Hopf fibration in physics

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Abstract

In 1931, while studying topological properties of spheres and maps between them, Heinz Hopf discovered a peculiar mapping between S^3 and S^2 which turned out to provide S^3 with a structure of a fibre bundle over S^2. While this seemed far removed from physics at the time, little by little physicists kept stumbling upon this structure lurking behind a wide array of distinct physical phenomena. In this seminar, we will first give a brief exposition of fibre bundles in general and then Hopf fibration in particular. We give its definition, explain how to visualize its fibres using stereographic projection and briefly mention some generalizations. In the second part, we turn to physics. First we summarize how Hopf's ideas about mappings between spheres sparked some ideas about the possible underlying structure of classical electrodynamics and then, using this, we obtain a topologically non-trivial solution of Maxwell's equations based on the Hopf fibration.

I. Introduction

It might seem surprising that such a seemingly extravagant mathematical structure as the Hopf fibration appears in physical context at all, but as is documented in [1] it appears in at least seven different contexts in physics. What is perhaps even stranger, most of these contexts are not related to gauge theory, where most physicists are accustomed at using fibre bundles. It appears, among others, in areas as diverse as general relativity, twistor theory, rigid body dynamics, 3-dimensional harmonic oscillator and even the Dirac monopole etc. But, among those, there is a system - a quantum 2-state system or the qubit - which, by its very simplicity, will serve us well in motivating the usefulness of the Hopf fibration in physics.
It is well known that physical states of the quantum systems correspond to equivalence classes of Hilbert space vectors. For a qubit any state vector can be written in the following form:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{1}$$

where α and β are complex numbers. Therefore, the Hilbert space of a qubit is isomorphic to $\mathbb{C}^2$. But, any state of the form $\lambda |\psi\rangle$ where $\lambda \in \mathbb{C}$ also represents the same state. If we use this equivalence relation to form a quotient space we see that the space of states corresponds to $\mathbb{C}P^1$, which we know is homeomorphic to S^2. So, how is all this related to Hopf fibration?
Let us attack the problem of constructing a state space for a qubit from a slightly different angle by first imposing normalization on our state vector:

$$\langle\psi|\psi\rangle = 1 \Rightarrow |\alpha|^2 + |\beta|^2 = 1 \tag{2}$$

which upon realification means restricting ourselves on subset S^3 of $\mathbb{R}^4$. There is still freedom of multiplication by a complex phase $e^{i\phi}$, but this can be eliminated by passing over to the density operator representation of pure states:

$$\rho = |\psi\rangle \langle\psi| \tag{3}$$

which is positive and satisfies both $\rho^2 = \rho$ and $Tr(\rho) = 1$. The set of physical states of a system is in one-to-one correspondence with the set of these density operators. This density operator can be represented by a matrix in a basis $(|0\rangle, |1\rangle)$ and, like any other 2x2 Hermitean matrix, it can be expanded in a basis consisting of identity and three Pauli matrices ($\mathbb{1}$, $\vec{\sigma}$) as:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{a} \cdot \vec{\sigma}) \tag{4}$$

where, due to $\rho^2 = \rho$ and $Tr(\rho) = 1$, $|\vec{a}| = 1$. We again find that the space of states of a qubit is a sphere S^2 (in this context it is called a Bloch sphere (Figure 1)).

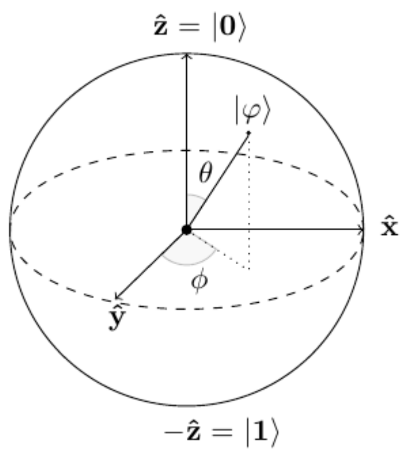

Figure 1: *Bloch sphere representation of quantum states*

But, besides this nice representations of quantum states, what we in fact obtained by this construction is a certain map (normalized $|\psi\rangle \mapsto \rho \mapsto \vec{a} \in S^2$) from the subset of normalized states in $\mathbb{C}^2 \cong \mathbb{R}^4$, namely S^3, to S^2. This map "projects down" circles $S^1 \subset S^3$ consisting of state vectors differing by a phase factor $e^{i\phi} \in U(1)$ to a distinct point on S^2 and therefore endows the S^3 with a fibre bundle structure. Fibration thus produced is, in fact, none other than Hopf fibration.

II. Few remarks about fibre bundles in general

Just as a manifold can be informally defined as a space which locally "looks like" (formally "is homeomorphic to") $\mathbb{R}^n$, so a fibre bundle can be viewed as a space which locally looks like a Cartesian product of two topological spaces. Although definitions vary widely across the literature we will settle for the one found in [2].

Fibre bundle is an ordered quadruple (E,π,B,F) (denoted as: $F \hookrightarrow E \rightarrow B$) consisting of 3 smooth manifolds, total space E, base space B and fibre F with a continuous and surjective projection map $\pi : E \rightarrow B$. Also, for every $b \in B$ there is a neighbourhood O_b and a homeomorphism (so called "local trivialization") $\psi_\beta : \pi^{-1}(O_b) \rightarrow O_b \times F$ which for every $x \in \pi^{-1}(O_b)$ satisfies the following condition:

$$(\pi_1 \circ \psi_\beta)(x) = \pi(x) \tag{5}$$

where $\pi_1((x, y)) = x$ is a canonical projection to the first factor.

