BTZ black hole

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(Dated: September 19, 2018)

In this paper the geometry of the chargeless spinning black hole in 2+1 dimensions, with a negative cosmological constant and without couplings to matter is analysed. It is shown that the black hole arises as a quotient space of three dimensional anti-de Sitter space by a discrete subgroup of its isometry group. A brief analysis of AdS space, as well as the $SO(2,2)$ group is given. The surface $r = 0$ is shown to be a singularity in the causal structure, meaning that continuing through it would result with closed timelike curves. Kruskal coordinates and Penrose diagrams are presented as well.

1. INTRODUCTION

The appeal of analysing a black hole in 2+1 dimensions is that it would provide a simpler insight into fundamental concepts regarding both classical and quantum gravity.

The 2+1 black hole displays similar properties to its 3+1 dimensional counterpart. It has an event horizon and in the rotating case an inner horizon and ergosphere as well. But unlike the 3+1 case, it is asymptotically anti-de Sitter and does not have a curvature singularity.

The structure of the paper is as follows: In the second section we discuss the physical properties of the black hole. Since any solution of Einstein's field equations in vacuum with negative cosmological constant in 2+1 dimensions has a constant negative curvature, it is locally isomorphic to anti-de Sitter. After we develop some theory on anti-de Sitter space and its isometry group, we show in section three that the black hole solution can be obtained as a quotient space of AdS_3 space by a discrete subgroup of its isometry group. Through this construction we show that the hypersurface $r = 0$ is not a curvature singularity, but rather a singularity in the causal structure. In section 4 we inspect the global properties of the black hole by exhibiting Kruskal coordinates and Penrose diagrams, both for a general spinning black hole and for its extreme states, which are the massless, nonrotating black hole ("vacuum") and the black hole with maximal angular momentum. Section 5 is devoted to concluding remarks.

2. BLACK HOLE

The metric of the BTZ black hole is given in "Schwarzschild" coordinates by

$$ds^2 = -(N^\perp)^2 dt^2 + (N^\perp)^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2, \tag{2.1}$$

where the so-called lapse and shift functions, respectively, are given by

$$N^\perp = \left(-M + \left(\frac{r}{l}\right)^2 + \left(\frac{J}{2r}\right)^2\right)^{1/2} \tag{2.2}$$

$$N^\phi = -\frac{J}{2r^2}, \tag{2.3}$$

with $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \phi < 2\pi$. l denotes the radius of curvature of anti-de Sitter space and is given by $l^{-2} = -\lambda$, where λ is the cosmological constant. M is the mass of the black hole, and J its angular momentum. The metric (1.1) is stationary and axially symmetric, with Killing vectors ∂_t and ∂_ϕ. The lapse function $N^\perp$ vanishes for two values of r given by

$$r_\pm = l\left[\frac{M}{2}\left(1 \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right)\right]^{1/2}, \tag{2.4}$$

whereas g_{00} vanishes for

$$r_{erg} = \sqrt{M}l. \tag{2.5}$$

These three special values of r obey

$$r_- \leq r_+ \leq r_{erg}. \tag{2.6}$$

In comparison with the 3+1 dimensional Kerr black hole, we can conclude that r_+, given by the coordinate singularity of the radial part of the metric, represents the outer event horizon, r_- the inner event horizon and the region between r_{erg} and r_+ is the ergosphere. In the ergosphere, all physical observers are dragged along by the rotation of the black hole. By observing the solutions (2.4), we see that for $|J| > Ml$ the event horizons disappear, leaving a metric that has a naked conical singularity at $r = 0$. At a conical singularity spacetime is inextendible, that is, it cannot be isometrically embedded into another larger spacetime as a proper subset, geodesics are incomplete, but curvature components do not diverge near the singularity. We would like to exclude those solutions from the physical spectrum. Therefore, the solutions that describe a black hole should satisfy the conditions

$$M > 0, \quad |J| \leq Ml. \tag{2.7}$$

By letting the black hole "disappear", we obtain the vacuum state. This is achieved by letting the horizon size go to zero, that is, letting $M \to 0$, which in turn, because of (2.7), requires $J \to 0$. The line element in this situation is

$$ds^2 = -(r/l)^2 dt^2 + (r/l)^{-2} dr^2 + r^2 d\phi^2. \tag{2.8}$$

As M grows negative, we obtain solutions that posses naked conical singularities, just as the 3+1 counterpart has curvature singularities for negative mass. As before, we exclude those solutions from the physical spectrum. As we reach the values $M = -1$ and $J = 0$, we obtain again a physical solution that has no singularities, as well as no horizon, namely, anti-de Sitter space with the line element

$$ds^2 = -(1 + (r/l)^2) dt^2 + (1 + (r/l)^2)^{-1} dr^2 + r^2 d\phi^2. \tag{2.9}$$

We see, therefore, that anti-de Sitter space represent a sort of "bound state", separated from the black hole solution by a mass gap of one unit. We cannot continuously deform this state into one of the black hole states because we would have to go through a series of naked singularities, which we have previously excluded from our configuration space.

3. BLACK HOLE AS ANTI-DE SITTER SPACE FACTORED BY A SUBGROUP OF ITS SYMMETRY GROUP

In the following discussion we show that the black hole arises as a quotient space of anti-de Sitter space by a discrete subgroup of its isometry group $SO(2,2)$. We show that the singularity obtained at $r = 0$ is a causal singularity, and not a curvature singularity. We also analyse the properties of anti-de Sitter space and the algebra $so(2,2)$.

3.1. Anti-de Sitter space in 2+1 dimensions

Anti-de Sitter space is a maximally symmetric pseudo-Riemannian manifold of constant negative scalar curvature. As such, it is an exact solution of Einstein's field equations for an empty universe with a negative cosmological constant. We can analyse some of its properties by embedding it in a four dimensional flat space of signature $(--++)$

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2, \tag{3.1}$$

with the constraint of a hyperboloid

$$-u^2 - v^2 + x^2 + y^2 = -l^2. \tag{3.2}$$

The embedding allows for an easier analysis due to simpler properties of flat space. We can construct another coordinate system that covers the entire manifold given by transformations

$$u = l\ \cosh\mu\ \sin\lambda, \quad v = l\ \cosh\mu\ \cos\lambda, \tag{3.3}$$
$$x = l\ \sinh\mu\ \sin\theta, \quad y = l\ \sinh\mu\ \cos\theta, \tag{3.4}$$

with $l\ \sinh\mu = \sqrt{x^2 + y^2}$ and $0 \le \mu < \infty$, $0 \le \lambda < 2\pi$. Inserting the transformations into the flat space metric gives

$$ds^2 = l^2 \left[-\cosh^2\mu d\lambda^2 + d\mu^2 + \sinh^2\mu d\theta \right]. \tag{3.5}$$

Since λ is an angle, anti-de Sitter space possesses closed timelike curves. To remedy this, one does not identify points λ and $\lambda + 2\pi$. The space thus obtained represents the universal covering of the original AdS_3 space, which is usually referred to as AdS_3 space. If this "unwrapped" λ is denoted by t/l, and we define $r = l\ \sinh\mu$, the line element is given by

$$ds^2 = -(1 + (r/l)^2) dt^2 + (1 + (r/l)^2)^{-1} dr^2 + r^2 d\phi^2, \tag{3.6}$$

which corresponds to the metric (2.1) with $M = -1$ and $J = 0$, as argued in the previous section.

