d}v \oint_{\mathcal{H}(v)} \left(\frac{\mathrm{d}\theta}{\mathrm{d}v} - \kappa\theta\right)\mathrm{d}S \\ &= -\frac{1}{8\pi}\oint_{\mathcal{H}(v)} \theta \mathrm{d}S\Big|_{-\infty}^{\infty} + \frac{\kappa}{8\pi}\int \mathrm{d}v \oint_{H(v)} \theta \mathrm{d}S\end{aligned}$$

Notice that v in $\mathcal{H}(v)$ is fixed as we integrate over $\mathcal{H}(v)$ surface so differentiation with respect to v can be pulled out. The black hole is stationary both before and after the perturbation. This implies $\theta(v = \pm\infty) = 0$ therefore the first terms vanish.

Substituting equation (64) gives us:

$$\delta M - \Omega_H \delta J = \frac{\kappa}{8\pi}\int \mathrm{d}v \oint_{\mathcal{H}(v)} \left(\frac{1}{\delta S}\frac{\mathrm{d}}{\mathrm{d}v}\delta S\right)\mathrm{d}S \tag{67}$$

On each hypersurface we have coordinate system denoted with $y^a = (v, \theta^A)$ where v is parameter along the curves, θ^A parameter across the curves (these are coordinates define by (106)). The area across the congruence, i.e. cross sectional area depicted in Figure 3, is here given by

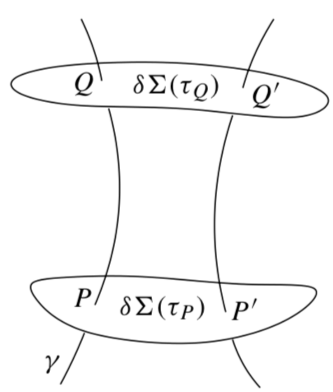

Figure 3. Congruence cross section changing as geodesics within the congruence diverge/converge. Note that the cross sectional area on this picture is denoted with Σ, which was our notation for $3D$ hypersurface, but in the context of this discussion we are considering S surface, $2D$ hypersurface (which is boundary of Σ) [1]

$\delta S = \sqrt{\sigma}d^2\theta$. Notice that since the coordinates are *comoving*, $d^2\theta$ does not change as the cross section evolves. With this in mind, going back to (67) we have:

$$\begin{aligned}\delta M - \Omega_H \delta J &= \frac{\kappa}{8\pi}\int \mathrm{d}v \oint_{\mathcal{H}(v)} \left(\frac{1}{\sqrt{\sigma}}\frac{\mathrm{d}}{\mathrm{d}v}\sqrt{\sigma}\right)\mathrm{d}S \\ &= \frac{\kappa}{8\pi}\oint_{\mathcal{H}(v)} \mathrm{d}S\Big|_{-\infty}^{\infty} \\ &= \frac{\kappa}{8\pi}\delta A\end{aligned} \tag{68}$$

□

VII. FIRST LAW WITH LAMBDA

The previous discussions were made in the light of vanishing cosmological constant but experimental observations suggest that the universe has a small, positive value of Λ. We consider a solution to Einstein's equations in D spacetime dimensions that describes a black hole with a Killing field and we will focus on the anti de Sitter black hole case, but methodology applies to de Sitter black holes as well. It has been shown[1920] that in order to extend the geometrical constructions to the case of nonvanishing cosmological constant (such as Komar integral and Smarr formula) there is a need for introducing quantity such as the Killing potential ω^{ab}:

$$\xi_b = \nabla^a \omega_{ab} \tag{69}$$

where ξ^b is Killing vector.
This will modify equation 59:

$$\frac{D-2}{8\pi G}\int_{\partial\Sigma} \mathrm{d}S_{ab}\left(\nabla^a\xi^b + \frac{2}{D-2}\Lambda\omega^{ab}\right) = 0 \tag{70}$$

VII.1. Scaling analysis

We will consider Euler's theorem for homogeneous functions (as this happens to provide a route between the first law of black hole mechanics and the Smarr formula for stationary black holes in the asymptotically flat case which). Gravitational action for nonvanishing Λ is given by:

$$S = \frac{1}{8\pi G}\int \mathrm{d}^D x\sqrt{-g}(R - 2\Lambda) \tag{71}$$

From this we see that the cosmological constant has dimension (length) $^{-2}$. Recall Euler's theorem for homogeneous functions:

$$f(\alpha^p x, \alpha^q y, \ldots, \alpha^r z) = \alpha^s f(x, y, \ldots, z) \tag{72}$$

which yields the scaling relation

$$sf(x,y,\ldots,z) = p\left(\frac{\partial f}{\partial x}\right)x + q\left(\frac{\partial f}{\partial y}\right)y + \cdots + r\left(\frac{\partial f}{\partial z}\right)z$$

upon taking the derivative with respect to α. From this we get:

$$rf(x,y) = p\left(\frac{\partial f}{\partial x}\right)x + q\left(\frac{\partial f}{\partial y}\right)y \tag{73}$$

The mass M of a static AdS black hole can be regarded as a function of the horizon area A and the cosmological constant Λ. These quantities scale as $M \propto l^{D-3}, A \propto l^{D-2}$ and $\Lambda \propto l^{-2}$ under a change in the length scale which combined with the Euler's theorem then implies that the mass of a static AdS black hole satisfies the (Smarr) formula

$$(D-3)M = (D-2)\left(\frac{\partial M}{\partial A}\right)A - 2\left(\frac{\partial M}{\partial \Lambda}\right)\Lambda \tag{74}$$

Setting $\Lambda = 0$ and $\partial M/\partial A = \kappa/8\pi G$, from the first law of black hole mechanics, gives the well-known (Smarr) formula for static, asymptotically flat black holes. The partial derivative $\partial M/\partial\Lambda$ in can be found by considering an extended version of the first law for AdS black holes that includes the effect of varying the cosmological constant

$$\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A + \left(\frac{\partial M}{\partial \Lambda}\right)\mathrm{d}\Lambda \tag{75}$$

VII.2. Derivation

As usual, we consider a family of spacelike surfaces Σ with unit timelike normal field n_a so that we have $(n_a n^a = -1)$. Let g_{ab} be the spacetime metric and h_{ab} the induced metric on Σ:

$$g_{ab} = h_{ab} - n_a n_b \tag{76}$$

We also have $h_{ab}n^b = 0$ by orthogonality. Foliating spacetime by a family of such hypersurfaces, the system can be taken to evolve along the vector field

$$\xi^a = Nn^a + N^a \tag{77}$$

where $N = -\xi^a n_a$ is lapse function and N^a shift function. We will consider Σ such that it extends from the horizon out to spatial infinity so that the boundary of Σ then has two components, an inner boundary at the horizon and an outer boundary at infinity. Recall equation 47, the full gravitational Hamiltonian is given by $\mathcal{H} = NH + N^a H_a$, where

$$\begin{aligned} H &\equiv -2G_{ab}n^a n^b = -R^{(d-1)} + \tfrac{1}{|h|}\left(\tfrac{\pi^2}{d-2} - \pi^{ab}\pi_{ab}\right) \\ H_b &\equiv -2G_{ac}n^a h^c_b = -2D_a\left(|h|^{-\frac{1}{2}}\pi^{ab}\right) \end{aligned}$$

where D_a is the covariant derivative operator with respect to h_{ab} on Σ, and $R^{(d-1)}$ its scalar curvature. Setting $8\pi T^a_b = -\Lambda g^a_b$ in equations above yields

$$H = -2\Lambda, \quad H_b = 0 \tag{78}$$

Those represent the constraint equations (recall the Note from chapter IV.2).