This definition is a top-down approach to fibre bundles. We are given a total space E with a projection map π which induces a **fibration** in such a way that E is now viewed as completely made up of distinct fibres, each corresponding to a point in base space B and each homeomorphic to a typical fibre F (for each $b \in B$ $\pi^{-1}(b) \cong F$). A fibre bundle which is globally homeomorphic to a product space $B \times F$ is called **trivial** . Although they are made of "patches" which are trivial, fibre bundles in general are not globally trivial.
The definition above gives us a way of decomposing the total space into a set of patches which have the structure of Cartesian products, but what about the other way around. What if we are given a base space B and a collection of open sets $O_\beta \subset B$? How do we assemble the original fibre bundle from pieces of the form $O_\beta \times F$?
Given a base manifold B with an open cover $\{O_\beta\}$, we assign to each $b \in B$ a homeomorphism $g(b) : F \rightarrow F$ such as to reproduce the effect of the original transition functions $\psi_\alpha \circ \psi_\beta^{-1}|_b$ on F. For each $b \in B$ $g(b)$ belongs to a group G - the so called **structure group** of a bundle. Structure group is not uniquely determined. Any group which contains the structure group as a subgroup is also a structure group. Now, before we patch them together, we form the simple union of local

trivializations:

$$\tilde{E} = \bigcup_{\beta} O_\beta \times F \tag{6}$$

On this space we introduce the following equivalence relation : $(b, f) \sim (b', f') \Leftrightarrow b = b'$ and $g(b)f = f'$. We can now "glue" together the total space E:

$$E = \tilde{E}/\sim \tag{7}$$

which as we defined consists of equivalence classes $[(b, f)]$. The projection map $\pi : E \to B$ is given naturally by $[(b, f)] \to b \in B$. Also, we can give inverses of local trivializations $\psi_\beta^{-1} : O_\beta \times F \to \pi^{-1}(O_\beta)$ by $(b, f) \to [(b, f)]$. A nice illustration of this construction using a Moebius strip (which has a structure group $\mathbb{Z}_2$) can be found in [3].

A **principal fibre bundle** $P(B, G)$ is a fibre bundle $F \hookrightarrow E \to B$ for which the fibre F is identical to the structure group G.

Also, a local **cross section** of a fibre bundle over an open set $O \subset B$ is a continuous mapping $s : O \to E$ which satisfies $\pi \circ s = id_B$ (it maps the point in the base space to the fibre "directly above it"). A cross section is called global if $O = B$. These two notions give us a nice criterion for deciding whether a given fibre bundle is trivial: A fibre bundle is trivial if and only if a principal bundle associated with it admits a global cross section.

III. Hopf fibration

The combination of words "Hopf fibration" should not, strictly speaking, be used in singular. There are actually four different bundles where total space, base space and fibre are all spheres of different dimensions: $S^0 \hookrightarrow S^1 \to \mathbb{R}P^1 \cong S^1$, $S^1 \hookrightarrow S^3 \to \mathbb{C}P^1 \cong S^2$, $S^3 \hookrightarrow S^7 \to \mathbb{H}P^1 \cong S^4$ and $S^7 \hookrightarrow S^{15} \to \mathbb{O}P^1 \cong S^8$, where $\mathbb{H}$ and $\mathbb{O}$ denote the rings of quaternions and octonions, respectively. These are all called Hopf fibrations.

While we will mostly be interested in the second case, let us try to get the general idea by first analyzing briefly the simplest Hopf fibration: $S^0 \hookrightarrow S^1 \to \mathbb{R}P^1 \cong S^1$. $\mathbb{R}P^1$ is just a quotient manifold $\mathbb{R}^2/\sim$ where the equivalence relation is defined as follows:

$$(x, y) \sim (x', y') \leftrightarrow (x', y') = \lambda(x, y) \tag{8}$$

for some $\lambda \neq 0$. As we can see, the total space is just the regular circle:

$$E = S^1 = \{(x, y) | x^2 + y^2 = 1\} \tag{9}$$

We define projection map: $\pi : S^1 \to \mathbb{R}P^1$ by:

$$\pi((x, y)) = [(x, y)] \in \mathbb{R}P^1 \tag{10}$$

From this we can immediately find the fibre:

$$F = \pi^{-1}([(x, y)]) = \{(x, y), (-x, -y)\} \tag{11}$$

The fibre is obviously homeomorphic to $\{-1, 1\}$ which is in fact the zero-sphere S^0. Now that we've shown that such a fibration of S^1 is possible, let's see if we can see how the S^1 is patched together (when viewed as a total space of a fibre bundle). First we give an open cover of the base space $\mathbb{R}P^1$:

$$O_1 = \{[(x, y)] \in \mathbb{R}P^1 | x \neq 0\} \tag{12}$$

and

$$O_2 = \{[(x, y)] \in \mathbb{R}P^1 | y \neq 0\} \tag{13}$$

On these open sets we introduce coordinates $\phi_i : O_i \to \mathbb{R}$ by following definitions:

$$\phi_1([(x, y)]) = \frac{y}{x} \tag{14}$$

and

$$\phi_2([(x, y)]) = \frac{x}{y} \tag{15}$$

We can now use these coordinates to define local trivializations $\psi_i : \pi^{-1}(O_i) \to \mathbb{R} \times F$ (where we've used $\mathbb{R}$ instead of O_i because they are homeomorphic):

$$\psi_1((x, y)) = \left(\frac{y}{x}, sgn(x)\right) \tag{16}$$

and

$$\psi_2((x, y)) = \left(\frac{x}{y}, sgn(y)\right) \tag{17}$$

To get the structure group, all we have to do is to compute the transition function $\psi_1 \circ \psi_2^{-1}$.

For the sake of brevity, we skip the computation and only give the result:

$$\psi_1 \circ \psi_2^{-1}(u,v) = \left(\frac{1}{u}, sgn(u)v\right) \qquad (18)$$

where $u \in \mathbb{R} - \{0\}$ and $v \in \{-1, 1\}$. On the fibre we have the following mapping:

$$v \longmapsto sgn(u) \cdot v \qquad (19)$$

It is clear now that the structure group is in this case $\mathbb{Z}_2$, which, as it happens to be, is homeomorphic to fibre $F = S^0$ so that $S^0 \hookrightarrow S^1 \to \mathbb{R}P^1 \cong S^1$ is, in fact, a principal bundle $P(S^1, \mathbb{Z}_2)$.

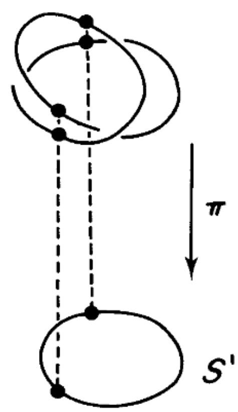

Figure 2: *A visual representation of $S^0 \hookrightarrow S^1 \to \mathbb{R}P^1 \cong S^1$. Taken from [3].*

A way to visually display this fibration is given in Figure 2. A circle is patched together by taking two circles $S^1 \times \{-1, 1\}$, "cutting" each of them at a single point and gluing them together using the procedure sketched in the previous section (the effect of $\mathbb{Z}_2$ can be seen in the way the upper and lower circle are glued together). That this bundle is non-trivial can be seen by the obvious impossibility of giving a global cross-section (continuity must be violated if we are to preserve single-valuedness).