3.2. Isometries

Anti-de Sitter space inherits the isometries of the embedding space that preserve the hyperboloid for the same reason that the isometries of S^2 are inherited from rotations in $\mathbb{R}^3$. The group of rotations and boosts in four dimensional space of signature $(--++)$ is $SO(2,2)$, so we expect that this group is the group of isometries of AdS_3 as well. On account of this argumentation one concludes that the Killing vectors of AdS_3 are

$$J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b}, \tag{3.7}$$

where $x^a = (v, u, x, y)$. Explicitly, we have 6 Killing vectors, as expected for a space with constant curvature

$$\begin{aligned} J_{01} &= v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \quad & J_{02} &= x\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}, \\ J_{03} &= y\frac{\partial}{\partial v} + v\frac{\partial}{\partial y}, \quad & J_{12} &= x\frac{\partial}{\partial u} + u\frac{\partial}{\partial x}, \\ J_{13} &= y\frac{\partial}{\partial u} + u\frac{\partial}{\partial y}, \quad & J_{23} &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \end{aligned} \tag{3.8}$$

By using transformations (3.3) and (3.4), we see that the vector $J_{01} = \partial_\lambda$ generates time translations (because of the "unwrapping" of the coordinate λ), and $J_{23} = \partial_\theta$ generates rotations in the $x - y$ plane. A general Killing vector can be written as a linear combination of the vectors (3.8), which is given by

$$\xi = \frac{1}{2}\omega^{ab} J_{ab}, \quad \omega^{ab} = -\omega^{ba}. \tag{3.9}$$

The vector is thus determined by an antisymmetric tensor in $\mathbb{R}^4$. These tensors can be classified according to their eigenvalues by means of the Jordan-Chevalley decomposition, given in the next subsection.

3.3. One-parameter subgroups of SO(2,2)

In order to obtain the black hole metric from the anti-de Sitter metric, one identifies points along an orbit generated by the action of a subgroup of $SO(2,2)$, that is, by constructing a quotient space. The subgroup is shown to be a discrete one-parameter subgroup defined by the exponential mapping of a Killing vector. Therefore, the task is to classify all one-parameter subgroups up to an equivalence relation. Two one-parameter subgroups $g(t)$ and $h(t)$ are equivalent if they are conjugate, i.e.

$$g(t) = k^{-1}h(t)k, \quad k \in SO(2,2). \tag{3.10}$$

Since one-parameter subgroups are obtained by exponentiation of generators of the Lie algebra $so(2,2)$, which are given by antisymmetric tensors ω^{ab}, we can instead analyse elements of $so(2,2)$. In that case, the equivalence relation from (3.10) translates into a transformation law for ω^{ab}

$$\omega' = k^T \omega k. \tag{3.11}$$

Therefore, we classify antisymmetric tensors under the equivalence relation (3.11).

Any linear operator M can be uniquely decomposed as a sum of a linear operator S that is diagonalizable over $\mathbb{C}$ (semi-simple) and a nilpotent operator N ($N^p = 0$ for some p) that commute. This is called the Jordan-Chevalley decomposition of M. The eigenvalues of S coincide with those of M. If the eigenvalues are non-degenerate, the operator M is diagonalizable and the operator N is identically zero. In that case, the operator M is completely determined by its eigenvalues (up to a similarity operation). In cases where some of the eigenvalues are repeated, the nilpotent part may not be zero and one requires information about it as well to fully characterize the operator M.

Utilizing this method one can classify the elements of $so(2,2)$. Since the matrix ω_{ab} is real and antisymmetric, there are restrictions on possible eigenvalues. Possible types of $so(2,2)$ operators according to their eigenvalues are the following:

Type I ($N = 0$)
- $\mathbf{I}_a$: 4 complex roots λ, $-\lambda$, λ^*, $-\lambda^*$ ($\lambda \neq \pm\lambda^*$).
- $\mathbf{I}_b$: 4 real roots λ_1, $-\lambda_1$, λ_2, $-\lambda_2$.
- $\mathbf{I}_c$: 4 imaginary roots λ_1, $-\lambda_1$, λ_2, $-\lambda_2$.
- $\mathbf{I}_d$: 2 real roots λ_1 and $-\lambda_1$, and two imaginary roots λ_2, $-\lambda_2$.

Type II ($N \neq 0, N^2 = 0$)
- $\mathbf{II}_a$: 2 real double roots, λ and $-\lambda$.
- $\mathbf{II}_b$: 2 imaginary double roots, λ and $-\lambda$.
- $\mathbf{II}_c$: 1 double root, zero, and 2 simple roots, λ and $-\lambda$ (real or imaginary).

Type III ($N^2 \neq 0, N^3 = 0$): one quadrupole root, zero.
Type IV ($N^3 \neq 0, N^4 = 0$): one quadrupole root, zero.

We can further subdivide type **III** according to the norm of its only non-null eigenvector, which can be ± 1, thus obtaining types $\mathbf{III}^{\pm}$.

We can define two quadratic Casimir invariants

$$I_1 = \omega^{ab}\omega_{ab}, \quad I_2 = \frac{1}{2}\epsilon^{abcd}\omega_{ab}\omega_{cd}. \tag{3.12}$$

Given that for all types of operators the eigenvalues are characterized by two independent real numbers, the knowledge of the two Casimir invariants (3.12) is equivalent to knowing the eigenvalues for any type of operator. Possible Killing vectors, as constructed in (3.9) are given in table (1), alongside the corresponding Casimir invariants. For a more detailed analysis, consult the appendix A in [1].

3.4. Identifications

Any Killing vector ξ defines a one-parameter subgroup of isometries by means of exponential mapping, which defines the mapping of points in anti-de Sitter

$$P \to e^{\alpha\xi} P. \tag{3.13}$$

If one takes α to be an integer multiple of 2π, the mapping (3.13) defines the discrete subgroup used to make the identifications necessary to construct the black hole metric. Since the identification subgroup is a subgroup of the isometry group of the anti-de Sitter space, the quotient space obtained by identifying points on the orbits generated by (3.13) inherits a well defined metric from the AdS_3 space with the same constant negative curvature, and is therefore a solution to Einstein field equations as well. The construction of the aforementioned quotient space results in identifying points on the same orbit, which creates closed curves out of lines connecting two points in the original space. In order for the quotient space to be a physical solution to Einstein equations, it must have a well defined causal structure, i.e. the newly-obtained closed curves must not be timelike or null, since they would imply time-travel. A necessary, and as will be shown, sufficient condition for the absence of closed timelike curves is that the Killing vector ξ be spacelike, i.e.

$$\xi \cdot \xi > 0. \tag{3.14}$$

Not all Killing vectors fulfil the condition (3.14) everywhere in the original anti-de Sitter space, specifically the one chosen to define the identification subgroup. Therefore, the regions where the vector ξ is timelike or null must be excluded from the AdS_3 space. Because the norm of Killing vectors is constant along their orbits, the resulting space, denoted by (AdS_3)', is also invariant under (3.13) and the quotient can still be taken.