Given $s^{(0)}_{ab}$ and $\pi^{ab}_{(0)}$ a solution to the Einstein equations with cosmological constant $\Lambda_{(0)}$ and with a Killing vector ξ^a, Hamiltonian evolution with respect to the Killing vector ξ^a implies that $-\dot{\pi}^{ab}_{(0)} = 0, \dot{s}^{(0)}_{ab} = 0$. Consider now perturbations from the background spacetime:

$$\begin{aligned} s_{ab} &= s^{(0)}_{ab} + h_{ab} \\ \pi^{ab} &= \pi^{ab}_{(0)} + p^{ab} \\ \Lambda &= \Lambda_{(0)} + \delta\Lambda \end{aligned} \tag{79}$$

to the spatial metric, the momentum and the cosmological constant respectively. It follow from the pertubation that the Hamilton's equation combines to form a derivative operator on Σ:

$$D_a B^a = N\delta H + N^a \delta H_a = -2N\delta\Lambda \tag{80}$$

where we inserted also constrants. We can rewrite this as:

$$D_a\left(B^a - 2\delta\Lambda\omega^{ab}n_b\right) = 0 \tag{81}$$

which can be recognize as a form of Gauss Law. We can rewrite the term involving cosmological constant by once again making use of the Killing potential ω^{ab} such that $N = -n_a\xi^a = -D_c\left(n_a\omega^{ca}\right)$. Substituting this and integrating we get:

$$\int_{\partial\Sigma}\mathrm{d}a_c\left(B^c + 2\omega^{cd}n_d\delta\Lambda\right) = 0 \tag{82}$$

Since we have chosen such Σ so that we have two boundary surface, we can write this in terms of the inner and outer boundaries of enclosed volume V:

$$\begin{aligned} &\int_{\partial\hat{V}_{\text{out}}}\mathrm{d}\mathcal{S}r_c\left(B^c - 2\delta\Lambda\omega^{cb}n_b\right) \\ &= \int_{\partial\hat{V}_{\text{in}}}\mathrm{d}\mathcal{S}r_c\left(B^c - 2\delta\Lambda\omega^{cb}n_b\right) \end{aligned} \tag{83}$$

where r_c is the unit normal respectively pointing into and out of the inner and outer boundaries. Notice that Killing potential is not unique, it is only defined up to a divergence-less term: if ω^{ab} solves $\xi_b = \nabla^a\omega_{ab}$, then so does $\omega^{ab} = \omega^{ab} + \eta^{ab}$ where eta is divergenceless $\nabla^a\eta_{ab} = 0$. We can use ω^{ab}_{AdS} the Killing potential of the background AdS spacetime (as we already stated, derivation is given for anti de Sitter black holes) to rewrite:

$$\omega^{cb} = \omega^{cb} - \omega^{cb}_{\text{AdS}} + \omega^{cb}_{\text{AdS}} \tag{84}$$

for the $\partial\hat{V}_{\text{out}}$ so that the integral yields

$$\begin{aligned} &\int_{\partial\hat{V}_{\text{out}}}\mathrm{d}\mathcal{S}r_c\left(B^c - 2\delta\Lambda\omega^{cb}_{\text{AdS}}n_b\right) \\ &= \int_{\partial\hat{V}_{\text{out}}}\mathrm{d}\mathcal{S}r_c\left(2\delta\Lambda\left(\omega^{cb} - \omega^{cb}_{\text{AdS}}\right)n_b\right) \\ &\quad + \int_{\partial\hat{V}_{\text{in}}}\mathrm{d}\mathcal{S}r_c\left(B^c - 2\delta\Lambda\omega^{cb}n_b\right) \end{aligned} \tag{85}$$

The possibility of adding a divergenceless term to the Killing potential implies that the integrals of ω^{ab} over the horizon and infinity are tied together and cannot be given separate interpretations (only their difference is meaningful).

Recall that we have picked an inner boundary at the horizon and an outer boundary at spatial infinity. In that context the respective variations in the total mass M and angular momentum J of the spacetime can be yield by respectively setting $\xi^a_{(t)} = (\partial_t)^a$ (time translations) and $\xi^a_{(\phi)} = (\partial_\varphi)^a$ (rotations):

$$\begin{aligned}16\pi\delta M = -\int_\infty \mathrm{d}\mathcal{S}r_c\left(B^c\left[\partial_t\right] - 2\delta\Lambda\omega^{cb}_{\mathrm{AdS}}n_b\right)\\ 16\pi\delta J = \int_\infty \mathrm{d}\mathcal{S}r_c B^c\left[\partial_\varphi\right]\end{aligned} \tag{86}$$

The $\omega^{cb}_{\mathrm{AdS}}$ term ensures δM is finite.

Recall from section V that the generators of the event horizon are given by the Killing vector $\xi^a = (\partial_t + \Omega_H\partial_\varphi)^a$ and $\kappa = \sqrt{-\frac{1}{2}\nabla^a\xi^b\nabla_a\xi_b}\Big|_{r=r_+}$ its surface gravity. With respect to this Killing field we yield third relation:

$$2\kappa\delta A = -\int_H \mathrm{d}\mathcal{S}r_c B^c\left[\partial_t + \Omega\partial_\varphi\right] \tag{87}$$

where A is area of the event horizon on which norm of ξ vanishes. Note that $\delta\Lambda$ is spacetime-independent. This enables us to define

$$V = \int_\infty \mathrm{d}\mathcal{S}r_c n_b\left(\omega^{cb} - \omega^{cb}_{\mathrm{AdS}}\right) - \int_H \mathrm{d}\mathcal{S}r_c n_b\omega^{cb} \tag{88}$$

and interpret the remaining terms in as $V\delta P$. Again $\omega^{cb}_{\mathrm{AdS}}$ term ensures us that V is finite. So we have recovered

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \frac{\Theta}{8\pi G}\delta\Lambda \tag{89}$$

where

$$\Theta = -\left[\int_{\partial\Sigma_\infty}\mathrm{d}S_{ab}\left(\omega^{ab} - \omega^{ab}_{\mathrm{AdS}}\right) - \int_{\partial\Sigma_h}\mathrm{d}S_{ab}\omega^{ab}\right] \tag{90}$$

Θ is determined by the difference between the integral of the renormalized Killing potential at infinity and the integral of the Killing potential on the horizon.