Now comes the time to entertain ourseleves with the Hopf fibration that is of interest to us in this seminar, namely $S^1 \hookrightarrow S^3 \to \mathbb{C}P^1 \cong S^2$. A schematic representation of this fibration is given in Figure 3.

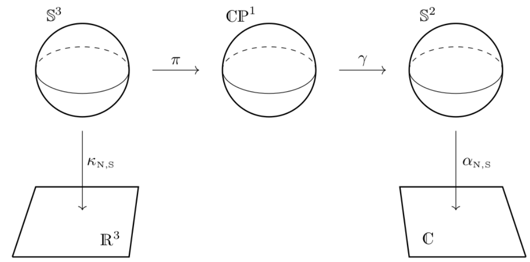

Figure 3: *Schematic representation of Hopf fibration. Taken from [2].*

The total space in this case is a 3-sphere S^3 defined by:

$$S^3 = \{(x, y, u, v) \in \mathbb{R}^4 | x^2 + y^2 + u^2 + v^2 = 1\} \qquad (20)$$

Although it is canonically defined as a subset of $\mathbb{R}^4$, it will be more convenient for our purposes to define it as a subset of $\mathbb{C}^2$:

$$S^3 = \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 = 1\} \qquad (21)$$

where $z = x + iy$ and $w = u + iv$. We now define the projection map $\pi : S^3 \to \mathbb{C}P^1$ as follows:

$$\pi((z, w)) = [(z, w)] \in \mathbb{C}P^1 \qquad (22)$$

which is none other tnan the projection of an element on S^3 on the corresponding equivalence class in $\mathbb{C}P^1 = \mathbb{C}^2/\sim$ with the equivalence relation:

$$(z, w) \sim (z', w') \Leftrightarrow (z', w') = \lambda(z, w) \qquad (23)$$

where $\lambda \in \mathbb{C}^x$. The fibre F is given, as usual, by the inverse image of the point in the base space under projection $F = \pi^{-1}([(z, w)])$. This corresponds to all the points on the 3-sphere which belong to the same equivalence class i.e. all the points on S^3 related to (z, w) by:

$$(z', w') = \lambda(z, w) \qquad (24)$$

Since both (z, w) and (z', w') belong to S^3 we have the following relation:

$$1 = |z'|^2 + |w'|^2 = |\lambda|^2(|z|^2 + |w|^2) = |\lambda|^2 \qquad (25)$$

$$\Rightarrow (z', w') = (e^{i\phi} z, e^{i\phi} w) \tag{26}$$

which implies that all the points (z', w') that get "projected down" to $[(z, w)]$ form a circle S^1. Therefore, $F = S^1$.
The structure group of $S^1 \hookrightarrow S^3 \to \mathbb{C}P^1 \cong S^2$ can be found by looking at transition functions between locally trivial patches. As usual we give an open cover of the base space $\mathbb{C}P^1$ in a way analogous to what we did in the example $S^0 \hookrightarrow S^1 \to \mathbb{R}P^1 \cong S^1$.

$$O_1 = \{[(z, w)] \in \mathbb{C}P^1 | z \neq 0\} \tag{27}$$

and

$$O_2 = \{[(z, w)] \in \mathbb{C}P^1 | w \neq 0\} \tag{28}$$

Local trivializations $\psi_i : \pi^{-1}(O_i) \to O_i \times F$ are given by:

$$\psi_1((z, w)) = \left(\frac{w}{z}, \frac{z}{|z|} \right) \tag{29}$$

and

$$\psi_2((z, w)) = \left(\frac{z}{w}, \frac{w}{|w|} \right) \tag{30}$$

where we again used coordinates $\frac{w}{z}$ and $\frac{z}{w}$ to label classes $[(z, w)]$ on O_1 and O_2, respectively. Note that $\frac{z}{|z|}$ can be written as a complex phase $e^{i\phi_z}$ as it belongs to a fibre S^1. To find the transition function $\psi_1 \circ \psi_2^{-1}$ we need to compute ψ_2^{-1}. If we label $\frac{z}{w}$ by r and $\frac{z}{|z|}$ by s, we have the following relations:

$$1 + |r|^2 = \frac{1}{|w|^2} \tag{31}$$

and

$$w = s|w| \quad z = rw \tag{32}$$

from which follows:

$$\psi_2^{-1}(r, s) = \left(\frac{rs}{\sqrt{1 + |r|^2}}, \frac{s}{\sqrt{1 + |r|^2}} \right) \tag{33}$$

Finally, transition function is given by:

$$\psi_1 \circ \psi_2^{-1}(r, s) = \left(\frac{1}{r}, \frac{r}{|r|} s \right) \tag{34}$$

The action on a fibre is obviously given by a multiplication by a complex phase: $s \to e^{i\phi} s$. Therefore, a structure group of $S^1 \hookrightarrow S^3 \to \mathbb{C}P^1 \cong S^2$ is a group $U(1)$ which is homeomorphic to S^1 and we once again find that the structure group is equal to the fibre. It follows then:

$$S^1 \hookrightarrow S^3 \to \mathbb{C}P^1 \cong P(S^2, U(1) \cong S^1) \tag{35}$$

IV. Visualizing Hopf fibration

In order to visualize the Hopf fibration we must find a way to "see" the total space S^3. Already in Figure 3 it is sketched how we might accomplish this. We shall use stereographic projections to identify the spheres S^3 and S^2 with the compactified 3-space $\mathbb{R}^3 \bigcup \{\infty\}$ and the compactified complex plane $\mathbb{C} \bigcup \{\infty\}$, respectively (the latter projection is not really necessary for visualization purposes, but we introduce it anyway for future convenience). Only then will we be able to see how the fibres are arranged and how they link with each other to form a (projected) total space.