FIG. 1. Classification of one-parameter subgroups of $SO(2,2)$. [1]

Type	Killing vector	$\frac{1}{4}I_1$	$\frac{1}{4}I_2$
$\mathbf{I}_a$	$b(J_{01}+J_{23})-a(J_{03}+J_{12})$	b^2-a^2	b^2+a^2
$\mathbf{I}_b$	$\lambda_1 J_{12}+\lambda_2 J_{03}$	$-\frac{1}{2}(\lambda_1^2+\lambda_2^2)$	$\lambda_1\lambda_2$
$\mathbf{I}_c$	$b_1 J_{01}+b_2 J_{23}$	$\frac{1}{2}(b_1^2+b_2^2)$	$b_1 b_2$
$\mathbf{II}_a$	$\lambda(J_{03}+J_{12})+J_{01}-J_{02}-J_{13}+J_{23}$	$-\lambda^2$	λ^2
	or		
	$\lambda(-J_{03}+J_{12})-J_{13}+J_{23}$ $(\lambda\neq 0)$	$-\lambda^2$	$-\lambda^2$
$\mathbf{II}_b$	$(b-1)J_{01}+(b-1)J_{23}+J_{02}-J_{13}$	b^2	b^2
$\mathbf{III}^+$	$-J_{13}+J_{23}$	0	0
$\mathbf{III}^-$	$-J_{01}+J_{02}$	0	0

In general, one can find geodesics that go from a region where $\xi\cdot\xi>0$ to a region where $\xi\cdot\xi<0$, that is, the space (AdS_3)' is geodesically incomplete. The border between these two regions, the surface $\xi\cdot\xi=0$ can be considered a true singularity in the quotient space, and it is a singularity in the causal structure because continuing past it would produce closed timelike curves. We see that in 2+1 dimensions without couplings to matter there are no possibilities for curvature singularities because the Riemann tensor is everywhere regular and given by

$$R_{\mu\nu\lambda\rho}=-l^{-2}(g_{\mu\lambda}g_{\nu\rho}-g_{\nu\lambda}g_{\mu\rho}). \tag{3.15}$$

Also, the Kretschmann scalar, $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$, often used to verify the presence of a curvature singularity, is identically zero everywhere.

3.5. Explicit construction of the black hole metric

In this subsection it is shown that the black hole metric can be obtained by making the identifications (3.13) with the Killing vector

$$\xi=\frac{r_+}{l}J_{12}-\frac{r_-}{l}J_{03}-J_{13}+J_{23}, \tag{3.16}$$

where J_{ab} are defined in (3.8). If one uses the definition (3.9), the corresponding antisymmetric tensor ω^{ab} is shown to have, by solving the equation $(\omega_{ab}-\lambda g_{ab})l^b=0$, 4 real eigenvalues, namely, $\pm r_+/l$ and $\pm r_-/l$. The corresponding Casimir invariants, as defined in (3.12) are

$$I_1=-\frac{2}{l^2}(r_+^2+r_-^2)=-2M,\quad I_2=-\frac{4}{l^2}r_+r_-=-2\frac{|J|}{l}. \tag{3.17}$$

Here it is evident, as well as from the form of the metric (1.1), that the black hole solution is characterized by only two parameters, namely the mass and angular momentum, as indicated by the no-hair theorem.

According to the classification made in the previous section, the Killing vector (3.16) is of type $\mathbf{I}_b$ when $r_+\neq r_-$, of type $\mathbf{II}_a$ when $r_+=r_-\neq 0$ and of type $\mathbf{III}^+$ when $r_+=r_-=0$. Firstly, we consider the non-extreme case where $r_+^2-r_-^2>0$. From the table (1) we see that the general vector of type $\mathbf{I}_b$ can be written as

$$\xi'=\frac{r_+}{l}J_{12}-\frac{r_-}{l}J_{03}, \tag{3.18}$$

which is obtained from ξ by an $SO(2,2)$ transformation. The norm of ξ' is given by

$$\xi'\cdot\xi'=\frac{r_+^2}{l^2}(u^2-x^2)+\frac{r_-^2}{l^2}(v^2-y^2), \tag{3.19}$$

which can also be written as

$$\xi'\cdot\xi'=\frac{r_+^2-r_-^2}{l^2}(u^2-x^2)+r_-^2, \tag{3.20}$$

using the constraint (3.2). The region $\xi'\cdot\xi'>0$ gives the inequality

$$\frac{-r_-^2l^2}{r_+^2-r_-^2}<u^2-x^2<+\infty. \tag{3.21}$$

This region can be divided into an infinite number of regions of three different types which are bounded by null surfaces $u^2-x^2=0$ or $v^2-y^2=0$. A hypersurface Σ is said to be null if its normal vector is a null vector (here normal means that it is orthogonal to all vectors in $T_p\Sigma\subset T_pM$). If a hypersurface is defined by setting a function f to a constant, normal vectors are constructed as

$$X^\mu=g^{\mu\nu}\nabla_\nu f. \tag{3.22}$$

To see that X^μ is orthogonal to any $V \in T_p\Sigma$, we take the derivative along the integral curve generated by V, $V^\nu\nabla_\nu$, which lies on the hypersurface Σ. Since the function f is constant along Σ, we have $V^\nu\nabla_\nu f = 0$, which gives $g_{\mu\nu}V^\nu X^\mu = 0$. We can therefore convince ourselves that the surfaces $u^2-x^2 = 0$ and $v^2-y^2 = l^2-(u^2-x^2) = 0$ are in fact null.

The regions into which we divide the region $\xi' \cdot \xi' > 0$ are:

Type I: Smallest connected regions with $u^2 - x^2 > l^2$ with y and u of definite sign. From the inequality one can see that these regions have no intersection with the $y = 0$ plane. We call regions of this type "the outer regions". The norm of the Killing vector (3.18) fulfils $r_+^2 < \xi' \cdot \xi' < +\infty$.

Type II: Smallest connected regions with $0 < u^2 - x^2 < l^2$ with u and v of definite sign. We call regions of this type "the intermediate regions". The norm of the Killing vector (3.18) fulfils $r_-^2 < \xi' \cdot \xi' < r_+^2$.

Type III Smallest connected regions with $\frac{-r_-^2 l^2}{r_+^2 - r_-^2} < u^2 - x^2 < 0$ with x and v of definite sign. From the inequality one can see that these regions have no intersection with the $x = 0$ plane. We call regions of this type "the inner regions", which exist only for $r_- \neq 0$. The norm of the Killing vector (3.18) fulfils $0 < \xi' \cdot \xi' < r_-^2$. These regions are shown in figure (2).

From figure (2) one can see that each region of type **I** has one region of type **II** in its future and one in its past. For $r_- \neq 0$, two situations can occur for each region of type **II**. It can have one region of type **II** and two regions of type **I** in its future as well as one region of type **II** and two regions of type **III** in its past, and vice versa. Each region of type **III** has one region of type **II** in its future as well as in its past. We can choose one of each contiguous regions of type **I**, **II** and **III** and parametrise them with parameters (t, r, ϕ) as follows (for definiteness it is assumed that $u, y > 0$ in **I**, $u, -v > 0$ in **II** and $x, -v > 0$ in **III**).