VIII. INTERPRETATION AND CONCLUSION

In light of black hole thermodynamics it would be preferable if we would be able to establish sensible connection between Θ and some thermdynamical variable - volume being the simplest guess, in the context of other laws of thermodynamics. To shed some light on this idea, let us firs reexpress these boundary integrals appearing in 90 as volume integrals. Let us suppose that the hypersurface Σ is orthogonal to the static Killing vector ξ^a so that $\xi^a = Nn^a$. As a generalisation of (27), we can express the full spacetime volume element

$$\sqrt{-g^{(D)}} = N\sqrt{g^{(D-1)}} \tag{91}$$

in terms of the intrinsic volume element $\sqrt{g^{(D-1)}}$ on Σ. The integral of ω^{ab} over the boundary $\partial\Sigma$ can then be rewritten as:

$$\begin{aligned}\int_{\partial\Sigma}\mathrm{d}S_{ab}\omega^{ab} &= \int_\Sigma \mathrm{d}^{D-1}x\sqrt{g^{(D-1)}}n_b\xi^b\\ &= -\int_\Sigma \mathrm{d}^{D-1}x\sqrt{-g^{(D)}}\end{aligned} \tag{92}$$

Notice that this corresponds to the volume between the black hole horizon and infinity:

$$V_{BH} \equiv \int_\Sigma \mathrm{d}^{D-1}x\sqrt{-g^{(D)}} \tag{93}$$

Similarly, the integral of $\omega^{ab}_{\mathrm{AdS}}$ over the boundary at infinity can be written as:

$$V_{\mathrm{AdS}} \equiv \int_{\partial\Sigma_\infty}\mathrm{d}S_{ab}\left(\omega^{ab}_{\mathrm{AdS}}\right) \tag{94}$$

Going back to 90 we have:

$$\Theta = V_{BH} - V_{\mathrm{AdS}} \tag{95}$$

If we associate the volume with $V = -\Theta$, this obviously gives us a measure of the volume excluded from the spacetime by the black hole horizon. If Θ is associated with the notion of volume, to yield equivalence with the 1. law of black hole thermodynamics (using the usual identification $T = \kappa/2\pi$ and $S = A/4G$ for the black hole temperature and entropy):

$$\delta M = T\delta S + V\delta P \tag{96}$$

we interpret the term $\Lambda/8\pi G$ as pressure. Notice that the variation of an AdS black hole mass isn't identified with the variation of total energy, as this would be in $\Lambda = 0$ case, rather the variation of enthapy $H = E + PV$. Does this makes sense? We cite article by David Kastor, Sourya Ray and Jennie Traschen: 'The identification of the AdS black hole mass as an enthalpy makes good physical sense. The mass of an AdS black hole is defined via an integral at infinity. However, between the black hole horizon and infinity is an infinite amount of energy density that needs to be subtracted off in some manner to get a finite result. Since the energy density in the cosmological constant is $\rho = +\Lambda/8\pi G$ adding a PV term naturally cancels out a ρV contribution to the energy.'[11]

IX. APPENDIX

IX.1. Hypersurface

A hypersurface $\Sigma = \Sigma_{(n)}$ is an n -dimensional subspace (submanifold) Σ of a $D = n+1$ dimensional space (spacetime manifold) $M = M_{(n+1)}, \Sigma \subset M$.

We describe the hypersurface in terms od embeddings:

$$\Psi : \quad \Sigma = \Sigma_{(n)} \hookrightarrow M = M_{(n+1)} \tag{97}$$

of Σ into M, specified by the map Ψ. Embedded hypersurface is a subspace of M:

$$\Sigma = \Sigma_{(n)} \subset M = M_{(n+1)} \tag{98}$$

specified e.g. by

$$\Sigma = \{x^\alpha \in M : \Phi(x^\alpha) = 0\} \tag{99}$$

for some real-valued function $\Phi(x)$ on M.

In terms of the embedding Ψ this can be phrased as the statement that the embedding Ψ can be used to pull back the function f on M to a function $\Psi^* f$ on Σ defined by

$$\begin{aligned} \Psi^* f : & \Sigma \to \mathbb{R} \\ & \left(\Psi^* f\right)(y) = f(\Psi(y)) \end{aligned}$$

When we're dealing with hypersurfaces we can put any coordinate system on them and it may or may not be simply related to the spacetime coordinates we use elsewhere. We will denote with x^α coordinates for spacetime as a whole (i.e. our manifold M) and by y^a intrinsic coordinates on Σ. In that description embedding f is given by parametric equations

$$f : \quad x^\alpha = x^\alpha\left(y^a\right) \tag{100}$$

which gives description of the curves contained in Σ.

If the hypersurface is not null, a unit normal n^α can be introduced by:

$$n^\alpha n_\alpha = \varepsilon \equiv \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike} \\ +1 & \text{if } \Sigma \text{ is timelike} \end{cases}$$

and we demand that n^α point in the direction of increasing $\Phi : n^\alpha \partial_\alpha \Phi > 0$. The vector $\partial_\alpha \Phi$ is normal to the hypersurface, because the value of Φ changes only in the direction orthogonal to Σ. Including the normalisation we get that n_α is given by

$$n_\alpha = \frac{\varepsilon \partial_\alpha \Phi}{\left|g^{\mu\nu}\partial_\mu \Phi \partial_v \Phi\right|^{1/2}}$$

if the hypersurface is either spacelike or timelike.

In the null case we have $g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$ equal to zero so above definition of the normal vector doesn't work. In the null case we let

$$k_\alpha = -\partial_\alpha \Phi$$

be the normal vector with the sign chosen so that k^α is future directed when Φ increases toward the future. Note that k^α is both orthogonal and tangent to the hypersurface since it is orthogonal to itself ($k^\alpha k_\alpha = 0$). In fact it can be shown that k^α is tangent to the null generators in Σ.

The coordinate system on the hypersurface can be specified in terms of normal vectors and tangent vectors to the hypersurface. Tangent vectors will be those which give displacements within the hypersurface:

$$e_a^\alpha = \frac{\partial x^\alpha}{\partial y^a} \tag{101}$$

Note that we have $e_a^\alpha n_\alpha = 0$ in the non-null case, and $e_a^\alpha k_\alpha = 0$ in the null case. The metric intrinsic to the hypersurface Σ is obtained by restricting the line element to displacements confined to the hypersurface:

$$\begin{aligned} \mathrm{d}s_\Sigma^2 &= g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\ &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}\mathrm{d}y^a\right)\left(\frac{\partial x^\beta}{\partial y^b}\mathrm{d}y^b\right) \\ &= h_{ab}\mathrm{d}y^a \mathrm{d}y^b \end{aligned} \tag{102}$$

where

$$h_{ab} = g_{\alpha\beta} e_a^\alpha e_b^\beta \tag{103}$$

is the induced metric of the hypersurface inherited from the spacetime metric (pullback of the metric on Σ). Notice that it is a scalar with respect to transformations $x^\alpha \to x^{\alpha'}$ of the spacetime coordinates, but it behaves as a tensor under transformations $y^a \to y^{a'}$ of the hypersurface coordinates.

We can also decompose the spacetime metric in terms of the basis (n_α, e_a^α):

$$g_{\alpha\beta} = \varepsilon n_\alpha n_\beta + h_{\alpha\beta} \tag{104}$$

where $h_{\alpha\beta} = h_{ab} e_\alpha^a e_\beta^b$.