First, we construct the mapping between $\mathbb{C}P^1$ and S^2. Combining the local chart $\psi_2 : O_2 \to \mathbb{C}$:

$$\psi_2([(z, w)]) = \frac{z}{w}, \quad w \neq 0 \tag{36}$$

with the inverse of a stereographic projection from the north pole $\alpha_N^{-1} : \mathbb{C} \to S^2$ defined by:

$$\alpha_N^{-1}(x + iy) = (2\sigma x, 2\sigma y, \sigma(x^2 + y^2 - 1)) \tag{37}$$

where we introduced the shorthand σ:

$$\sigma = \frac{1}{x^2 + y^2 + 1} \tag{38}$$

we obtain the mapping $\gamma : O_2 \to S^2$ as $\gamma = \alpha_N^{-1} \circ \psi_2$ whose action is given by:

$$\gamma([(z, w)]) = (2\xi Re(\bar{z}w), 2\xi Im(\bar{z}w), \xi(|z|^2 - |w|^2)) \tag{39}$$

where we again introduced the shorthand ξ to stand for:

$$\xi = \frac{1}{|z|^2 + |w|^2} \tag{40}$$

This mapping can be extended to the whole of $\mathbb{C}P^1$ by defining γ to map $[z, 0]$ to the north pole which represents the "point at infinity". Since $|z|^2 + |w|^2 = 1$ for points (z, w) on the S^3 it follows:

$$(\gamma \circ \pi)(z, w) = (2Re(\bar{z}w), 2Im(\bar{z}w, |z|^2 - |w|^2) \tag{41}$$

Combining this with the stereographic projection α_N whose inverse we defined above and an inverse projection $\kappa_N^{-1} : \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$ which is defined as follows:

$$\kappa_N^{-1}(\vec{x}) = \left(\frac{2\vec{x}}{|\vec{x}|^2 + 1}, \frac{|\vec{x}|^2 - 1}{|\vec{x}|^2 + 1} \right) \tag{42}$$

we obtain $\phi = \alpha_N \circ \gamma \circ \pi \circ \kappa_N^{-1} : \mathbb{R}^3 \to \mathbb{C}$ whose action is given by:

$$\phi(\vec{x}) = \frac{2(x_1 + ix_2)}{2x_3 + i(|\vec{x}|^2 - 1)} \tag{43}$$

where we used $\vec{x}$ to label elements of $\mathbb{R}^3$ and written each mapping in terms of real components. This map is just what we wanted - a "projected projection" which allows us to "fibrate" $\mathbb{R}^3 \bigcup \{\infty\}$.

Before we discuss the full-blown structure of fibres in the Hopf fibration, let us say a few words about its building blocks. Each pair of S^1 fibres in the Hopf fibration forms the so-called **Hopf link** (Figure 4).

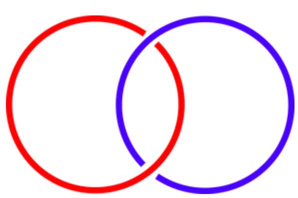

Figure 4: *A Hopf link*

Informally, a **link** is an assembly of knots with mutual entanglements, where a **knot** is just a smooth embedding of a circle S^1 in $\mathbb{R}$. Each link is characterized by the so called **linking number** which, in loose terms, is none other than a number of times one component of a link intersects an oriented surface spanned by another component of a link. A Hopf link, obviously, has a linking number 1. Stated in more computational terms, the linking number for the two closed curves $\vec{c_1}(t)$ and $\vec{c_2}(t)$ in $\mathbb{R}^3$ can be computed using the Gauss linking integral:

$$L_{c_1 c_2} = \frac{1}{4\pi} \int dt_1 dt_2 \frac{d\vec{c_1}}{dt_1} \cdot \frac{\vec{c_1} - \vec{c_2}}{|\vec{c_1} - \vec{c_2}|^3} \times \frac{d\vec{c_2}}{dt_2} \tag{44}$$

Note that this integral can be computed for the case $c_1(t) = c_2(t) = c(t)$. In such a case it gives the measure of knottedness (self-linking) of a curve.

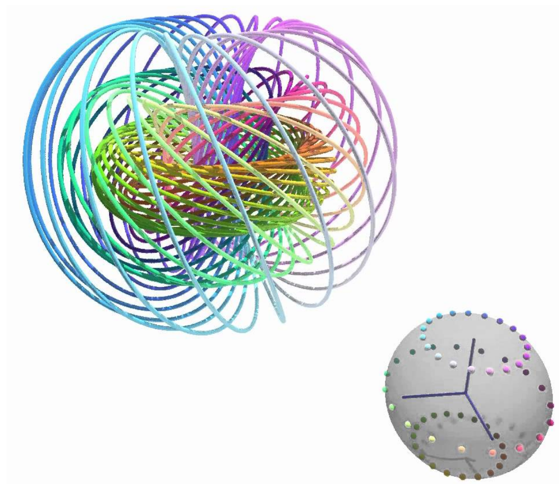

Figure 5: *A graphic representation of fibres corresponding to points on S^2 aligned along the "parallels".*

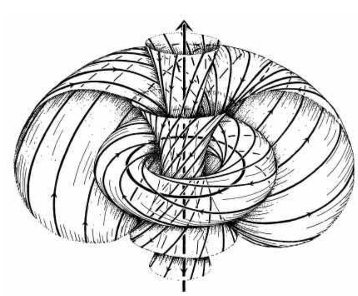

Figure 6: *A display of the way in which fibres are grouped in toruses which subsequently fill the $\mathbb{R}^3 \bigcup \{\infty\}$.*

In Figure 5 and Figure 6 we see the way distinct fibres corresponding to points arranged along parallels of S^2 form the nested tori which ultimately fill the entire $\mathbb{R}^3 \bigcup \{\infty\}$.

Each tori is made up of circles (fibres) in such a way that no two circles cross and each circle is linked to every other one. Each circle in such a configuration wraps once around each circumference of the torus. By nesting such tori into one another, the whole of three dimensional space, including the point at $r = \infty$ ($\mathbb{R}^3 \bigcup \{\infty\} \cong S^3$) can be filled with linked circles. There are two 'special' fibres: the circle of unit radius that corresponds to the infinitely thin torus, and the straight line, or circle of infinite radius, that corresponds to an infinitely large torus. The Hopf map maps such circles in $\mathbb{R}^3 \bigcup \{\infty\} \cong S^3$ to points on the sphere $S^2 \cong \mathbb{C} \bigcup \{\infty\}$. Each circle is projected to a point, each torus onto a (parallel) circle on S^2. The circular and straight special fibres are mapped to the north and south pole, respectively.

V. Topological electromagnetism

Even tough we are primarily interested in the construction of the solution of the Maxwell's equations based on the Hopf fibration - the so called Hopf-Ranada solution - in this section we are going to sketch the broader context in which such a solution was first found by Ranada [9].

While investigating the connection between topology and electromagnetism, Ranada discovered that a subset of radiation solutions (those which satisfy $\vec{E} \cdot \vec{B} = 0$) to Maxwell's equations in vacuum is associated with the set of smooth maps $S^3 \longmapsto S^2$. He named these solutions **admissible fields**. Additionally, he showed that any radiation field is locally equal to an admissible field and can therefore be patched together from such fields. These discoveries encouraged Ranada to put forth the idea that standard electromagnetisam is just the "linearized" (not an approximate) version of an underlying topological theory based on mappings $S^3 \longmapsto S^2$. This set of ideas he provisionally called **topological electromagnetism**. Let us sketch briefly the train of thought involved here.