Region I $r > r_+$:

$$\begin{aligned} u &= \sqrt{A(r)}\, \cosh\tilde{\phi}(t,\phi), \\ x &= \sqrt{A(r)}\, \sinh\tilde{\phi}(t,\phi), \\ y &= \sqrt{B(r)}\, \cosh\tilde{t}(t,\phi), \\ v &= \sqrt{B(r)}\, \sinh\tilde{t}(t,\phi). \end{aligned} \tag{3.23}$$

Region II $r_- < r < r_+$:

$$\begin{aligned} u &= \sqrt{A(r)}\, \cosh\tilde{\phi}(t,\phi), \\ x &= \sqrt{A(r)}\, \sinh\tilde{\phi}(t,\phi), \\ y &= -\sqrt{-B(r)}\, \cosh\tilde{t}(t,\phi), \\ v &= -\sqrt{-B(r)}\, \sinh\tilde{t}(t,\phi). \end{aligned} \tag{3.24}$$

Region III $0 < r < r_-$:

$$\begin{aligned} u &= \sqrt{-A(r)}\, \cosh\tilde{\phi}(t,\phi), \\ x &= \sqrt{-A(r)}\, \sinh\tilde{\phi}(t,\phi), \\ y &= -\sqrt{-B(r)}\, \cosh\tilde{t}(t,\phi), \\ v &= -\sqrt{-B(r)}\, \sinh\tilde{t}(t,\phi). \end{aligned} \tag{3.25}$$

The functions $A(r)$, $B(r)$, $\tilde{t}(t,\phi)$ and $\tilde{\phi}(t,\phi)$ are defined as

$$\begin{aligned} &A(r) = l^2\left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right), \quad B(r) = l^2\left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2}\right), \\ &\tilde{t}(t,\phi) = \frac{1}{l}(r_+\frac{t}{l} - r_-\phi), \quad \tilde{\phi}(t,\phi) = \frac{1}{l}(-r_-\frac{t}{l} + r_+\phi). \end{aligned} \tag{3.26}$$

By inserting (3.23)-(3.26) into the flat space metric, one obtains

$$ds^2 = -(N^\perp)^2 dt^2 + (N^\perp)^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2, \tag{3.27}$$

with $-\infty < t, \phi < +\infty$. The coordinate ϕ in this case is not periodic. By inserting the transformations in (3.18), one obtains

$$\xi' = \frac{\partial}{\partial\phi}. \tag{3.28}$$

If we construct the quotient space with the identification group as defined in (3.13), that implies the identification

$$\phi \to \phi + 2k\pi, \tag{3.29}$$

because the operator $e^{\alpha\xi'}$ resembles the operator of rotations by the angle ϕ. Therefore, the identification (3.29) recovers the black hole metric, as claimed at the beginning of this subsection.

As we have stated previously, we have only taken one of each three regions to construct the metric (3.27), therefore the coordinate system t, r, ϕ does not cover the entire region $\xi' \cdot \xi' > 0$. One would have to repeat each of the regions an infinite number of times to cover the entire region $\xi' \cdot \xi' > 0$. This occurs because we have used the universal cover of anti-de Sitter space for the construction of the quotient space.

It is worthwhile to consider the extreme case where $r_+ = r_-$. The above derivation cannot be repeated in this case because the Killing vectors (3.16) and (3.18) are not of the same type, according to the classification in subsection (3.3). The vector ξ is of type $\mathbf{II}_a$, and ξ' is of type $\mathbf{I}_b$ with doubly degenerate roots. Hence, we cannot map one to the other by means of a $SO(2,2)$ transformation. Nevertheless, we can argue that the identifications defined by ξ yield the extreme black hole metric without explicitly constructing coordinate transformations that bring ξ into ∂_ϕ. Firstly, we note that the metric (3.27) is regular even if we set $r_- = r_+$, that is, even when $J = Ml$. In the construction of the black hole metric

