Tidal Work

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Tidal work, or tidal heating, is an interesting phenomenon in planet-moon systems in which tidal flexing by the planet on a moon pumps energy into the moon, which rises tides and is dissipated as heat. It is the source of geological activity on Jupiter's moon Io, as well as source of the heat required to keep oceans on Jupiter's Europa and Saturn's Enceladus liquid. Here, we use quasilocal and pseudotensor approaches to theoretically assess tidal work.

I. INTRODUCTION

Tides have been an interesting phenomenon to people long before there was an explanation. The aboriginal Yolngu people correctly noticed that the tides were related to the position of the Moon. However, they attributed this to Moon filling with water and emptying out.[1] The first correct explanation for tides came from Newton.[2] Newton attributes tides to the tidal forces.

Tidal forces are apparent forces an observer attached to the center of a body notices in the presence of a non-uniform external gravitational field, e.g. a moon, a parent planet or star. These forces have an apparent effect of stretching the body in one direction and squeezing it in the other.

Numerous effects occur under the stretching and squeezing of tidal forces. Firstly, it causes tides, i.e. two tidal bulges, under the presence of a liquid on the surfaces of the body because the liquid is displaced easily compared to the rigid surface of a body. Secondly, it can create vortexes in the presence of a layer of liquid under the surface of the body which dissipate energy as heat and can produce magnetic fields in the body if the liquid is ferromagnetic, as is the case for Earth. The dissipation into heat also occurs as the body rotates under the tidal bulges and it slows down or speeds up the rotation of the body until tidal locking is achieved, which is a state in which the same side of the moon faces the parent planet all the time. Finally, the tidal forces can be large enough to cause disintegration of the object if the object is at the Roche limit, which created Saturn's rings.[3]

The tidal forces can input or extracts energy from the system and that is called tidal work. In this work we study tidal work as the change of mass-energy of the system due to changes of mass distribution of the object in a non-uniform external gravitational field.

II. PREPARATION

In order to derive an expression for tidal work we will consider isolated body and an external gravitational field, i.e. external universe curvature. This can represent a far-enough-away body and a planet, a star or a black hole. *Isolated* means that the radius of curvature of external universe $\mathcal{R}$ and the length scale on which it changes $\mathcal{L}$ are a lot bigger than the size of the body R. That is, $R/\mathcal{R} \ll 1$ and $R/\mathcal{L} \ll 1$. We will also consider the system to be *slow-moving*, which means that the timescale $\mathcal{T}$ of changes to mass-energy M and current moments is much larger than the body, i.e. $R/\mathcal{T} \ll 1$. We are using natural units such that $G = c = 1$.

Furthermore, to derive an expression for tidal work we examine the rate of change of mass-energy $\mathrm{d}M/\mathrm{d}t$ and identify the term which corresponds to tidal work. In order to do that we use the following expression[4]:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\oint \mathrm{d}^2 S_j \, t^{0j} \tag{1}$$

where $t^{\mu\nu}$ is an energy-momentum pseudotensor of gravitational field and $d^2 S_j = n_j r^2 \mathrm{d}\Omega$ is the surface element of 2-sphere of radius r such that $r/\mathcal{R} \ll 1$, $r/\mathcal{L} \ll 1$ and $M/r \ll 1$. This means that the frame of reference we are using is a local asymptotic rest frame of the isolated body.

Immediately a question is raised. Since $t^{\mu\nu}$ is a pseudotensor, does the choice of mathematics we use to arrive at the expression for (1) impact the expression? In other words, is the expression for $\mathrm{d}M/\mathrm{d}t$ ambiguous? The answer to this question is yes, but we will argue that the expression for the tidal work part of it is unambiguous.