Consider a complex scalar field $\phi(\vec{r})$ with a well defined value as $r \to \infty$ (limit does not depend on the direction). Such a field $\phi : \mathbb{R}^3 \to \mathbb{C}$ can, by identfying $\mathbb{R}^3 \bigcup \{\infty\}$ with S^3 and $\mathbb{C} \bigcup \{\infty\}$ with S^2 via stereographic projection, be viewed as a map $\phi : S^3 \to S^2$. In this way, S^3 comes to represent the compactified physical 3-space with only one "point at infinity".

Given a map $\phi : S^3 \to S^2$, the inverse images $\phi^{-1}(a)$ and $\phi^{-1}(b)$ of any two points $a, b \in S^2$ are, in general, two closed curves in S^3 and therefore they form what we previously defined as a link. Its linking number, as we defined it above, is just equal to the number of times one curve intersects the oriented surface bounded by another one, where intersections in the negative sense are counted with a minus sign (Figure 7).

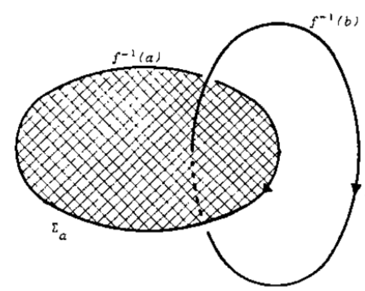

Figure 7: *The curves $\phi^{-1}(a)$ and $\phi^{-1}(b)$ (here denoted as $f^{-1}(a)$ and $f^{-1}(b)$) forming, in this case, a Hopf link with a linking number 1. Taken from [8].*

This linking number does not depend on the particular pair of points $a, b \in S^2$ chosen, since by moving them continuously to any other pair $a', b' \in S^2$ the inverse images can neither untie or tie further since, for this to occur, they would have to intersect and that would mean that the same point in the domain is mapped to two distinct points in the codomain. For the same reason, if the map ϕ evolves continuously in time, the linking number does not depend on time either. This gives us a convenient way to classify the maps $\phi : S^3 \to S^2$ into homotopy classes, labeled by this linking number, which, in this context, is called the

Hopf index and labeled as γ.
So, what does all this has to do with electromagnetic fields? Although Ranada in [8] seems somewhat unclear at certain points about what he is trying to accomplish, the bottom line is this. He managed to show that given two maps $\phi, \theta : S^3 \to S^2$ he could, by pulling back with ϕ and θ the normalized area 2-form from S^2, which in stereographic coordinates $(z, \bar{z})$ is given by (see the next section for details):

$$\epsilon = \frac{1}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 + \bar{z}z)^2} \tag{45}$$

and adding a time dimension (because of covariance), construct two 2-forms, which when written in terms of components are given by:

$$\Phi_{\mu\nu} = \frac{1}{2\pi i} \frac{\partial_\mu \bar{\phi} \partial_\nu \phi - \partial_\nu \bar{\phi} \partial_\mu \phi}{(1 + \bar{\phi}\phi)^2} \tag{46}$$

$$\Theta_{\mu\nu} = \frac{1}{2\pi i} \frac{\partial_\mu \bar{\theta} \partial_\nu \theta - \partial_\nu \bar{\theta} \partial_\mu \theta}{(1 + \bar{\theta}\theta)^2} \tag{47}$$

Furthermore, by requiring that Θ and Φ be dual to each other:

$$\Phi_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \Theta^{\rho\sigma} \quad \Theta_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \Phi^{\rho\sigma} \tag{48}$$

he ensured that both Φ and Θ satisfy Maxwell's equations:

$$\partial_\mu \Phi^{\mu\nu} = 0 \quad \partial_\mu \Theta^{\mu\nu} = 0 \tag{49}$$

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu \Phi_{\rho\sigma} = 0 \quad \epsilon^{\mu\nu\rho\sigma} \partial_\nu \Theta_{\rho\sigma} = 0 \tag{50}$$

The second pair is satsified automatically by virtue of equations (46) and (47). The first pair is satisfied by virtue of the second pair and the duality conditions (48). The only thing left to ensure is that if Θ and Φ start out as dual, they remain so for all time. This is accomplished by defining the lagrangian of the theory to have the same form as in standard electodynamics:

$$L = -\frac{1}{8} (\Phi_{\mu\nu} \Phi^{\mu\nu} + \Theta_{\mu\nu} \Theta^{\mu\nu}) \tag{51}$$

Equations of motions are obtained by varying this lagrangian subject to a constraint of duality (48) using the method of Lagrange multipliers. What these considerations show is that there is a formal relation between mappings $S^3 \longmapsto S^2$ and electromagnetic fields. Φ seems to represent the electromagnetic 2-form F and Θ its dual $*F$ (strictly speaking, in making this identification we should insert a factor $\sqrt{a}$ where a is an action constant intended to convert the expressions to physical units). If we take this identification seriously, electric and magnetic one forms can be extracted in the usual way using contraction:

$$E = -i_u \Phi \quad B = i_u \Theta \tag{52}$$

which when written in terms of components in the rest frame of the observer ($u^\mu = (1, 0, 0, 0)$) has the form:

$$E_i = -\Phi_{0i} \quad B_i = \Theta_{0i} = \frac{1}{2} \epsilon_{ijk} \Phi^{jk} \tag{53}$$

The fields obtained in this way can be shown to be of the form:

$$\vec{B} = -g(\phi, \bar{\phi}) \nabla \bar{\phi} \times \nabla \phi \tag{54}$$

$$\vec{E} = -g(\theta, \bar{\theta}) \nabla \bar{\theta} \times \nabla \theta \tag{55}$$

Let us now summarize what we have obtained:

(1) Maps from S^3 to S^2 can be classified into homotopy classes which can be labeled by a Hopf index (linking number of the level curves) $\gamma = 0, \pm 1, \pm 2, ...$
(2) Corresponding to the pair of functions $\phi, \theta : S^3 \to S^2$, which are not independent because of the duality condition and for that reason have orthogonal level curves, corresponds a radiation field given by an electromagnetic 2-form in (46). Resulting electric and magnetic fields are tangent to the level curves of ϕ and θ, respectively (and therefore satisfy, as required, the condition $\vec{E} \cdot \vec{B} = 0$).
(3) The "admissible fields" obtained this way can therefore also be classified according to the Hopf index $\gamma = \pm 1, \pm 2, ...$ ($\gamma = 0$ case is forbidden because it corresponds to the field with the zero energy density).
(4) Since electric and magnetic fields are tangent to the level curves of ϕ and θ, a solution with $\gamma = n$ corresponds to a field in which

two distinct electric/magnetic field lines are interlinked with each other with a linking number γ. This γ is a topological constant of the motion.
(5) Equations of motion are highly nonlinear at the level of ϕ and θ; however they can be transformed exactly into linear Maxwell's equations for $\vec{E}$ and $\vec{B}$ (there is no truncation involved). For details see [10].