For the null case first we have to notice one subtile thing. In the spacelike (timelike) case we had one normal vector and three tangent. Recall that the k^α is orthogonal to itself. Therefore it will not be a good normal vector since one of the tangent vectors will coincide with it. So we have to construct good normal vector to the null hypersurface. We will denote it with N^α and define it with:

$$\begin{aligned} N^\alpha N_\alpha &= 0 \\ N^\alpha k^\alpha &= -1 \\ N^\alpha e_\alpha^a &= 0 \end{aligned} \tag{105}$$

where the first relation is null property, second is normalisation and third is orthogonality to tangent vectors. Our null basis now consists of two spacelike vectors e^a_α and two null vectors k^α and N^α where N^α is orthogonal vector and others are tangent to the null Σ.

Now we will pick the intrinsic coordinate system on the hypersurface. We will choose it so that it is adapted to the network of null curves lying on our null hypersurface. On each curve we have parameter λ running along the curve and parameter θ_A being its constant label. Note that parameter θ_A will be a parameter running across curves and we will have two of those since Σ is here 3D. Our intrinsic coordinate system is then:

$$y^a = \left(\lambda, \theta^A\right) \tag{106}$$

Going back to equation (101), our tangent vectors are now specified:

$$k^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A}, \quad e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda \tag{107}$$

where $k^\alpha e^\alpha_A = 0$. Since one of the tangent vectors is now null, the induced metric is now 2D:

$$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B \tag{108}$$

We have the decomposition of the spacetime metric:

$$g_{\alpha\beta} = -k_\alpha N_\beta - N_\alpha k_\beta + \sigma_{AB} e^A_\alpha e^B_\beta \tag{109}$$

IX.2. Integration on the hypersurface

Since integration means adding up and since we cannot add vectors defined in defferent points in the context of curved spacetime, we know that integration will have to be defined on scalars. We start by introducing a notion of directed surface element (such that it has a sensible null limit):

$$\mathrm{d}\Sigma_\mu = \varepsilon_{\mu\alpha\beta\gamma} e^\alpha_1 e^\beta_2 e^\gamma_3 \mathrm{d}^3 y \tag{110}$$

where $\varepsilon_{\mu\alpha\beta\gamma} = \sqrt{-g}[\mu\alpha\beta\gamma]$ is the Levi-Civita tensor. Permutation symbol is defined by:

$$[\alpha\beta\gamma\delta] = \begin{cases} +1 \text{ if } \alpha\beta\gamma\delta \text{ is an even permutation of } 0123 \\ -1 \text{ if } \alpha\beta\gamma\delta \text{ is an odd permutation of } 0123 \\ \quad 0 \text{ if any two indices are equal} \end{cases}$$

For timelike and spacelike case:

$$\mathrm{d}\Sigma_\alpha = \varepsilon n_\alpha \mathrm{d}\Sigma$$

thus, apart from a factor $\varepsilon = \pm 1, \mathrm{d}\Sigma_\alpha$ is a directed surface element on Σ as we would construct it for Euclidean case. But for the null case we get:

$$\mathrm{d}\Sigma_\mu = k^v \mathrm{d}S_{\mu\nu} \mathrm{d}\lambda \tag{111}$$

where $\mathrm{d}^3 y = \mathrm{d}\lambda \mathrm{d}^2\theta$, and

$$\mathrm{d}S_{\mu v} = \varepsilon_{\mu v\beta\gamma} e^\beta_2 e^\gamma_3 \mathrm{d}^2\theta \tag{112}$$

an element of 2D surface in the transverse subspace which can also be expressed as:

$$\mathrm{d}S_{\alpha\beta} = 2k_{[\alpha} N_{\beta]} \sqrt{\sigma} \mathrm{d}^2\theta \tag{113}$$

where N_α is the auxiliary null vector field, relation (86) and σ 2D induced metric defined by (108). To recap what we've done here notice that we started from a null hypersurface described in terms of $y^a = (\lambda, \theta_A)$ i.e. λ (parameter along the null generators) and θ_A parameters. Then for $\lambda = const.$ we get 2D surface in the transverse subspace. We know that σ_{AB} is metric in that transverse subspace and $\sqrt{\sigma}\mathrm{d}^2\theta$ a surface element in it. Since this is 2D subspace embedded in 4D we have to have to use two vectors to give that surface element direction, and that is exactly the purpose of $k_{[\alpha} N_{\beta]}$ in the above equation.

If we go back with this into relation (111) we get:

$$\mathrm{d}\Sigma_\alpha = -k_\alpha \sqrt{\sigma} \mathrm{d}^2\theta \mathrm{d}\lambda \tag{114}$$

Apart from a minus sign, the surface element is directed along k_α, the normal to the null hypersurface, $\sqrt{\sigma}\mathrm{d}^2\theta$ is an element of 2D surface area with the direction transverse to the generators and $\mathrm{d}\lambda$ is infinitesimal element for the integration along the generators.

Now we can state the Gauss's theorem in 4D:

$$\int_V \nabla_\alpha A^\alpha \sqrt{-g} \mathrm{d}^4 x = \oint_{\partial V} A^\alpha \mathrm{d}\Sigma_\alpha \tag{115}$$

and Stokes's theorem:

$$\int_V \nabla_\beta B^{\alpha\beta} \mathrm{d}\Sigma_\alpha = \frac{1}{2} \oint_{B^{\alpha\beta}} \mathrm{d}S_{\alpha\beta} \tag{116}$$

Notice that Stokes's theorem works only for $B^{\alpha\beta}$ antisymmetric because $\mathrm{d}S_{\alpha\beta}$ is antisymmetric (meaning it would give us zero for any symmetric $B^{\alpha\beta}$).

IX.3. Killing vectors and Lie derivatives

Lie derivative is useful tool which allows us to do the differentiation without introducing the connection. The other feature of Lie derivative is it's power to encodes the information that the certain tensor is invariant under the flow of diffeomorphism. By flow of diffeomorphism we mean: shift the point at which a tensor is evaluated by pushing it forward and then transform (pull back) the coordinates so that the shifted point has the same coordinate labels as the old point, meaning: $T(P_0) \mapsto \bar{T}(P) \equiv \phi_\lambda T(P_0)$ such that the coordinate values are unchanged: $\bar{x_\mu}(P) = x_\mu(P_0)$. Since a diffeomorphism maps a manifold back to itself, under a diffeomorphism a rank (m, n) tensor is mapped to another rank (m, n) tensor. We can apply it to the components of one form:

$$\bar{V}_\mu(P_0) \equiv V_\alpha(P)\frac{\partial x^\alpha}{\partial \bar{x}^\mu}(P), \qquad \text{where } \bar{x}^\mu(P) = x^\mu(P_0)$$

Starting with V_α at point P_0 with coordinates $x^\mu(P_0)$, we push the coordinates forward to point P, we evaluate V_α there, and then we transform the basis back to the coordinate basis at P with new coordinates $\bar{x}^\mu(P)$.