A related problem is also present. Since we are not working with asimptotically flat spacetime here, but only locally asimptotically flat spacetime, the mass-energy does not have a precisely defined value. In fact[4], the value of mass-energy is defined up to $\Delta M \sim \mathcal{Q}_{ij}\mathcal{E}^{ij}$. Here $\mathcal{E}_{ij} = R_{i0j0}$ is the tidal field of external universe and $\mathcal{Q}_{ij}$ is the mass quadrupole moment of the isolated body defined as:

$$\mathcal{Q}_{ij} = \int \mathrm{d}^3 x \, \rho \left(x_i x_j - \frac{1}{3} r^2 \right)$$

Lastly, we will expand the metric g_{ab} in a dimensionless parameter ε around Minkowski metric η_{ab}, with signature convention $(-, +, +, +)$, to get:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \epsilon^2 k_{\alpha\beta} + \mathcal{O}(\varepsilon^3) \tag{2}$$

where h_{ab} contains terms linear in the parameters M, $\mathcal{Q}$ and $\mathcal{E}$ while k_{ab} contains products of any two of them. All parameters will be considered linear in ε.

We present here a derivation of tidal work on the body by utilising Einstein pseudotensor, known also as canonical pseudotensor. It is derived using formula for energy-momentum tensor of a classical field with Lagrangian density L and field variable χ which can be a tensor of any rank[5]:

$$T_\mu{}^\nu = \frac{\partial L}{\partial \partial_\nu \chi} \partial_\mu \chi - L \delta_\mu{}^\nu \tag{3}$$

The variable is the metric tensor g_{ab} and the Lagrangian L is[6]:

$$L = \frac{1}{16\pi} \sqrt{-g}\, g^{\alpha\beta} \left(\Gamma^\gamma_{\alpha\beta} \Gamma^\sigma_{\gamma\sigma} - \Gamma^\sigma_{\alpha\gamma} \Gamma^\gamma_{\beta\sigma} \right)$$

where g is the determinant of the metric.

Therefore equation (3) becomes:[7]

$$\sqrt{-g}\, t_\mu{}^\nu = \frac{\partial L}{\partial \partial_\nu g_{\alpha\beta}} \partial_\mu g_{\alpha\beta} - L \delta_\mu{}^\nu \tag{4}$$

The Einstein pseudotensor is therefore:

$$\sqrt{-g}\, t_\mu{}^\nu = \frac{1}{16\pi} \left[\left(\Gamma^\nu_{\alpha\beta} - \delta_\beta{}^\nu \Gamma^\sigma_{\alpha\sigma} \right) \partial_\mu \left(g^{\alpha\beta} \sqrt{-g} \right) - \delta_\mu{}^\nu L \right]$$

III. CALCULATION

To get the accuracy of ε^2 we need to expand the metric to the order ε because we have $\sqrt{-g}\, g_{ab}$ terms only. We therefore have:

$$\sqrt{-g} = 1 + \frac{1}{2} \varepsilon h + \mathcal{O}(\varepsilon^2) \tag{5}$$

where $h = h^\alpha{}_\alpha$.

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \varepsilon h^{\alpha\beta} \tag{6}$$

We thus have:

$$\begin{aligned} \sqrt{-g}\, t_\mu{}^\nu = & \frac{\varepsilon^2}{16\pi} \Bigg[\frac{1}{2} \partial_\mu h\, \partial_\alpha h^{\nu\alpha} - \frac{1}{4} \partial_\mu h\, \partial^\nu h \\ & - \partial_\mu h^{\alpha\beta}\, \partial_\beta h^\nu{}_\alpha + \frac{1}{2} \partial_\mu h^{\alpha\beta}\, \partial^\nu h_{\alpha\beta} \\ & - \frac{1}{4} \partial_\mu h\, \partial^\nu h + \frac{1}{2} \partial_\alpha h\, \partial_\mu h^{\alpha\nu} - \delta_\mu{}^\nu \Bigg(\frac{1}{2} \partial_\gamma h\, \partial_\alpha h^{\gamma\alpha} \\ & - \frac{1}{4} \partial_\gamma h\, \partial^\gamma h - \frac{1}{2} \partial^\gamma h^{\alpha\sigma} \partial_\sigma h_{\alpha\gamma} + \frac{1}{4} \partial^\gamma h^{\alpha\sigma}\, \partial_\gamma h_{\alpha\sigma} \Bigg) \Bigg] \end{aligned}$$