As we presented it, this seems as "merely" a way to generate topologically non-trivial solutions to Maxwell's equations, but these ideas may have a wider scope. It can be shown that every radiation field is locally identical to an admissible field (for a precise sense of this statement see [8]). Therefore, electromagnetism can be seen to emerge from an underlying topological non-linear theory that differs only globally from standard theory. There are many difficulties with this, but there is much to recommend it, too. Prime example of this is charge quantization, which emerges in this theory from simple topological considerations.

VI. Hopf-Ranada solution

In the last section we sketched (without delving into too much detail) how a topologically non-trivial solution of Maxwell's equations can be constructed which has an arbitrary linking number $\gamma = \pm 1, \pm 2, \ldots$. We've shown that such solutions are based on maps $S^3 \longmapsto S^2$, where S^3 is meant to represent the compactified physical 3-space and S^2 the compactified complex plane. In this section we detail the construction of perhaps the simplest instance of such solutions based on the projecton map in the Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow S^2$. Since fibres (level curves of the projection map) are in this case linked once, that will also be the case with the electric and magnetic field lines.
To construct such a solution we must use the pullback of a volume form on S^2. Given a m-manifold (M, g_{ab}) with metric g_{ab}, a **volume form** $\epsilon \in \Omega^m(M)$ can be given using a coordinate induced basis as:

$$\epsilon = \sqrt{|g|} \quad dx^1 \wedge \ldots . \wedge dx^m \tag{56}$$

where g is a metric determinant. In spherical polar coordinates (θ, ϕ), a normalized volume form (volume form divided by the area of the sphere) is given by:

$$\tilde{\epsilon} = \frac{1}{4\pi} \sin\theta d\theta \wedge d\phi \tag{57}$$

Before we can pull this form back on the physical space using our "projected projection" $\phi = \alpha_N \circ \gamma \circ \pi \circ \kappa_N^{-1} : \mathbb{R}^3 \rightarrow \mathbb{C}$ introduced earlier, we must first pull this form to a complex plane using the inverse stereographic projection α_N^{-1} defined earlier in (37).

First, we express the volume form using the cartesian coordinates (x, y, z) of the space $\mathbb{R}^3$ in which the sphere is embedded. Using $x^2 + y^2 + z^2 = 1$ and $(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ we obtain:

$$\tilde{\epsilon} = \frac{dx \wedge dy}{4\pi z} \tag{58}$$

Now, we can do the pullback:

$$\epsilon = (\alpha_N^{-1})^* \tilde{\epsilon} \in \Omega^2(\mathbb{C}) \tag{59}$$

where, for the sake of clarity, we repeat the definition of α_N^{-1}:

$$\alpha_N^{-1}(a + ib) = \Big(2\sigma a, 2\sigma, \sigma(a^2 + b^2 - 1)\Big) \tag{60}$$

where:

$$\sigma = \frac{1}{a^2 + b^2 + 1} \tag{61}$$

Using the properties of pullback we get:

$$\Rightarrow \epsilon = \frac{d(x \circ \alpha_N^{-1}) \wedge d(y \circ \alpha_N^{-1})}{4\pi (z \circ \alpha_N^{-1})} \tag{62}$$

which, after expanding each one-form simplifies to:

$$\epsilon = -\frac{1}{\pi} \frac{da \wedge db}{(a^2 + b^2 + 1)^2} \tag{63}$$

Now we express this in terms of $z = a + ib$ and $\bar{z} = a - ib$ by first noting:

$$dz \wedge d\bar{z} = (da + idb) \wedge (da - idb) \tag{64}$$

$$\Rightarrow dz \wedge d\bar{z} = -2ida \wedge db \tag{65}$$

It now follows:

$$\epsilon = \frac{1}{2\pi i}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \in \Omega^2(\mathbb{C}) \qquad (66)$$

As prescribed in the previous section we now use our Hopf projection map $\phi = \alpha_N \circ \gamma \circ \pi \circ \kappa_N^{-1} : \mathbb{R}^3 \to \mathbb{C}$ whose action is given by:

$$\phi(\vec{x}) = \frac{2(x_1 + ix_2)}{2x_3 + i(|\vec{x}|^2 - 1)} \qquad (67)$$

to construct our electromagnetic 2-form on the physical space:

$$F = -\sqrt{a}\phi^*\epsilon = -\frac{\sqrt{a}}{2\pi i}\frac{d(z\circ\phi)\wedge d(\bar{z}\circ\phi)}{(1+|z\circ\phi|^2)^2} \qquad (68)$$

where we've introduced a factor $\sqrt{a}$ for dimensional reasons.

$$\Rightarrow F = -\frac{\sqrt{a}}{2\pi i}\frac{d\phi\wedge d\bar{\phi}}{(1+|\phi|^2)^2} \qquad (69)$$

The "dual" map $\theta : \mathbb{R}^3 \bigcup\{\infty\} \to \mathbb{C}\bigcup\{\infty\}$ with orthogonal level lines which "generates" $*F$ turns out to be given by a cyclic permutution of coordinates in ϕ:

$$\theta(\vec{x}) = \frac{2(x_2 + ix_3)}{2x_1 + i(|\vec{x}|^2 - 1)} \qquad (70)$$

$$\Rightarrow *F = \sqrt{a}\theta^*\epsilon = \frac{\sqrt{a}}{2\pi i}\frac{d\theta\wedge d\bar{\theta}}{(1+|\theta|^2)^2} \qquad (71)$$