Now, recall that any vector field has a unique set of integral curves whose tangent vector is $\xi^\mu = \frac{\partial \xi^\mu}{\partial \lambda}$. Using the integral curves of a vector field we can shift a curve $x^\mu(\tau)$ to a new curve $y^\mu(\tau)$. The shift is called pushforward but when it defines a continuous one to one mapping of the space back to itself then we call it diffeomorphism.

Thus, a general diffeomorphism may be obtained from the infinitesimal diffeomorphism with pushforward $y^\mu = x^\mu + \xi^\mu d\lambda$. The corresponding coordinate transformation is (to first order in $d\lambda$)

$$\bar{x}^\mu = x^\mu - \xi^\mu d\lambda$$

so that $\bar{x}^\mu(P) = x^\mu(P_0)$. This yields (in the x^μ coordinate system)

$$\begin{aligned}&\bar{V}_\mu(x) \equiv V_\alpha(x+\xi d\lambda)\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\\ &= V_\mu(x) + \left[\xi^\alpha \partial_\alpha V_\mu(x) + V_\alpha(x)\partial_\mu \xi^\alpha\right] d\lambda + O(d\lambda)^2\end{aligned}$$

Here we can expand the inverted Jacobian

$$\frac{\partial \bar{x}^\mu}{\partial x^\alpha} = \delta^\mu_\alpha - \frac{\partial_\alpha x i^\mu}{d\lambda}$$

to first order in $d\lambda$,

$$\frac{\partial \bar{x}^\mu}{\partial x^\alpha} = \delta^\alpha_\mu + \partial_\mu \xi^\alpha d\lambda + O(d\lambda)^2.$$

In a similar manner, the infinitesimal diffeomorphism of the metric gives

$$\begin{aligned}&\bar{g}_{\mu\nu}(x) \equiv g_{\alpha\beta}(x+\xi d\lambda)\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\\ &= g_{\mu\nu}(x) + \left[\xi^\alpha\partial_\alpha g_{\mu\nu}(x) + g_{\alpha\nu}(x)\partial_\mu\xi^\alpha + g_{\mu\alpha}(x)\partial_\nu\xi^\alpha\right] d\lambda\\ &+ O(d\lambda)^2\end{aligned}$$

Lie derivative is rate of change with respect to the infinitesimal diffeomorphism $\overline{\mathrm{T}} \equiv \phi_{\Delta\lambda}\mathrm{T}$:

$$\begin{aligned}&\mathfrak{L}_\xi \mathrm{T} \equiv \lim_{\Delta\lambda\to 0}\frac{\phi_{\Delta\lambda}\mathrm{T}(x) - \mathrm{T}(x)}{\Delta\lambda}\\ &\text{with } \bar{x}^\mu(P) = x^\mu(P_0) = x^\mu(P) - \xi^\mu\Delta\lambda + O(\Delta\lambda)^2\end{aligned} \tag{117}$$

This discussion was highly referring to[16].

It is straightforward to check that the Lie derivatives of $V_\mu(x)$ and $g_{\mu\nu}(x)$ are:

$$\mathfrak{L}_\xi V_\mu(x) = \xi^\alpha\partial_\alpha V_\mu + V_\alpha\partial_\mu\xi^\alpha, \tag{118}$$

$$\mathfrak{L}_\xi g_{\mu\nu}(x) = \xi^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu\xi^\alpha + g_{\mu\alpha}\partial_\nu\xi^\alpha \tag{119}$$

Here the first term $\xi^\alpha\partial_\alpha$ corresponds to the pushforward - shifting a tensor to another point in the manifold. The remaining terms arise from the coordinate transformation back to the original coordinate values.
One can check that Lie derivative is actually a tensor (despite that suspicious appearance of partial derivatives). We can actually write equation (34) in terms of covariant derivatives, though it obscures than the point that connections are not needed here.
Lie derivative is well defined for any tensor field and it tensor of the same rank. So generally:

$$\begin{aligned}\mathfrak{L}_X T^{a_1\ldots a_k}_{b_1\ldots b_\ell} &= X^c\nabla_c T^{a_1\ldots a_k}_{b_1\ldots b_\ell} +\\ &+ \sum_{i=1}^{\ell} T^{a_1\ldots a_k}_{b_1\ldots c\ldots b_\ell}\nabla_{b_i}X^c\\ &- \sum_{j=1}^{k} T^{a_1\ldots c\ldots a_k}b_1\ldots b_\ell\nabla_c X^{a_j}\end{aligned} \tag{120}$$

The Lie derivative of a vector field is an antisymmetric object known also as the commutator or Lie bracket:

$$(\mathfrak{L}_X Y)^a = [X,Y]^a = X^b\nabla_b Y^a - Y^b\nabla_b X^a \tag{121}$$

Vector field is said to be *Killing vector field* if it satisfies the relation:

$$\mathfrak{L}_\xi g_{ab} = \nabla_a\xi_b - \nabla_b\xi_a = 0 \tag{122}$$

This equation is known as Killing's equation.

Theorem If the Lagrangian is invariant under the diffeomorphism generated by a vector field ξ^μ, then $\tilde{p}(\vec{\xi}) = p_\mu\xi^\mu$ is conserved along curves that extremize the action, i.e. for trajectories obeying the equations of motion.

If the spacetime has a Killing vector ξ^μ, then $p_\mu\xi^\mu$ is conserved along any geodesic. The existence of a Killing vector represents a symmetry. Spacetimes with Killing vectors have a conserved 4-vector energy current: $j_\nu = \xi_\mu T^{\mu\nu}$. Local stress-energy conservation $\nabla_\nu T^{\mu\nu} = 0$ implies:

$$\nabla_\mu j^\mu = 0 \tag{123}$$

which can be integrated over a volume to give the usual form of an integral conservation law:

$$\int_{\Sigma_1} j^\alpha \mathrm{d}\Sigma_\alpha + \int_{\Sigma_2} j^\alpha \mathrm{d}\Sigma_\alpha = 0 \tag{124}$$

for any closed hypersurface Σ. Here we supposed that Σ_1 and Σ_2 are two spacelike hypersurfaces extended to infinity where integration is performed (and where $j^\alpha = 0$).

IX.4. Black hole

We define a black hole as part of spacetime from which no future directed timelike or null line can escape to arbitrarily large distance into the outer asymptotic region (which will in the case of asymptotically flat spacetimes be null infinity and timelike infinity for asymptotically anti-de Sitter spacetimes). If we denote by Ξ^+ the future asymptotic region of a spacetime $(\mathcal{M}, g^{\mu\nu})$ the black hole region $\mathcal{B}$ is defined as

$$\mathcal{B} \equiv \mathcal{M} - I^- \left(\Xi^+\right)$$

where I^- denotes the chronological past. The region $I^-\left(\Xi^+\right)$ is the set of points for which it is possible to construct a future directed timelike line, it is usually referred to as the domain of outer communication.