Now we introduce the de Donder gauge:

$$\partial_\nu h^{\mu\nu} = \frac{1}{2} \partial^\mu h \tag{7}$$

This reduces the equation to:

$$\begin{aligned} \sqrt{-g}\, t_\mu{}^\nu = & \frac{\varepsilon^2}{16\pi} \Bigg[- \partial_\mu h^{\alpha\beta}\, \partial_\beta h^\nu{}_\alpha + \frac{1}{2} \partial_\mu h^{\alpha\beta}\, \partial^\nu h_{\alpha\beta} \\ & - \frac{1}{4} \partial_\mu h\, \partial^\nu h + \frac{1}{2} \partial_\alpha h\, \partial_\mu h^{\alpha\nu} \\ & + \delta_\mu{}^\nu \left(\frac{1}{2} \partial^\gamma h^{\alpha\sigma} \partial_\sigma h_{\alpha\gamma} - \frac{1}{4} \partial^\gamma h^{\alpha\sigma}\, \partial_\gamma h_{\alpha\sigma} \right) \Bigg] \end{aligned}$$

Furthermore, we only need to evaluate the $\sqrt{-g}\, t_0{}^j$ terms. In order to evaluate these terms we need the following identities:[8]

$$h_{00} = 2\frac{M}{r} + 3 \frac{\mathcal{Q}_{ij} x^i x^j}{r^5} - \mathcal{E}_{ij} x^i x^j \tag{8}$$

$$h_{0j} = -2 \frac{\dot{\mathcal{Q}}_{ja} x^a}{r^3} - \frac{10}{21} \dot{\mathcal{E}}_{ab} x^a x^b x_j + \frac{4}{21} \dot{\mathcal{E}}_{ja} x^a r^2 \tag{9}$$

$$h_{ij} = \delta_{ij} \left(2\frac{M}{r} + 3 \frac{\mathcal{Q}_{kl} x^k x^l}{r^5} - \mathcal{E}_{kl} x^k x^l \right) \tag{10}$$

We will ignore all terms except $\mathcal{E}\dot{\mathcal{Q}}$ and $\mathcal{Q}\dot{\mathcal{E}}$ because only they contribute to tidal work. This can easily be seen since the tidal work must arise through coupling of the quadrupole moment and an external tidal field. Dimensional analysis fixes the choice to the two mentioned ones.

First we notice that, because we are calculating the non-diagonal $t_0{}^j$ term, anything proportional to $\delta_\mu{}^\nu$ vanishes. This means that equation for $\sqrt{-g}\, t_0{}^j$ is reduced to:

$$\begin{aligned} \sqrt{-g}\, t_0{}^j = & \frac{\varepsilon^2}{16\pi} \Big(- \partial_0 h^{\alpha\beta}\, \partial_\beta h^j{}_\alpha + \frac{1}{2} \partial_0 h^{\alpha\beta}\, \partial^j h_{\alpha\beta} \\ & - \frac{1}{4} \partial_0 h\, \partial^j h + \frac{1}{2} \partial_\alpha h\, \partial_0 h^{\alpha j} \Big) \end{aligned}$$

Furthermore, because we only consider first time derivatives of $\mathcal{E}$ and $\mathcal{Q}$ with no derivatives of coordinates we see that $\partial_0 h^{0j}$ will be ignored and our equation reduces to:

$$\begin{aligned} \sqrt{-g}\, t_0{}^j = & \frac{\varepsilon^2}{16\pi} \Big(- \partial_0 h^{kl}\, \partial_l h^j{}_k + \frac{1}{2} \partial_0 h^{00}\, \partial^j h_{00} \\ & + \frac{1}{2} \partial_0 h^{kl}\, \partial^j h_{kl} - \frac{1}{4} \partial_0 h\, \partial^j h + \frac{1}{2} \partial_k h\, \partial_0 h^{kj} \Big) \end{aligned}$$