It's not too hard to check that this expression satisfies Maxwell's equations $dF = 0$ and $d*F = 0$. Using the commutativity of exterior derivative and pullback we get:

$$dF = d(-\sqrt{a}\phi^*\epsilon) = -\sqrt{a}\phi^*(d\epsilon) = 0 \qquad (72)$$

$$d*F = d(\sqrt{a}\theta^*\epsilon) = -\sqrt{a}\theta^*(d\epsilon) = 0 \qquad (73)$$

where the last equality in both equations follows from the fact that $d\epsilon = 0$, since $d\epsilon$ is a 3-form on a 2-dimensional space, which makes it automatically zero. Using $d\phi = \partial_\mu\phi dx^\mu$ and $(\alpha\wedge\beta)_{\mu\nu} = \alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu$ we can express the components of F as:

$$F_{\mu\nu} = -\frac{\sqrt{a}}{2\pi i}\frac{\partial_\mu\phi\partial_\nu\bar{\phi} - \partial_\nu\phi\partial_\mu\bar{\phi}}{(1+|\phi|^2)^2} \qquad (74)$$

For reasons of convenience and to be consistent with the notation of the last section, we abbreviate this expression by defining:

$$g(\phi,\bar{\phi}) = -\frac{\sqrt{a}}{2\pi i}\frac{1}{(1+|\phi|^2)^2} \qquad (75)$$

Thus, we can now write:

$$F_{\mu\nu} = g(\phi,\bar{\phi})(\partial_\mu\phi\partial_\nu\bar{\phi} - \partial_\nu\phi\partial_\mu\bar{\phi}) \qquad (76)$$

Since we are interested in the field lines, we must extract $\vec{E}$ and $\vec{B}$ fields from an electromagnetic 2-form (and its dual). We already showed how to do this in the previous section. To avoid having to take time derivatives (since, so far, we've said nothing about time dependence of ϕ and θ) we utilize following formulas.

$$B_i = \frac{1}{2}\epsilon_{ijk}F^{jk} \quad E_i = -\frac{1}{2}\epsilon_{ijk}(*F)^{jk} \qquad (77)$$

Using the expressions for F and $*F$ and the relation $(\vec{a}\times\vec{b})^i = \epsilon^i_{kl}a^k b^l$ we find as expected:

$$\vec{B} = -g(\phi,\bar{\phi})\nabla\bar{\phi}\times\nabla\phi \qquad (78)$$

$$\vec{E} = -g(\theta,\bar{\theta})\nabla\bar{\theta}\times\nabla\phi \qquad (79)$$

Finally, by expressing everything in terms of dimensionless spacetime coordinates (T, X, Y, Z) related to the physical ones by a factor λ we can compute the expressions (78) and (79) to find the Hopf-Ranada fields:

$$\vec{B}(\vec{r},0) = \frac{4\sqrt{a}\lambda^2}{\pi(1+|\vec{r}|^2)^3}[2(Y-XZ)\hat{e}_x - \\ -2(X+YZ)\hat{e}_y + (X^2+Y^2-Z^2-1)\hat{e}_z] \qquad (80)$$

$$\vec{E}(\vec{r},0) = \frac{4\sqrt{a}\lambda^2}{\pi(1+|\vec{r}|^2)^3}[(1+X^2-Y^2-Z^2)\hat{e}_x + \\ +2(XY-Z)\hat{e}_y + 2(Y+XZ)\hat{e}_z] \qquad (81)$$

VII. Time evolution and conserved quantities

We close this treatment by making a few sketchy remarks about the physical aspects of these solutions. The fields (80) and (81) we found represent the intial time values of a Cauchy problem. Despite the fact that the evolution of a function ϕ which "generates" our fields is governed by a highly non-linear equation stemming from lagrangian (51), the evolution of fields can be found fairly painlessly by Fourier decomposition. We will not go through the procedure here (the derivation can be found in [11]). It suffices for our purposes to note that the time dependent functions ϕ, θ can be written in closed form (using our dimensionless coordinates):

$$\phi(\vec{r}, t) = \frac{(AX - TZ) + i(AY + T(A-1))}{(AX + TZ) + i(A(A-1) - TY)} \tag{82}$$

$$\theta(\vec{r}, t) = \frac{(AY + T(A-1)) + i(AX + TZ)}{(AX - TZ) + i(A(A-1) - TY)} \tag{83}$$

where $A = \frac{1}{2}(X^2 + Y^2 + Z^2 - T^2 + 1)$. This time evolution is depicted in Figure 8. Both magnetic and electric field lines are pictured. At time t = 0, the Hopf fibration can be clearly recognized. Special straight fibres corresponding to the south pole of S^2 under projection are aligned along the x and y axes and their cross product determines the direction of propagation. The same sets of fibres are shown at times t = 0.5 and 1. The fibration can be seen to locally rotate about the z axis as well as expanding and deforming, with the structure remaining centred on the centre of energy of the knot. The rotation and deformation seen at these times slows down at subsequent times. The electric and magnetic field lines remain orthogonal at all times as required by construction.

Noether currents corresponding to spacetime symmetries can also be computed. The evolution of the energy density $\frac{1}{2}(\vec{E}^2 + \vec{B}^2)$ is given in the Figure 9.

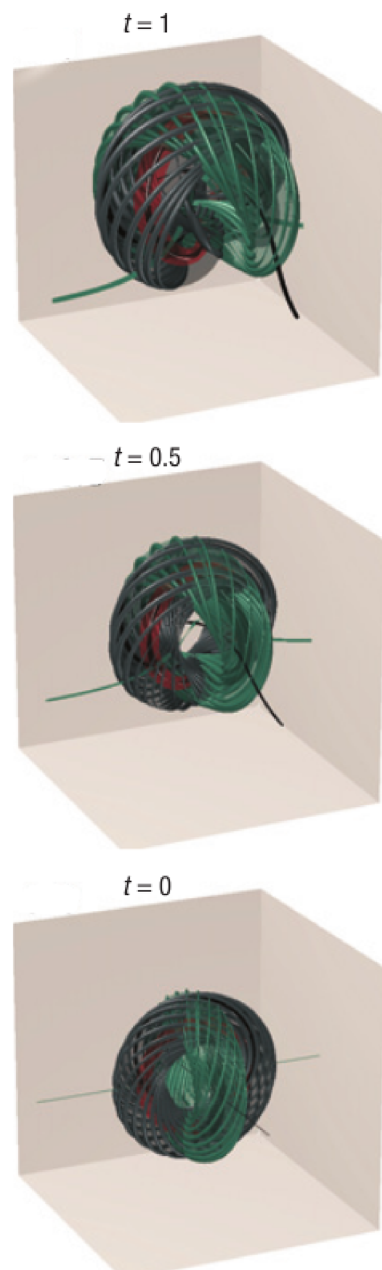

Figure 8: *Time evolution of Hopf-Ranada field. Taken from [7].*

As we can see, it is initially spherical and then subsequently propagates along the z-axis and expands like an "opening umbrella". When rescaled by the energy density, momentum $\vec{E} \times \vec{B}$ and angular momentm density