We define the event horizon $\mathcal{H}$ of a black hole as the boundary $\delta\mathcal{B}$. Let us denote $J^-(U)$ the causal past of a set of points $U \subset \mathcal{M}$ and $\bar{J}^-(U) = (J^-(U) \cup$ its accumulation points) - meaning $\bar{J}^-(U) \equiv$ topological closure of $J^-(U)$. We have $I^-(U) \subset J^-(U)$. The event horizon of $\mathcal{M}$ is then defined as the boundary of the closure of the causal past of the future asymptotic region of spacetime Ξ^+

$$\mathcal{H} \equiv \bar{J}^- \left(\Xi^+\right) - J^- \left(\Xi^+\right)$$

The event horizon is a concept defined with respect to the entire causal structure of $\mathcal{M}$, in the physical context, the event horizons are null hypersurfaces (with some interesting properties).

IX.5. Short overview of differential geometry

In this section we provide short overview of, for the discussion useful, differential geometry notions. The reader should look at it as a catalogue of useful definitions which, if not familiar, should be looked up in more detail (see the references[7,8,5,9,10] and one being on croatian[2]) in order to fully understand the above discussion.

Let us review the process of adding mathematical structures to form mathematical context of general relativity. Almost every mathematical story starts from the notion of a set, which is object with no structure. In this case we then proceed by introducing the concept of open sets of our set - meaning, we introduce a topology and promote the set to a topological space to enable us to have a good notion of continuity (the inverse image of any open set is open) and a notion of homeomorphisms between topological spaces.

Definition(Topological space)
Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X, called open subsets, satisfying:
(1) X and $\emptyset$ are open.
(2) The union of any family of open subsets is open.
(3) The intersection of any finite family of open subsets is open.

A pair $(X,\, \mathcal{T})$ consisting of a set X together with a topology $\mathcal{T}$ on X is called a topological space.

In this context there is no notion of differentiability or differentiation so we proceed to introduce a concept of a manifold by demanding that each open set look like a region of $\mathbf{R}^n$ making it topological space that locally looks like Euclidean space. A smooth manifold is a manifold M for which this resemblance is sharp enough to permit the establishment of partial differentiation along with all the essential features of calculus on M.

Definition(Manifold)
Suppose M is a topological space. We say that M is a topological manifold of dimension n or a topological n-manifold if it has the following properties:
(1) M is a Hausdorff space: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$. (2) M is second-countable: there exists a countable basis for the topology of M . (3) M is locally Euclidean of dimension n: each point of M has a neighborhood that is homeomorphic to an open subset of R^n

A manifold is very flexible and powerful structure, and comes equipped naturally with a tangent bundle, tensor bundles, the ability to take exterior derivatives, and so forth. We will not go into all of the details here as we assume that this is just recap of already familiar notions.

Often we will refer to 4D spacetime as being our manifold. It is useful to also introduce the concept of submanifold, as we will often work with just a subset of spacetime manifold:

Definition(Submanifold)
Let M be a smooth manifold of dimension n. A subset $N \subset M$ is called a k -dimensional submanifold of M if for each $x \in N$, there is a chart (U, u) for M with $x \in U$ such that $u(U \cap N) = u(U) \cap \mathbb{R}^k$. We view $\mathbb{R}^k \subset \mathbb{R}^n$ as the subspace of points for which the last $n-k$ coordinates are zero.

To get a good relationship between our submanifold and whole spacetime we will also have to define the notion of smooth mapping between manifolds

Definition(Smooth mapping between manifolds) Let M^m and N^n be manifolds. A mapping $\phi : M \to N$ is smooth provided that for every coordinate system ξ in M and η in N the coordinate expression $\eta \circ \phi \circ \xi^{-1}$ is Euclidean smooth (and defined on an open

set of R^m).
Explicitly, if U and V are the domains of ξ and η, then for all $p \in \phi^{-1}(V) \cap U$ the coordinates $y^j(\phi p), 1 \le j \le n$, depend smoothly on the coordinates $x^1(p), \ldots, x^m(p)$

Some mapping are special because they are structure preserving. Generally we call those maps isomorphisms but in the context of smooth manifold we refer to those as diffeomorphisms.

Definition(Diffeomorphism) A diffeomorphism $\phi : M \to N$ is a smooth mapping that has an inverse mapping which is also smooth.

Now we want to introduce the notion of coordinate system. In curved spacetime it is not possible to construct good coordinate system to cover the whole manifold. We will have to introduce instead entire collection of coordinate systems to cover M and we will demand that their covering is compatible with the condition of smoothness. Notion of a charts allows us the introduction of local coordinates on an open set U ⊂ M such that a point p ∈ U corresponds to in a point $\phi(p) \in R^n$. A topological manifold topological possesses a covering by open sets U_a with charts $C_a = (U_a, \phi_a)$.

Definition(Chart and Atlas) Let M be a set. A chart (ψ, V) of M is a bijective map ψ of $V \subseteq M$ onto an open subset U of $\mathbb{R}^n, \psi : V \to U$. Two charts $(\psi_1, V_1), (\psi_2, V_2)$ are called $(\mathcal{C}^\infty-)$ compatible if $\psi_1(V_1 \cap V_2)$ and $\psi_2(V_1 \cap V_2)$ are open in $\mathbb{R}^n$ and the change of charts $\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ is a $\mathcal{C}^\infty$ -diffeomorphism (note that this condition is symmetric in ψ_1, ψ_2).
A $\mathcal{C}^\infty$ -atlas of M is a family $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ of pairwise compatible charts such that $M = \bigcup_{\alpha \in A} V_\alpha$. Two atlasses $\mathcal{A}_1, \mathcal{A}_2$ are called equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ itself is an atlas of M, i.e., if all charts of $\mathcal{A}_1 \cup \mathcal{A}_2$ are compatible. An differentiable manifold is a set M together with an equivalence class of atlasses. Such an equivalence structure is called a differentiable (or $\mathcal{C}^\infty$ -) structure on M.

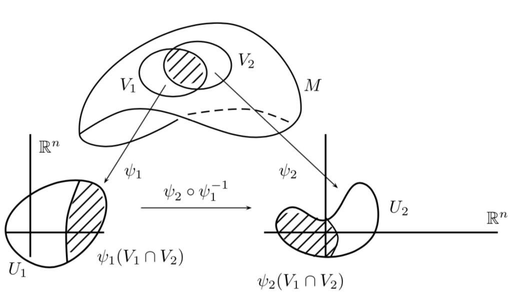

Figure 4. Charts covering open sets V1 and V2 on M [8]

If we are to introduce the differentiation we have to consider also concepts of tangent bundles. The basic idea of differentiation is to find a linear approximation of a map in the neighborhood of a point. For the case of maps between open subsets of R^n and R^m, these approximations are just linear maps between these vector spaces. In the case of submanifolds, one first has to define appropriate vector spaces on which such linear approximations can be defined, and usually one will obtain different spaces for different points. Let us consider the situation where $M \subset \mathbb{R}^n$ and where all these spaces can be realized as linear subspaces of the ambient Rn as depicted on IX.5.