Now, because $h_{ij} \propto \delta_{ij}$ we have a further reduction (we drop the Einstein summation convention):

$$\begin{aligned} \sqrt{-g}\, t_0{}^j = & \frac{\varepsilon^2}{16\pi} \Bigg(- \partial_0 h^{jj}\, \partial_j h^j{}_j + \frac{1}{2} \partial_0 h^{00}\, \partial^j h_{00} \\ & + \sum_{i=1}^{3} \frac{1}{2} \partial_0 h^{ii}\, \partial^j h_{ii} - \frac{1}{4} \partial_0 h\, \partial^j h + \frac{1}{2} \partial_j h\, \partial_0 h^{jj} \Bigg) \end{aligned}$$

Finally, because $h_{jj} = h_{00}$ then $h = 2h_{00}$ and, because we thus only have $\partial_0 h^{00} \partial_j h^{00}$ combinations our equation becomes:

$$\sqrt{-g}\, t_0{}^j = \frac{\varepsilon^2}{16\pi} \partial_0 h_{00} \partial^j h_{00} \tag{11}$$

After some algebra (see Appendix A) we have:

$$\begin{aligned} \sqrt{-g}\, t_0{}^j = - & \frac{\varepsilon^2}{4\pi} \Bigg[\frac{3}{2} \mathcal{Q}^j{}_a \dot{\mathcal{E}}_{bc} \frac{x^a x^b x^c}{r^5} \\ & + \frac{3}{2} \dot{\mathcal{Q}}_{ab} \mathcal{E}^j{}_c \frac{x^a x^b x^c}{r^5} - \frac{15}{4} \mathcal{Q}_{ab} \dot{\mathcal{E}}_{bc} \frac{x^a x^b x^c x^d x^j}{r^7} \Bigg] \end{aligned}$$

Because we already have ε^2 we can set $\sqrt{-g}=1$. Since our choice of region was such that tidal field is constant and multipole moments do not vary spatially we are left with solving the integrals of the form:[9]

$$\oint \mathrm{d}\Omega\, n^{a_1} n^{a_2} \dots n^{a_p} = \frac{4\pi}{2p+1} \delta^{(a_1\, a_2} \dots \delta^{a_{p-1},\, a_p)}$$

Finally, setting $\varepsilon = 1$, we get:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{1}{2}\mathcal{E}_{jk}\frac{\mathrm{d}\mathcal{Q}^{jk}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{10}\mathcal{E}_{jk}\mathcal{Q}^{jk}\right) \tag{12}$$

IV. DISCUSSION

Calculations similar to the presented one are possible for different choices of pseudotensors. For example, a calculation using the Landau-Lifshitz pseudotensor, related to Einstein pseudotensor as:

$$\begin{aligned}(-g)t[LL]^{\mu\nu} =& (-g)g^{\mu\rho}t[E]_\rho{}^\nu \\ &- \frac{1}{16}\frac{g_{\rho\alpha}}{\sqrt{-g}}\left(\sqrt{-g}\, g^{\mu\rho}\right) \times \\ &\partial_\lambda\left[-g\left(g^{\nu\alpha}g^{\sigma\lambda} - g^{\sigma\alpha}g^{\nu\lambda}\right)\right]\end{aligned}$$

yealds:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{1}{2}\mathcal{E}_{jk}\frac{\mathrm{d}\mathcal{Q}^{jk}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{10}\mathcal{E}_{jk}\mathcal{Q}^{jk}\right)$$

We see that different pseudotensors yield different final forms. The question remains, is this a problem? The answer is no, because the interpretations of each of the terms is different.

The first term represents the change in self-energy of the isolated body. That is a physical observable and is therefore unchanged[6] regardless of which pseudotensor we use. It is unambiguous.