$\vec{r} \times (\vec{E} \times \vec{B})$ can expressed as:

$$\vec{P} = \left(0, 0, \frac{1}{2}\right) \quad \vec{L} = \left(0, 0, \frac{1}{2}\right) \tag{84}$$

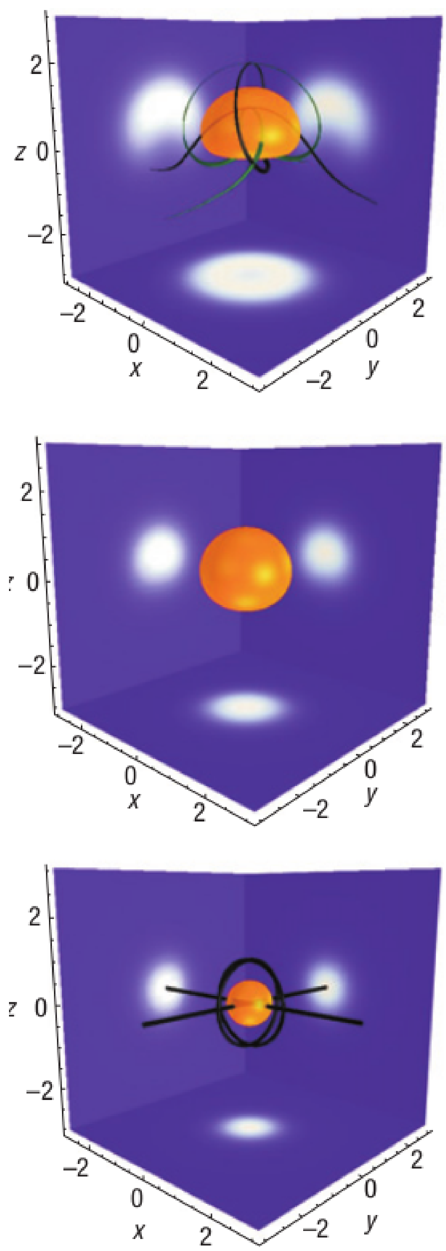

Figure 9: *Time evolution of the energy density. Taken from [7].*

But among conserved quantities, there is one which is perhaps the most interesting to us, given the topological nature of our solution. It is called helicity, but it is not to be confused generally with the helicity as used in elementary particle physics which measures the projection of spin along the axis of motion, although in this case it can be related to it (see [11]). What concerns us here is the connection between topological and physical properties.

In section V we made a point that if our field lines had a linking number (Hopf index) γ, and if the functions ϕ and θ evolved continuously, then the linking number would stay the same for all times (since the field lines could neither untie nor tie further without violating single-valuedness of ϕ and θ). Therefore, perhaps unsurprisingly, Hopf index is a (topological) constant of the motion and consequently, fields that start out in one homotopy class remain in it forever. But, remarkably, this constant can be shown to be equal (up to a dimensional constant) as an integral over physical field variables, namely **helicity**.

We've already shown that the pullback of a volume form of S^2 by ϕ is a closed form ($d(\phi^*\epsilon) = 0$). But because of the cohomological properties of S^3 it is also exact and, therefore, there exists a function g such that $\phi^*\epsilon = dg$. According to Whitehead's theorem [10], a Hopf index of a mapping $\phi : S^3 \to S^2$ can be expressed as the following integral:

$$\gamma(\phi) = \int_{S^3} g \wedge \phi^*\epsilon \tag{85}$$

After expressing our 2-form in terms of components in stereographic coordinates as before:

$$\phi^*\epsilon = \frac{1}{2}(\phi^*\epsilon)_{ij} dx^i \wedge dx^j \tag{86}$$

and defining a vector $\vec{b}$ as:

$$b_i = \frac{1}{2}\epsilon_{ijk}(\phi^*\epsilon)^{jk} \tag{87}$$

we can, after some manipulation write:

$$\gamma(\phi) = \int d^3r(\vec{a} \cdot \vec{b}) \tag{88}$$

where $\vec{a}$ is defined by $\vec{b} = \nabla \times \vec{a}$. It is clear from our previous constructions that $\vec{b}$ and $\vec{a}$, as we defined them here, differ from magnetic field $\vec{B}$ and vector potential $\vec{A}$ only by a factor of $\sqrt{a}$, which is just a constant, introduced for dimensional reasons. Therefore:

$$h_m = \int d^3 r \vec{A} \cdot \vec{B} = \gamma(\phi) a \tag{89}$$

and analogously:

$$h_e = \int d^3 r \vec{C} \cdot \vec{E} = \gamma(\theta) a \tag{90}$$

where h_m and h_e are electric and magnetic helicities and $\vec{E} = \nabla \times \vec{C}$. Therefore, helicity, as a physical property, is directly related to the Hopf index (linking number) which makes it both quantized (for "admissible fields") and conserved.
The name "helicity" comes from fluid dynamics where it is defined as measuring a projection of vorticiy $\nabla \times \vec{v}$ on velocity $\vec{v}$. In our case, magnetic field plays the role of vorticity of vector potential. Hopf-Ranada solution clearly has both helicities 1 (we forget dimensions for the moment). Using this "basic" solution, solutions with arbitrary helicities $h_e, h_m \in \mathbb{Z}$ can easily be constructed by using, instead of ϕ and θ defined in (67) and (70), their m-th and n-th powers ϕ^n and θ^m.

VIII. Conclusion

To do justice to Hopf fibration, both in the context of pure mathematics and its physical applications, is impossible in a somewhat limited treatment like we attempted in this seminar. Instead, what we tried to do is to motivate the consideration of Hopf fibration in physics by a simple and familiar example and then to introduce the minimal mathematical apparatus needed for us to appreciate and use this beautiful structure to construct some surprising solutions of the well known equations of classical electrodynamics. We've sketched how Hopf fibration fits in the broader context of the theory of fibre bundles and tried to exhibit its geometrical structure. Then we used it (and also some of Hopf's broader ideas about mappings) to show that there is a class of topologically nontrivial solutions of Maxwell's equations and we explicitly constructed simplest of these. We tried to highlight the most important aspects, but still much more can be said, given the expanding interest in these solutions following the article of Irvine [7]. Among the things we ignored are the important questions of experimental realizations and also, the manifold generalizations of these solutions that were discevered through the years. In the end, we apologize for the somewhat misleading title, since to even scratch the surface of possible applications of Hopf fibration in physics would require a much bigger seminar.

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