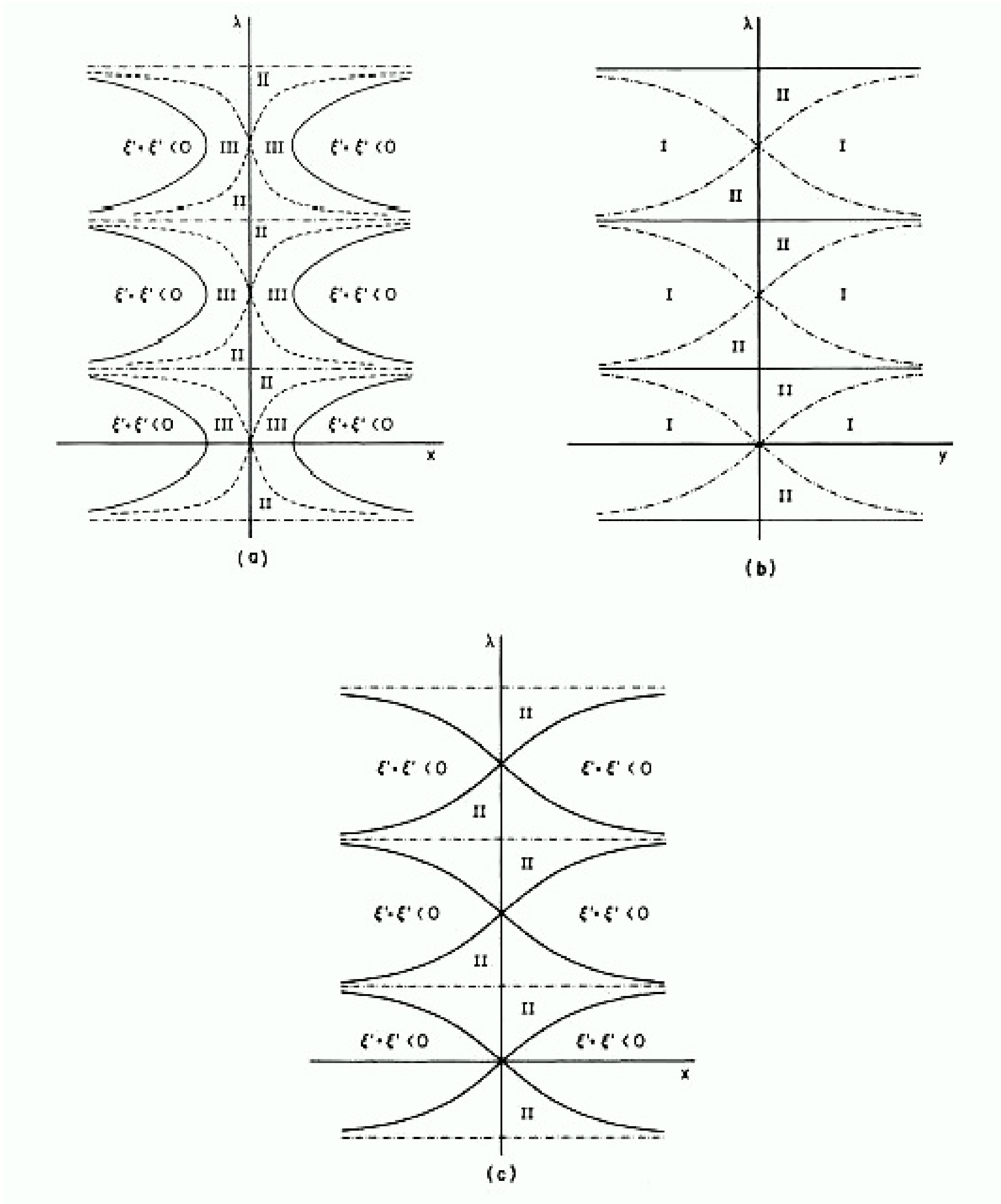

FIG. 2. (a) Section with surface $y=0$ when $r_- \neq 0$. The solid lines are the curves $\xi' \cdot \xi' = 0$, $y=0$, which are timelike (normal vector is spacelike). The dotted lines are $\xi' \cdot \xi' = r_-^2$ $(u^2 - x^2 = 0)$, bounding regions **II** and **III**. The lines formed by dots and segments are $\xi' \cdot \xi' = r_+^2$, $y=0$.
(b) Section with surface $x=0$ when $r_- \neq 0$. The surface $x=0$ has $\xi' \cdot \xi' > 0$ everywhere for $r_- \neq 0$. The horizontal solid lines are the curves $\xi' \cdot \xi' = r_-^2$, $x=0$. The lightlike lines formed by dots and segments are $\xi' \cdot \xi' = r_+^2$. The region $\xi' \cdot \xi > 0$, $x=0$ splits into disconnected components separated by the horizontal lines and containing two regions **I** and two regions **II**.
(c) Section with surface $y=0$ where $r_- = 0$. The solid lines have $\xi' \cdot \xi' = 0$, $y=0$. The lines formed by dots and segments have $\xi' \cdot \xi' = r_+^2$. The region $\xi' \cdot \xi' > 0$ splits into disconnected components separated by the horizontal lines with each component consisting of two regions **II** (and two regions **I**, not seen in this figure since they have no intersection with $y=0$). Regions **III** have disappeared because of $r_- = 0$. Note that the Killing vector ξ' is now tangent to the lightlike curves $u^2 - x^2 = 0$, $y=0$. In all three figures λ is the "time" coordinate defined in (3.3). [1]

we have observed the case when $r_+^2 - r_-^2 > 0$ and before ϕ was identified as an angle, it described a portion of the original anti-de Sitter space, and would so even in the limit $r_+ - r_- \to 0$. Secondly, ∂_ϕ is a Killing vector regardless of the values of r_+ and r_-. Therefore, its Casimir invariants remain the same as defined in (3.17). By looking at table (1) and comparing the corresponding Casimir invariants, one can see that the vector ∂_ϕ is either of type $\mathbf{I}_b$ with coincident roots or type $\mathbf{II}_a$. To convince ourselves that it is of type $\mathbf{II}_a$, we look at the norm of the vector ξ, which is r_+^2, and the norm of ∂_ϕ, which is r^2. The first is a constant, which corresponds to the type $\mathbf{I}_b$ for degenerate roots, whereas the norm of a type $\mathbf{II}_a$ vector is not a constant in general. Therefore, the vector ∂_ϕ is of type $\mathbf{II}_a$ and corresponds to the vector (3.16).

We should also verify that our construction has no closed causal curves, that is, that there are no non-spacelike, future-directed curves joining a point and its image generated by $e^{2k\pi\xi}$ in the region $\xi \cdot \xi > 0$ of anti-de Sitter space. Since the surfaces $r = r_+$ and $r = r_-$ are null, a causal curve which leaves through either one of it can never re-enter. Since we have required that the vector ξ be spacelike, the image of a point generated by ξ has to remain in the same region as the starting point, because they are bounded by null surfaces. Therefore, we can inspect the regions of type **I**, **II** or **III** separately.

In each of the regions the metric takes the form

$$ds^2 = -(N^\perp)^2 dt^2 + (N^\perp)^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2, \tag{3.30}$$

where $-\infty < \phi < \infty$. Consider a causal curve $t(\lambda)$, $r(\lambda)$ and $\phi(\lambda)$, where the parametrization is such that the tangent vector $(dt/d\lambda, dr/d\lambda, d\phi/d\lambda)$ does not vanish for any value of λ. For a curve to be causal, it has to satisfy the condition

$$-(N^\perp)^2 \left(\frac{dt}{d\lambda}\right)^2 + (N^\perp)^{-2} \left(\frac{dr}{d\lambda}\right)^2 + +r^2 \left(N^\phi \frac{dt}{d\lambda} + \frac{d\phi}{d\lambda}\right)^2 \le 0. \tag{3.31}$$

The identification joins points (t_0, r_0, ϕ_0) and $(t_0, r_0, \phi_0 + 2k\pi)$. For some value of λ, $dt/d\lambda$ is zero since we return to the same point in time. If $(N^\perp)^2 > 0$, it follows from (3.31) that $dr/d\lambda = d\phi/d\lambda = 0$, leading to a contradiction. On the other hand, if $(N^\perp)^2 < 0$, the fact that $dr/d\lambda = 0$ implies that $dt/d\lambda = d\phi/d\lambda = 0$, which is also a contradiction.

If we were to keep the region $\xi \cdot \xi \le 0$, we could exit and re-enter the regions of type **III** through the surface $\xi \cdot \xi = 0$, which is timelike (has spacelike normal vector) for $J \ne 0$ (for $J = 0$ it is null, so it wouldn't be possible). If an observer is located at $r > r_+$, for them it doesn't really matter if we include the region $\xi \cdot \xi \le 0$ because $r = r_+$ is the event horizon in any case, and an outside observer cannot probe spacetime inside the event horizon.

3.6. Smoothness of the quotient space

In this section we are concerned with the question of smoothness, that is, whether or not the quotient space inherits the smoothness of anti-de Sitter space. We verify this by proving that the quotient space is a Hausdorff manifold.

Quotient spaces are Hausdorff manifolds if and only if the action of the identification subgroup $H = \{exp(2k\pi\xi), k \in \mathbb{Z}\}$ is properly discontinuous, i.e., if the following properties hold:

(i) Each point $Q \in AdS$ has a neighbourhood U such that $(exp(2k\pi\xi))(U) \cap U = \emptyset$ for all $k \in \mathbb{Z}$, $k \ne 0$;

(ii) If $P, Q \in AdS$ do not belong to the same orbit of H (meaning that there is no k for which $(exp(2k\pi\xi))(P) = Q$), then there are neighbourhoods B and B' of P and Q respectively such that $(exp(2k\pi\xi))(B) \cap B' = \emptyset$ for all $k \in \mathbb{Z}$

Firstly, we define the Euclidean norm on $\mathbb{R}^4$ as

$$[(u-u')^2 + (v-v')^2 + (x-x')^2 + (y-y')^2]^{1/2}. \tag{3.32}$$

The norm of the Killing vector

$$\xi = \frac{r_+}{l}\left(u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}\right) - \frac{r_-}{l}\left(v\frac{\partial}{\partial y} + y\frac{\partial}{\partial v}\right) \tag{3.33}$$

is bounded from bellow by r_- (we have used the condition (3.2)).