The second term represents the change in interaction energy. It is easy to see that, since the only variables in the system are $\mathcal{Q}$, $\mathcal{E}$ and M, the interaction energy must be of the form $\mathcal{Q}_{jk}\mathcal{E}^{jk}$. It is also easy to see why there is no $\mathcal{Q}_{jk}\dot{\mathcal{E}}^{jk}$ term, since tidal force does no work if there is no displacement on the body. The change in the interaction energy is a reversible term, i.e. it is the full derivative in time coordinate. We can think of it as the potential energy which is the reason why its ambiguity is not a problem, the potential energy is always ambiguous.

V. CONCLUSION

We managed to derive an expression for tidal work on an isolated body in a slow-changing external gravitational field. We also made physical arguments as to the derived terms and their physicality and showed that the tidal work part is unambiguous.

However, we only considered slow-moving and isolated body and thus dropped the higher order terms. We leave higher-order approximations for further research. Further research is also needed in systems with quickly-varying fields such as neutron star binaries and mergers and, of course, black holes.

Lastly, experimental methods measuring tidal heating need various models of the body[10] and are presently inconsistent with measurements made on Jupiter's moon Io.[11]

VI. APPENDIX

A. Calculation of $\sqrt{-g}t_0{}^j$

We substitute the h_{00} to begin the calculation:

$$\begin{aligned}\sqrt{-g}\, t_0{}^j &= \frac{\varepsilon^2}{16\pi}\partial_0 h_{00}\partial^j h_{00} \\ &= \frac{\varepsilon^2}{16\pi}\partial_0\left(2\frac{M}{r} + 3\frac{\mathcal{Q}_{ab}x^a x^b}{r^5} - \mathcal{E}_{ab}x^a x^b\right) \times \\ &\times \partial^j\left(2\frac{M}{r} + 3\frac{\mathcal{Q}_{kl}x^k x^l}{r^5} - \mathcal{E}_{kl}x^k x^l\right)\end{aligned}$$

First we ignore the M term and distribute the time derivative to $\mathcal{Q}$ and $\mathcal{E}$ while distributing the spatial derivative to x:

$$\begin{aligned}\sqrt{-g}\, t_0{}^j = \frac{\varepsilon^2}{16\pi}&\left(3\frac{\dot{\mathcal{Q}}_{ab}x^a x^b}{r^5} - \dot{\mathcal{E}}_{ab}x^a x^b\right) \times \\ &\times\left[3\frac{\mathcal{Q}_{kl}}{r^5}\left(\delta^{kj}x^l + \delta^{lj}x^k - 5\frac{x^k x^l x^j}{r^2}\right)\right. \\ &\left. - \mathcal{E}_{kl}\left(\delta^{kj}x^l + \delta^{lj}x^k\right)\right]\end{aligned}$$

Now renaming indexes k and l properly:

$$\begin{aligned}\sqrt{-g}\, t_0{}^j &= \frac{\varepsilon^2}{16\pi}\left(3\frac{\dot{\mathcal{Q}}_{ab}x^a x^b}{r^5} - \dot{\mathcal{E}}_{ab}x^a x^b\right) \times \\ &\times\left[3\frac{\mathcal{Q}_{kl}}{r^5}\left(2\delta^{kj}x^l - 5\frac{x^k x^l x^j}{r^2}\right) - 2\mathcal{E}_{kl}\delta^{kj}x^l\right]\end{aligned}$$

We multiply out but only keep the terms $\dot{\mathcal{Q}}\mathcal{E}$ and $\dot{\mathcal{E}}\mathcal{Q}$, we also contract over j:

$$\begin{aligned}\sqrt{-g}\, t_0{}^j = \frac{\varepsilon^2}{16\pi}&\left(\frac{-6\dot{\mathcal{Q}}_{ab}\mathcal{E}^j{}_l}{r^5}x^a x^b x^l - \frac{6\mathcal{Q}^j{}_l\dot{\mathcal{E}}_{ab}}{r^5}x^a x^b x^l\right. \\ &\left. + \frac{15\mathcal{Q}_{kl}\dot{\mathcal{E}}_{ab}}{r^7}x^a x^b x^k x^l x^j\right)\end{aligned}$$

Finally, proper renaming of k, l, a and b and extracting -4 from the parentheses we get the result.

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