Electric/Magnetic duality using differential geometry

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December 7, 2019

Abstract

The general concept of duality is of great importance since dualities are known to be deeply connected with non-perturbative Physics. In contrast to symmetries, which are related to intrinsic properties of one single theory, dualities relate two (or more) seemingly different theories with each other. However, the first hint for an underlying duality is the observation that two different physical theories possess the same symmetries. The first such example ever is classical Electromagnetism, described by Maxwell equations in 4 dimensions. The remarkable symmetry of these equation under the exchange of the electric and the magnetic fields lead to the discovery of the prototype example of duality, which is the Electric/Magnetic one. Here we are going to discuss this duality in the simplest setting, using tools from differential geometry.

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1 Introduction

In this small essay I will try to explain the very basic ideas behind the Electric/Magnetic duality, without getting any technical about it. To achieve this, I will only analyze in depth the case of differential form duality since it already captures all of the essential features and also does not require much knowledge, either on the general research field or on specific mathematical technicalities. It is my belief that the master student audience for which this note is intended will benefit much more from learning the basic concept than from having to endure a long analysis at higher-loop orders. Besides the differential form case, I will also briefly mention the case of linearized gravity because of its physical importance.

A general free gauge field A propagating in D-dimensional Minkowski space always lives in an *irreducible* representation of the Lorentz group $SO(D-1,1)$. However, in any covariant gauge theory involving such gauge fields, the physical field A_{phys} with the true propagating degrees of freedom is found after gauge fixing. The field equations for the physical field will then reduce to the free wave equation $\Box A_{phys} = 0$ and no gauge invariance remains in the theory.

In the physical theory, the field should not only be $SO(D-1,1)$-irreducible but also irreducible under the little group $SO(D-2) \subset SO(D-1,1)$. This basically means that the physical field will have to be fully traceless with respect to the $SO(D-2)$-invariant metric. In addition, it will be represented by a tensor with the same number of indices (and index symmetries) as in the pre-gauge-fixed case, but now these indices will run from 1 to $D-2$. These being said, an arbitrary physical theory corresponding to a particular irreducible tensor representation of $SO(D-2)$ can arise from a number of different covariant gauge theories. The key feature here is that different irreducible representations of $SO(D-1,1)$ may correspond to equivalent irreducible representations of $SO(D-2)$, after full gauge fixing.

In the simplest setting, one considers a gauge theory for a p-form and another gauge theory for a $(D-p-2)$-form, which are both irreducible representations of $SO(D-1,1)$. These gauge fields have different number of components, namely $\binom{D}{p}$ and $\binom{D}{D-p-2}$. However, after gauge fixing, the physical field in the first case is a p-form representation of

$SO(D-2)$ with $\binom{D-2}{p}$ components, while in the second case it is a $(D-p-2)$-form representation of $SO(D-2)$ with $\binom{D-2}{D-p-2}$ components. Therefore the degrees of freedom of the two physical fields end up being the same, which was to be expected since they are related by Hodge duality. Thus, the initial covariant gauge theories are physically equivalent, or electric/magnetic duals of one another.

A quite illustrative diagram depicting Electric/Magnetic duality is the following:

$$\begin{tikzcd} SO(D-1,1)\{R\} \arrow[r, leftrightarrow] \arrow[d] & SO(D-1,1)\{R'\} \arrow[d] \\ SO(D-2)\{R\} \arrow[r, equal] & SO(D-2)\{R'\} \end{tikzcd}$$

At the top level we have two covariant dual (indicated by the blue double-arrow) gauge theories involving two gauge fields in different irreducible representations R, R' of $SO(D-1,1)$. The black arrows indicate full gauge fixing, leading to two physical theories at the bottom level. These theories propagate the same number of degrees of freedom, which implies that the representations R, R' are equivalent (indicated by the red double-line) when seen as irreducible representations of the little group $SO(D-2)$.

2 The prototype example of EM duality

Consider the Maxwell theory in 4-dimensional Minkowski space. This is a covariant gauge theory describing a free 1-form connection A_1, by means of its 2-form curvature $F_2 := dA_1$. The equations dictating the dynamics of A_1 are the well-known Maxwell equations

$$dF_2 = 0\,, \qquad d\star F_2 = 0\,, \tag{2.1}$$

which can be seen to possess a remarkable symmetry. Indeed, they are invariant under the transformation

$$F_2 \;\mapsto\; \star F_2\,. \tag{2.2}$$

To see this, note that the Hodge operator $\star$ squares to -1 when acting on a 2-form due to the signature of the Minkowski metric.

This symmetry of the physical laws leads to a very interesting observation. First of all, we can introduce another 2-form $G_2 \equiv \star F_2$. This 2-form solves the Maxwell equations as a consequence of the above symmetry, thus

$$dG_2 = 0\,, \qquad d \star G_2 = 0\,. \tag{2.3}$$

The first equation above, which is the so-called Bianchi identity, implies that G_2 is also a curvature. Indeed, using the Poincaré lemma one can locally identify G_2 with the curvature of another 1-form connection

$$G_2 := dB_1\,. \tag{2.4}$$

Thus, the Maxwell theory in $D = 4$ can be described by two different 1-form connections A_1 and B_1. We will refer to these as the electric and, respectively, the magnetic dual representations of Maxwell theory. This is the prototype example of Electric/Magnetic duality. However, the fact that both the electric and the magnetic descriptions are 1-forms may seem quite strange or, even, redundant. This is the so-called *self-duality* of the Maxwell theory and it is only present in $D = 4$.