Let Q_0 be a point of AdS with coordinates (u_0, v_0, x_0, y_0) that satisfies $u_0^2 + v_0^2 - x_0^2 - y_0^2 = l^2$. Its successive images Q_n are given by

$$u_n = (\cosh\, n\alpha)u_0 + (\sinh\, n\alpha)x_0, \tag{3.34}$$

$$x_n = (\sinh\, n\alpha)u_0 + (\cosh\, n\alpha)x_0, \tag{3.35}$$

$$v_n = (\cosh\, n\beta)v_0 - (\sinh\, n\beta)y_0, \tag{3.36}$$

$$y_n = -(\sinh\, n\beta)u_0 + (\cosh\, n\beta)y_0, \tag{3.37}$$

with $n \in \mathbb{Z}$, $\alpha = 2\pi r_+/l$, $\beta = 2\pi r_-/l$. The Euclidean distance $d_E(Q_0, Q_n)$, $(n \ne 0)$ is shown to be bounded from bellow

$$\begin{aligned}
&(u_n - u_0)^2 + (x_n - x_0)^2 + (v_n - v_0)^2 + (y_n - y_0^2) \\
&\ge |(u_n - u_0)^2 - (x_n - x_0)^2| + |(v_n - v_0)^2 - (y_n - y_0^2)| \\
&= 2(\cosh\, n\alpha - 1)|u_0^2 - x_0^2| + 2(\cosh\, n\beta - 1)|v_0^2 - y_0^2| \\
&\ge 2(\cosh\, \beta - 1)[|u_0^2 - x_0^2| + |v_0^2 - y_0^2|] \\
&\ge 2(\cosh\, \beta - 1)|u_0^2 - x_0^2 + v_0^2 - y_0^2| \\
&= 2l^2(\cosh\, \beta - 1),
\end{aligned} \tag{3.38}$$

which gives $d_E(Q_0, Q_n) \ge l\sqrt{2(\cosh\, \beta - 1)} > 0$, $(n \ne 0)$. We see that the bound does not depend on Q_0.

Let P be another point of AdS with coordinates $(\tilde{u_0}, \tilde{v_0}, \tilde{x_0}, \tilde{y_0})$. From the third step in the previous calculation one can see that the distance between P_n and

Q_0 goes to infinity as $n \to \pm\infty$. Therefore, there is a minimal "distance of approach" of the orbit of P_0 to Q_0 (it may even be zero, that is, there can be a k for which $P_k = Q_0$) and it varies continuously as one varies P_0 continuously.

Let U be the open ball centred around Q_0 with radius $r < \frac{1}{2}\sqrt{2(\cosh\beta - 1)}$. The image of any point inside this ball by $exp(2k\pi\xi)$, $k \neq 0$ cannot be inside the ball according to the bound calculated in (3.38). This proves **(i)**.

To prove **(ii)**, we look at a point P_0 that cannot be mapped to Q_0 by any $exp(2k\pi\xi)$. In the open ball U there can be either one image of P_0 or none at all. In the latter case, points sufficiently close to P_0 would also not have an image in U, which automatically proves **(ii)**. Assuming that there is one image of P_0 in U, say P_k, we observe an open ball B' centred around P_k that is completely in U. All the images of the points in B' lie outside U. Let B" be an open ball around Q_0 such that $B" \cap B' = \emptyset$. Then $B = (exp(-2k\pi\xi))(B')$ and B" satisfy **(ii)**.

For mathematical simplicity we have used the simpler version of the Killing vector ξ, (3.18), which is appropriate for $|J| < Ml$. It would be simple to verify the result holds also for the case $J = Ml$. It would also be worthy to note that the calculation would fail if there was no angular momentum because then the vector $\xi = r_+/l\,(u\partial_x + x\partial_u)$ would vanish along the line $u = x = 0$, which is the line of fixed points. Therefore, the bound on the distance would be zero and both **(i)** and **(ii)** would fail if we were to choose the point Q_0 on the line $u = x = 0$.

3.7. Killing vectors

We have started our construction of the black hole metric from the anti-de Sitter space that possesses 6 Killing vectors, the maximum for a three-dimensional manifold. By inspecting the metric of the black hole, we have concluded that it possesses two Killing vectors, ∂_t and ∂_ϕ, which commute. One might wonder if there are other Killing vectors. In order for an AdS vector field η to induce a well defined vector field in the quotient space, η must be invariant under the transformations of the identification subgroup, that is

$$(e^{2\pi\xi})^*\eta = \eta, \tag{3.39}$$

so that we do not end up with a multivalued vector field at a single point. Using the definition of a pushforward on a vector field, (3.39) turns into

$$e^{2\pi\xi}\eta e^{-2\pi\xi} = \eta, \tag{3.40}$$

that is

$$[e^{2\pi\xi}, \eta] = 0. \tag{3.41}$$

We can perform the aforementioned Jordan-Chevalley decomposition of ξ

$$\xi = s + n, \tag{3.42}$$

where s is the semi-simple part, n is the nilpotent part, and the two commute. For $\exp(2\pi\xi)$ the semi-simple part is $\exp(2\pi s)$, and the nilpotent part is $\exp(2\pi s)[\exp(2\pi n) - 1]$. If a matrix commutes with $\exp(2\pi\xi)$, it must separately commute with $\exp(2\pi s)$ and $\exp(2\pi n)$ because the semi-simple and nilpotent parts of a matrix can be expressed as polynomials of that matrix. If a matrix commutes with $\exp(2\pi s)$, whose eigenvalues are real and positive (because eigenvalues of ξ and s are real, as discussed previously), it also commutes with $\log(\exp(2\pi s)) = 2\pi s$. We can also express n as a polynomial of $[\exp(2\pi n) - 1]$, which means that n also commutes with η. In conclusion, we have the following result

$$[s, \eta] = 0, \quad [n, \eta] = 0 \Rightarrow [\xi, \eta] = 0. \tag{3.43}$$

We have therefore reduced the problem of finding all Killing vectors to finding all elements of $so(2,2)$ that commute with ξ. In order to do that, we first perform the decomposition $so(2,2) = so(2,1) \oplus so(2,1)$ (most easily seen by observing the commutation relations of the Killing vectors (3.8)). The vector ξ then takes the form

$$\xi = \xi^+ + \xi^-, \tag{3.44}$$

where we denote the self-dual part as ξ^+, and the anti-self-dual as ξ^-. We can perform the same decomposition for η. Equation (3.43) is equivalent to

$$[\xi^+, \eta^+] = 0, \quad [\xi^-, \eta^-] = 0, \tag{3.45}$$

because the self-dual and anti-self-dual parts commute, given that they belong to different subspaces of $so(2,2)$. The three dimensional algebra $so(2,1)$ is given by the commutation relations

$$\begin{aligned} [L_3, L_\pm] &= \pm L_\pm, \\ [L_+, L_-] &= -L_3. \end{aligned} \tag{3.46}$$

Therefore, the only elements that commute with a non-zero element of $so(2,1)$ must be multiples of that element. Since both $\xi^\pm$ are non-zero for the black hole, we have

$$\eta^+ = \alpha\xi^+, \quad \eta^- = \beta\xi^-, \quad \alpha, \beta \in \mathbb{R}. \tag{3.47}$$

In conclusion, the general Killing vector is a linear combination of the two Killing vectors we have already found, ∂_t and ∂_ϕ.

4. GLOBAL STRUCTURE

The next step is to observe the global properties of the 2+1 black hole by constructing Kruskal coordinates and Penrose diagrams. We will find some similarities with the 3+1 dimensional counterpart, which we have anticipated at the beginning of this paper.

4.1. Kruskal coordinates

In this section we wish to maximally extend the spacetime of the black hole to cover the entire manifold. We see that the line element

$$ds^2 = -(N^\perp)^2 dt^2 + (N^\perp)^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2, \tag{4.1}$$

has a coordinate singularity for $(N^\perp)^2 = 0$, that is for $r = r_\pm$. We therefore try to construct a coordinate patch around those values in order to bring the metric (4.1) to the form

$$ds^2 = \Omega^2(du^2 - dv^2) + r^2(d\phi + N^\phi dt)^2, \tag{4.2}$$

where $t = t(u, v)$. We see that the metric (4.2) has no singularities of any kind.