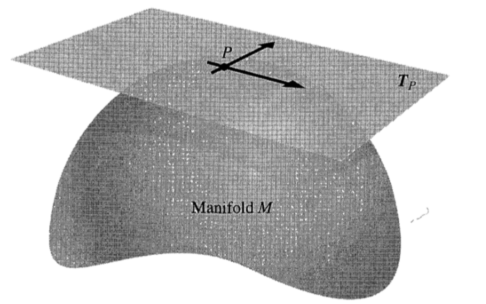

Figure 5. Linear subspace of R_n representing tangent space T_M [6]

Let us first simply define a tangent vector:

Definition(Tangent vector space) Let p be a point of a manifold M. A tangent vector to M at p is a real valued function $v : C_p^\infty(M) \to \boldsymbol{R}$ such that it satisfies:
(1) $\boldsymbol{R}$ -linear: $v(af + bg) = av(f) + bv(g)$, and
(2) Leibnizian: $v(fg) = v(f)g(p) + f(p)v(g)$ for all $a, b \in \boldsymbol{R}$ and $f, g \in C_p^\infty(M)$ At each point $p \in M$ let $T_p(M)$ be the set of all tangent vectors to M at p The usual definitions of functional addition and scalar multiplication make $T_p(M)$ a vector space over the real numbers $\boldsymbol{R}$. Explicitly,

$$(v + w)(f) = v(f) + w(f)$$
$$(av)(f) = av(f) \quad \text{for all } f \in \mathfrak{F}(M), \quad a \in \boldsymbol{R}$$

and $T_p(M)$ is called the tangent space to M at p. To define partial differentiation on a manifold, we move the function f back to Euclidean space using a coordinate system (chart), and then take the usual partial derivatives.

Now the notion of tangent bundle may be more comprehensible:

Definition(Tangent bundle and tangent map) For a smooth submanifold $M \subset \mathbb{R}^n$ we define the tangent bundle $TM \subset \mathbb{R}^n \times \mathbb{R}^n$ of M as the subset $\{(x, v) : x \in M, v \in T_x M\}$. Let $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ be submanifolds and let $f : M \to N$ be a smooth map. Then we define the tangent map $Tf : TM \to TN$ of f by $Tf(x, v) := (f(x), T_x f(v))$

It is interesting to see some properties following from: *Proposition*(1) For a smooth submanifold $M \subset \mathbb{R}^n$ of dimension k, the tangent bundle TM is a smooth submanifold of $\mathbb{R}^{2n}$ of dimension $2k$. The first projection $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ induces a smooth map $p : TM \rightarrow M$ (2) For a smooth map $f : M \rightarrow N$ between submanifolds, the tangent map $Tf : TM \rightarrow TN$ is smooth, too, and it satisfies $p \circ Tf = f \circ p$ (3) For smooth maps $f : M \rightarrow N$ and $g : N \rightarrow P$ between submanifolds, we have the chain rule $T(g \circ f) = Tg \circ Tf$

Proof. (1) Take a point $(x, v) \in TM$. Then we know that there is an open subset $U \subset \mathbb{R}^n$ with $x \in U$ and a smooth function $F : U \rightarrow \mathbb{R}^{n-k}$ such that $M \cap U = F^{-1}(\{0\})\mathbb{R}^n$. Now we define $\tilde{U} := U \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ and consider the smooth map $\tilde{F} : \tilde{U} \rightarrow \mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$ by $\tilde{F}(y, w) := (F(y), DF(y)(w))$. Now $\tilde{F}(y, w) = (0, 0)$ is equivalent to $y \in F^{-1}(0) = U \cap M$ and $w \in \ker(DF(y)) = T_y M$ and thus to $(y, w) \in \tilde{U} \cap TM$. Assuming this, we compute

$$D\tilde{F}(y, w)\,(v_1, v_2) = \big(DF(y)\,(v_1)\,, D^2F(y)\,(w, v_1) + DF(y)\,(v_2)\big)$$

This readily implies that $D\tilde{F}(y, w)$ is surjective, which completes the proof of the first part. The second part is clear, since the first projection is a global smooth extension of p.
(2) Let us again take $(x, v) \in TM$. Then by assumption, there is an open subset $U \subset \mathbb{R}^n$ and a smooth map $\tilde{f} : U \rightarrow \mathbb{R}^m$ such that $\tilde{f}\big|_{U \cap M} = f|_{M \cap U}$. Similarly as above, we define a smooth map $\varphi : U \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ by $\varphi(y, v) := (\tilde{f}(y), D\tilde{f}(y)(v))$. For $(y, w) \in (U \times \mathbb{R}^n) \cap TM$, we then get $\varphi(y, w) = (f(y), T_y f(v)) = Tf(y, w),$. Thus φ is a smooth extension of Tf on an open neighborhood of (x, v) and smoothness follows. The last claim is obvious from the definition of Tf.
(3) This is an obvious consequence of the chain rule.
$\square$
(this proof is taken from[7])

We then proceeded to put a metric on the manifold. This will result in a manifold with metric (semi-Riemannian manifold). Independently of the metric we could introduce a connection, allowing us to take covariant derivatives but once we have a metric there is automatically a unique torsion-free metric-compatible connection. Likewise we could introduce an independent volume form, although one is automatically determined by the metric. In principle there is nothing to stop us from introducing more than one connection, or volume form, or metric, on any given manifold. In general relativity we do have a physical metric, which determines volumes and the covariant derivative, and the independence of these notions is not a crucial feature. Again we note that here we will not introduce all of the definitions but only definitions for crucial objects which are explicitly used in our article.

Metric is a special type of bilinear form so recall:

Definition (Bilinear forms). Let V be a finite dimensional vector space. A bilinear form on V is an $\mathbb{R}$ -bilinear mapping $b : V \times V \rightarrow \mathbb{R}$. It is called symmetric if

$$b(v, w) = b(w, v) \quad \text{for all } v, w \in V$$

A symmetric bilinear form is called (i) Positive (negative) definite, if $b(v, v) > 0 (< 0)$ for all $0 \neq v \in V$, (ii) Positive (negative) semidefinite, if $b(v, v) \geq 0 (\leq 0)$ for all $v \in V$, (iii) nondegenerate, if $b(v, w) = 0$ for all $w \in V$ implies $v = 0$. Finally we call b (semi) definite if one of the alternatives in [(i) (resp. (ii)) hold true. Otherwise we call b indefinite.

Definition(Metric) A semi-Riemannian metric tensor g_{ab} (or metric, for short) on a smooth manifold M is a smooth, symmetric and nondegenerate (0, 2)-tensor field g on M of constant index.
g smoothly assigns to each point p ∈ M a symmetric nondegenerate bilinear form $g(p) = g_p : T_pM \times T_pM \mapsto R$ such that the index r_p of g_p is the same for all p.

Definition (Index) We define the index r of a symmetric bilinear form b on V *by* $r := \max\{\dim W \mid W$ subspace of V with $b|_W$ negative definite $\}$. By definition we have $0 \leq r \leq \dim V$ and $r = 0$ iff b is positive semidefinite.