Firstly, we start with r_+. The Kruskal coordinates areound r_+ are defined by

Patch K_+:

$$r_- < r < r_+ \begin{cases} U_+ = \sqrt{A(r)} \sinh a_+ t, \\ V_+ = \sqrt{A(r)} \cosh a_+ t, \end{cases} \tag{4.3}$$

$$r_+ < r < \infty \begin{cases} U_+ = \sqrt{-A(r)} \cosh a_+ t, \\ V_+ = \sqrt{-A(r)} \sinh a_+ t, \end{cases} \tag{4.4}$$

with

$$A(r) = \left(\frac{-r + r_+}{r + r_+}\right) \left(\frac{r + r_-}{r - r_-}\right)^{r_-/r_+}, \tag{4.5}$$
$$a_+ = \frac{r_+^2 - r_-^2}{l^2 r_+}.$$

The angular coordinate on K_+, denoted by ϕ_+, is chosen so that

$$N^\phi(r_+) = 0. \tag{4.6}$$

By inserting the coordinate transformations into (4.1), we recover the metric (4.2) with the factor

$$\Omega^2(r) = \frac{(r^2 - r_-^2)(r + r_+)^2}{a_+^2 r^2 l^2} \left(\frac{r - r_-}{r + r_+}\right)^{r_-/r_+}, \tag{4.7}$$

for $r_- < r < \infty$. Because of the condition (4.6), the term $N^\phi dt$ remains regular at $r = r_+$.

Analogously, we construct the coordinate patch around $r = r_-$ as

Patch K_-:

$$0 < r < r_- \begin{cases} U_- = \sqrt{B(r)} \cosh a_- t, \\ V_- = \sqrt{B(r)} \sinh a_- t, \end{cases} \tag{4.8}$$

$$r_- < r < r_+ \begin{cases} U_+ = \sqrt{-B(r)} \sinh a_- t, \\ V_+ = \sqrt{-B(r)} \cosh a_- t, \end{cases} \tag{4.9}$$

with

$$B(r) = \left(\frac{-r + r_-}{r + r_-}\right) \left(\frac{r + r_+}{-r + r_+}\right)^{r_+/r_-}, \tag{4.10}$$
$$a_- = \frac{r_-^2 - r_+^2}{l^2 r_-}.$$

In this case we choose the coordinate ϕ_- so that $N^\phi(r_-) = 0$. We again obtain the metric (4.2) with the factor

$$\Omega^2(r) = \frac{(r_+^2 - r^2)(r + r_-)^2}{a_-^2 r^2 l^2} \left(\frac{r_+ - r}{r_+ + r}\right)^{r_+/r_-}, \tag{4.11}$$

for $0 < r < r_+$.

We denote the overlap of the patches (the region $r_- < r < r_+$) as K. We can see that the Kruskal patches would not exist in the case $r_+ = r_-$, and that we would only need one patch, K_+ to cover the entire space when there is only one root $J = 0$. As in the 3+1 case, we obtain the maximal extension by glueing together an infinite number of K_+ and K_- patches. We refrain from illustrating Kruskal diagrams and move forward to constructing Penrose diagrams, as they are more illuminating.

4.2. Penrose diagrams

Penrose diagrams are frequently used to illustrate the causal structures of spacetimes containing black holes. The goal of Penrose diagrams is to represent the entire spacetime on a finite graph, all the while keeping light cones at 45° (in a unit-independent form this condition means that we want coordinates, for example T and R, in which null rays satisfy $dT/dR = \pm 1$). The spatial dimension is represented horizontally, and the temporal vertically. The Penrose metric is thus locally conformally equivalent to the black hole metric, meaning that they differ by a multiplicative factor, given by the square of a smooth, real-valued function.

In the case of $r_+ \neq r_-$ we achieve this by the change of coordinates

$$U + V = \tan\left(\frac{p + q}{2}\right) \quad U - V = \tan\left(\frac{p - q}{2}\right). \tag{4.12}$$

The inverse transformation is conventionally defined so that the inverse tangent lies between $-\pi/2$ and $\pi/2$.

Consider first the case $J = 0$, i.e. $r_- = 0$. From (4.12), (4.3) and (4.4) one can see that $r = \infty$ is mapped into the lines $p = \pm\pi/2$, the singularity $r = 0$ is mapped into the lines $q = \pm\pi/2$, and the horizon $r = r_+$ is mapped into $p = \pm q$. Both Kruskal and Penrose diagrams are shown in figure (3).

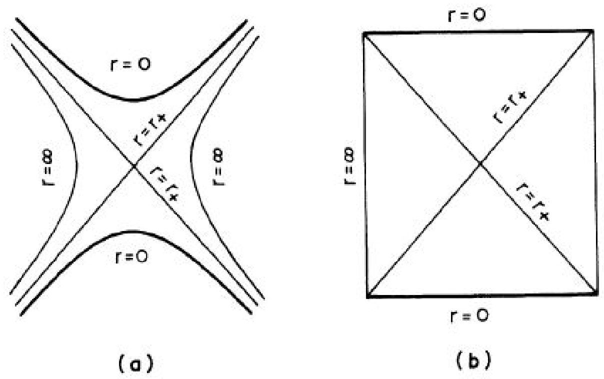

FIG. 3. (a) Kruskal and (b) Penrose diagrams for the case $J = 0$. [1]

Next one might consider the rotating case. We perform the same coordinate transformations (4.12), but now for both patches K_+ and K_- separately. Therefore, we get a Penrose diagram for each patch. Both diagrams contain the overlapping region K and those parts should be identified. As we have stated before, we need to glue together an infinite number of patches to get the maximal extension of the black hole. The starting black hole metric covered only the region K and had one region of type **III** in figure (4.a) or **I** in figure (4.b). We now include both regions in each diagram so that the maximal extension is given in (4.c).

4.3. Extreme cases $M = 0$ and $M = |J|/l$

Firstly, we inspect the $M = 0$ case. The metric is

$$ds^2 = -(r/l)^2dt^2 + (r/l)^{-2}dr^2 + r^2d\phi^2. \qquad (4.13)$$

We now define dimensionless null coordinates (meaning that the corresponding one-form has zero norm)

$$u = t/l - l/r, \quad v = -t/l - l/r. \qquad (4.14)$$

The metric in these coordinates is

$$ds^2 = r^2dudv + r^2d\phi^2. \qquad (4.15)$$

We now define the Penrose coordinates as

$$u = \tan\left(\frac{p+q}{2}\right), \quad v = \tan\left(\frac{p-q}{2}\right). \qquad (4.16)$$

The relationship between r, p and q is

$$r = -l\frac{\cos p + \cos q}{\sin p}, \qquad (4.17)$$

and the metric takes the form

$$ds^2 = l^2\frac{dp^2 - dq^2}{\sin^2 p} + r^2d\phi^2. \qquad (4.18)$$

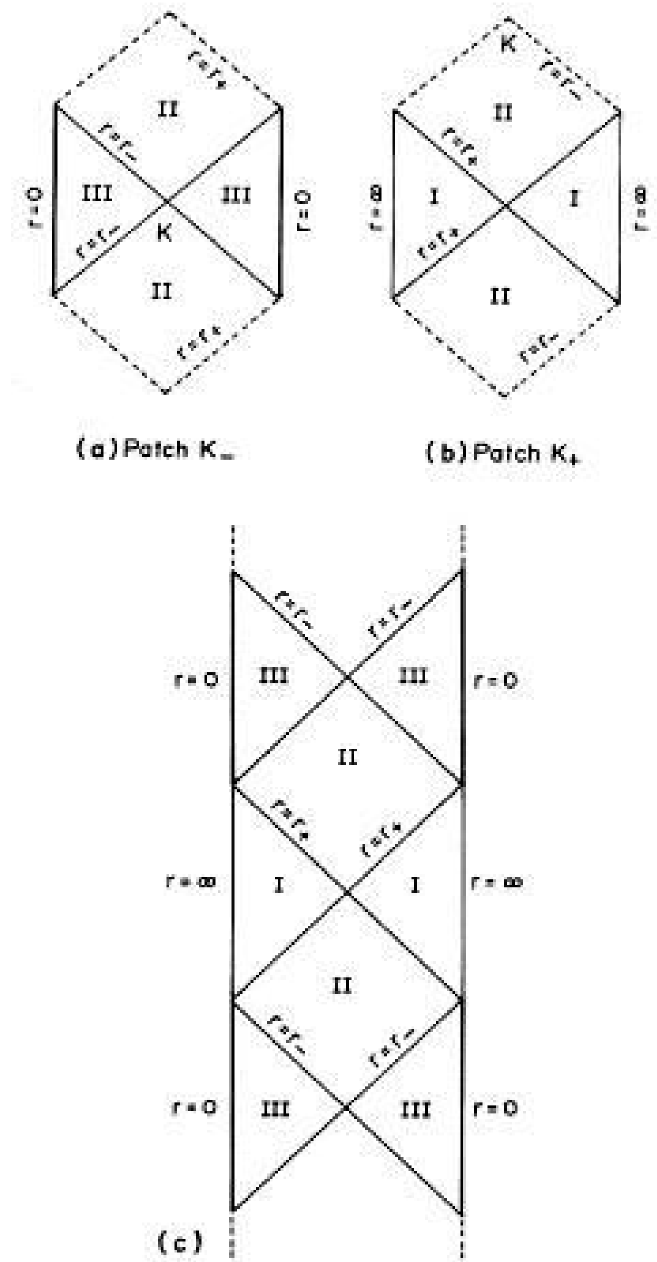

FIG. 4. Penrose diagrams for $J \neq 0$: (a) Patch K_-; (b) Patch K_+; (c) Maximally extended black hole solution. [1]

From (4.17) we see that the origin is mapped to segments of the lines $p = \pi \pm q$ running from $p = 0$ to $p = \pi$, while spacelike infinity is mapped to $p = \pi$. The corresponding Penrose diagram is depicted in figure (5.a).

Now we look into the case $M = |J|/l$, that is, $r_- = r_+$. The metric is

$$ds^2 = -\frac{(r^2 - r_+^2)^2}{r^2l^2}dt^2 + \frac{r^2l^2}{(r^2 - r_+^2)^2}dr^2 + r^2(N^\phi dt + d\phi)^2, \qquad (4.19)$$

where $r = r_+ = l\sqrt{M/2}$ is the horizon. In this case we define the null coordinates

$$u = t + r^*, \quad v = -t + r^*, \tag{4.20}$$

where r^* is the so-called tortoise coordinate, defined as

$$r^* = \int \frac{dr}{(N^\perp)^2} = \frac{-rl^2}{2(r^2 - r_+^2)} + \frac{l^2}{4r_+} \ln \left| \frac{r - r_+}{r + r_+} \right|. \tag{4.21}$$

We define the Penrose coordinates in the same way as in the previous case, (4.16), and we obtain the line element

$$ds^2 = \frac{4(N^\perp)^2 l^2 (dp^2 - dq^2)}{(\cos\, p + \cos\, q)^2} + r^2 (N^\phi dt + d\phi)^2. \tag{4.22}$$

The relationship between r, p and q is given by

$$\frac{\cos\, p + \cos\, q}{\sin\, p} = \frac{-rl}{2(r^2 - r_+^2)} + \frac{l}{4r_+} \ln \left| \frac{r - r_+}{r + r_+} \right|. \tag{4.23}$$

From the previous relation one can see that the the lines $r = r_+$ are at $\pm 45^\circ$, $r = 0$ is at $p = (k\pi)^+$ (meaning that we approach that value from above), and the spacelike infinity is at $p = (k\pi)^-$ (meaning that we approach that value from below). We divide the spacetime into regions $0 < r < r_+$ and $r_+ < r < \infty$. Both Penrose diagrams are triangles with line segments corresponding to $r = r_+$ at $\pm 45^\circ$ and vertical lines corresponding to $r = 0$ in the first case and $r = \infty$ in the second case. To achieve the maximal extension, one must glue an infinite number of these triangles together, which we glue along the lines $r = r_+$. In order to do that, the two types of regions must have a different determination of the arctangent in (4.16), that is, they must take values in different intervals. For example, we can choose that the first region is bounded by $p = 0$ $(r = 0)$, $p + q = \pi$ and $p - q = \pi$, whereas the other region is bounded by $p = \pi$ $(r = \infty)$, $p + q = \pi$ and $p - q = -\pi$. Once this is done, one can safely cross the line $r = r_+$ because the zero of $N^\perp$ is cancelled by the zero in the denominator of the metric (4.22). The Penrose diagram is depicted in figure (5.b).

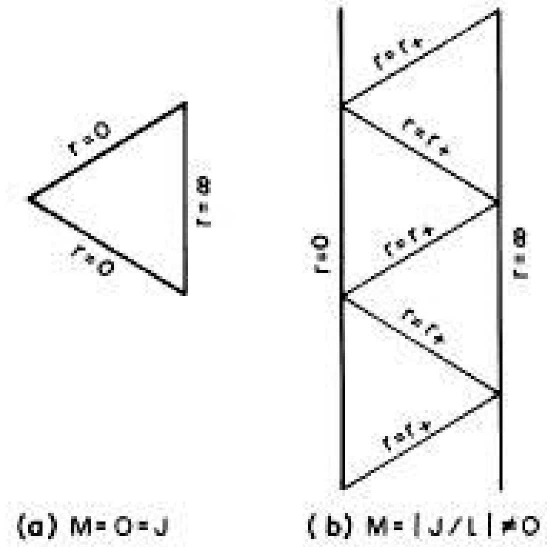

FIG. 5. Penrose diagrams for the extreme cases: (a) $M = 0$; (b) $M = |J|/l$. [1]

5. CONCLUSION

In this paper we have analysed a 2+1 dimensional spinning black hole with no charge that is a solution to Einstein's field equations in vacuum with a negative cosmological constant. We have shown that it can be constructed as a quotient space of anti-de Sitter space by a discrete subgroup of its isometry group $SO(2,2)$. The solution does not display a curvature singularity, unlike the 3+1 dimensional black hole, and is also not asymptotically Minkowski, but rather anti-de Sitter. Nevertheless, it displays some properties similar to the 3+1 black hole which makes it worthwhile the analysis, mostly as a tool for better intuition, due to the simpler calculus. For the same reason, the hope is that it can provide a better insight into quantum gravity as well.

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