Integration The main aim is to recover calculus and other physicists tools for analysing physical situations. So of course we need integration also. We already mentioned the process of taking an integral in GR (**??**). To state it clearly, integral over an n-dimensional region $\Sigma \subseteq M$ manifold is a map from n-form field to the real numbers. The main thing to clarify in order to understand this procedure is the notion of the volume form:

Definition(Volume Elements) A volume element on an n -dimensional semi-Riemannian manifold M is a smooth n -form ω such that $\omega\,(e_1, \ldots, e_n) = \pm 1$ for every frame on M
Intuitively a volume element on an n -dimensional scalar product space V is a function ω that assigns to n vectors $v_1, \ldots, v_n \in V$ the volume of the parallelepiped with these vectors as sides. (Thus $\omega\,(v_1, \ldots, v_n) = 0$ if the vectors are linearly dependent, that is, if the parallelepiped collapses.)

Volume elements always exist at least locally.
Lemma. On the domain U of a coordinate system ξ there is a volume element ω_ξ such that $\omega_\xi\,(\partial_1, \ldots, \partial_n) = |\det\,(g_{ij})|^{1/2}$
Proof. For vector fields $V_1, \ldots, V_n$ on U write $V_j = \sum V_j^i \partial_i$ and define

$$\omega_\xi\,(V_1, \ldots, V_n) = \det\left(V_j^i\right) |\det\,(g_{ij})|^{1/2}$$

Properties of determinants show that this uniquely defines ω_ξ as an n -form on U If $V_1, \ldots, V_n$ is a frame field,

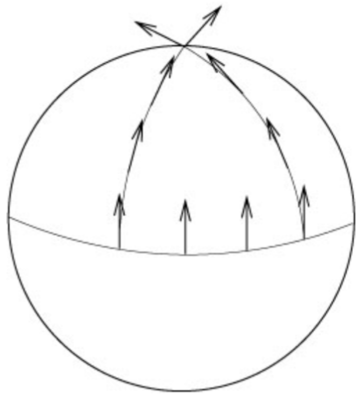

Figure 6. Upon parallel transporting along a geodesic triangle on the sphere, starting vector will end up in a different tangent vector space reflecting the non commutativity of covariant derivatives. [9]

then

$$\delta_{ij}\varepsilon_j = \langle V_i, V_j\rangle = \left\langle \sum V_i^r \partial_r, \sum V_j^s \partial_s \right\rangle = \sum V_i^r g_{rs} V_j^s$$

Taking determinants gives $(-1)^v = \left(\det\left(V_j^i\right)\right)^2 \det\left(g_{ij}\right)$, hence $\omega_\xi\left(V_1, \ldots, V_n\right) = \det\left(V_j^i\right)\left|\det\left(g_{ij}\right)\right|^{1/2} = \pm 1$
$\square$
(this proof is taken from[5])
In the notation of differential forms, $\omega_\xi = |g|^{1/2} dx^1 \wedge \cdots \wedge dx^n$, where $|g| = \left|\det\left(g_{ij}\right)\right|$

Other notions we have referred to...
We used the notion of a pullback a lot in previous discussion when we talked about the hypersurfaces (3D) and the induced metric on them. We then said that the metric was 'inherited' from 4D metric of our spacetime manifold where by 'inherited' we actually meant that it was pullback of it. Same notion was used while discussing embeddings. Pullback isn't reserved for embeddings, it works for any two manifolds with a smooth mapping between them. Here we provide rigorous definition:
Definition(Pullback) Let M, N be manifolds, and $F: M \to N$ smooth. For $\omega \in \mathcal{T}_k^0(N)$, the pullback of ω under F is defined as $F^*\omega(p) := \left(T_p F\right)^*(\omega(F(p)))$. For $X_1, \ldots, X_k \in T_p M$ we therefore have

$$F^*\omega(p)\left(X_1, \ldots X_k\right) = \omega(F(p))\left(T_p F\left(X_1\right), \ldots, T_p F\left(X_k\right)\right)$$

In particular, $F^* f = f \circ F$ for $f \in \mathcal{C}^\infty(N) = \Omega^0(N)$

We also talked about pushforwards in section IX.3. This notion generally interests us when we want to move on the manifold from one tangent space to the other.
Definition(Pushforward)
Let M and N be smooth manifolds and $F: M \to N$ smooth mapping. Then in every point $p \in M$ we define mapping

$$F_{*,p}: T_p M \to T_{F(p)} N$$

as

$$\left(F_{*,p}\left(\left.X\right|_p\right)\right)(f) \equiv \left.X\right|_p\left(F^* f\right) = \left.X\right|_p(f \circ F)$$

for every $f \in C_{F(p)}^\infty(M)$.

Curvature The main tool for describing curvature is Riemannian curvature tensor. To motivate the definition of the curvature tensor on consider the parallel transport of a vector along a curve. If we parallel transport a vector along a closed curve in the plane then upon returning to the starting point we end up with the same vector as we have started with, however if we parallel transport a vector along a closed curve on the sphere we will end up at a different vector (as depicted on 6). Difference between the starting vector and the final vector can be expressed in terms of the non commutativity of covariant derivatives.

Definition(Riemannian curvature tensor) Riemannian curvature tensor is $R: X(M) \times X(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

In chapter IV we dealt with the notion of foliation. Here is the formal definition.

Definition(Foliation)
By a p-dimensional, class C^r foliation of an m-dimensional manifold M we mean a decomposition of M into a union of disjoint connected subsets $\left\{L_\alpha\right\}_{\alpha \in A}$, called the leaves of the foliation, with the following property: Every point in M has a neighborhood U and a system of local, class C^r coordinates $x = \left(x^1, \cdots, x^m\right): U \to R^m$ such that for each leaf L_α, the components of $U \cap L_\alpha$ are described by the equations $x^{p+1} =$ constant $, \cdots, x^m =$ constant.

We shall denote such a foliation by $F = \left\{L_\alpha\right\}_{\alpha \in A}$ It will often be more natural to refer to the codimension $q = m - p$ of F rather than to its dimension p.

Note that every leaf of F is a p-dimensional, embedded submanifold of M.

Local coordinates with the property mentioned in definition are said to be distinguished by the foliation. If x and y are two such coordinate systems defined in an open set $U \subset M$, then the functions giving the change of coordinates $y_i = y_i\left(x^1, \cdots, x^m\right)$ must satisfy the equations

$$\partial y_i / \partial x_j = 0$$

for $1 \leqq j \leqq p < i \leqq m$ in U. Hence, choosing a covering of M by distinguished local coordinates gives rise to a G-structure on M where $G \subset GL(m, R)$ is the group of matrices with zeros in the lower left $(m-p) \times p$ block. That is, G is the subgroup of $GL(m, R)$ which preserves the linear subspace $\boldsymbol{R}^p = \left\{\left(x^1, \cdots, x^p, 0, \cdots, 0\right)\right\} \subset \boldsymbol{R}^m$.
(this definition and comment are entirely from[21])

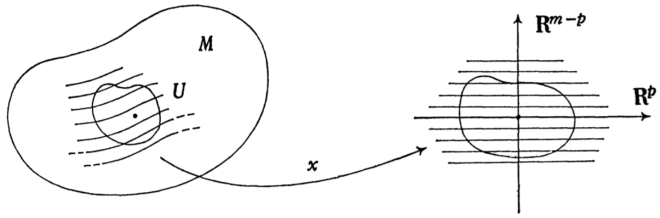

Figure 7. Depict of foliation on some manifold M [21]

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