We now want to give an example where no self-duality is present. If one considers the Maxwell theory in $D = 5$, then the Maxwell equations (2.1) are unchanged and so is their symmetry (2.2). The difference now is that the Hodge dual of F_2 is no longer a 2-form, but a 3-form $G_3 \equiv \star F_2$. This means that the Bianchi identity on G_3 will render it a curvature of a 2-form connection $G_3 := dB_2$. Thus, we see that 5D classical electromagnetism can be described using either the 1-form A_1 (electric) or the 2-form B_2 (magnetic) connection.

3 General differential form duality

This discussion can be easily generalized to any gauge theory describing the dynamics of a free p-form connection in D dimensions. We will denote the p-form connection in the electric representation by A_p and the corresponding curvature will be the $(p+1)$-form $F_{p+1} := dA_p$. The generalized Maxwell equations are then given by

$$dF_{p+1} = 0\,, \qquad d \star F_{p+1} = 0\,, \tag{3.1}$$

which are again invariant under the transformation

$$F_{p+1} \mapsto \star F_{p+1}\,. \tag{3.2}$$

We now set $G_{D-p-1} \equiv \star F_{p+1}$ and note the Bianchi identity $dG_{D-p-1} = 0$. Using the Poincaré lemma, we introduce the magnetic representation B_{D-p-2} by locally identifying $G_{D-p-1} := dB_{D-p-2}$. This is the most general example involving differential forms. As already mentioned, the underlying reason for this duality is that, while A_p and B_{D-p-1} are in different representations of the Lorentz group $SO(D-1,1)$, they correspond to equivalent representations of the little group $SO(D-2)$ after full gauge fixing.

Let us now explain what the parent theory approach is. A natural way to relate dual theories is in terms of a parent Lagrangian. This is typically of first order in derivatives and contains two independent fields such that integrating out each of them leads to the two dual second order theories. Parent Lagrangians are simple to construct and analyze for differential form dualities and we are, hence, going to present this approach for the aforementioned p-form duality in D dimensions. The parent Lagrangian we construct is of the form

$$\mathcal{L} = F_{p+1} \wedge (\star F_{p+1} + dB_{D-p-2})\,, \tag{3.3}$$

and it involves the two independent differential form fields F_{p+1} and B_{D-p-2}. It is also of first order in derivatives, in contrast to the standard Lagrangian kinetic terms. For completeness, we also note that the corresponding action can be obtained by integration over the D-dimensional spacetime as $\mathcal{S} = \int d^D x\, \mathcal{L}$. It is easy to check that the field equations for the two fields read as

$$dF_{p+1} = 0\,, \qquad \star F_{p+1} = -\frac{1}{2}\, dB_{D-p-2}\,. \tag{3.4}$$

The first equation corresponds to the *Bianchi identity* on F_{p+1} and can be locally solved by Poincaré lemma to give

$$F_{p+1} := dA_p\,. \tag{3.5}$$

Plugging this back into the parent Lagrangian gives the standard second order kinetic term for the "electric" p-form field

$$\mathcal{L}_1 = dA_p \wedge \star\, dA_p\,. \tag{3.6}$$

Note that the second term in the parent Lagrangian is a total derivative since the exterior derivative d squares to zero, i.e.

$$dA_p \wedge dB_{D-p-2} = d(A_p \wedge dB_{D-p-2}) + (-1)^p A_p \wedge d^2 B_{D-p-2} = d(A_p \wedge dB_{D-p-2}) + 0\,,$$

and is therefore unimportant at the classical level. This is however surely not true at the quantum level, but we will always discard any such surface term since we are working with classical theories.

The second field equation, which is also called the *duality relation*, can be easily solved in terms of F_{p+1} since the Hodge star operator $\star$ squares to $(-1)^{D(p+1)+p}$ when acting on a $(p+1)$-form. Thus, we get

$$\star\, F_{p+1} = -\frac{1}{2}\, dB_{D-p-2} \quad \Rightarrow \quad F_{p+1} = \frac{(-1)^{D(p+1)+p+1}}{2} \star dB_{D-p-2} \tag{3.7}$$

and substituting in the parent Lagrangian leads to the second order kinetic term for the "magnetic" $(D-p-2)$-form field

$$\mathcal{L}_2 = \frac{1}{4}\, dB_{D-p-2} \wedge \star\, dB_{D-p-2}\,. \tag{3.8}$$

We observe that the first order parent Lagrangian $\mathcal{L}$ is on-shell equivalent to the two second order Lagrangians $\mathcal{L}_1$, $\mathcal{L}_2$ and, thus, dictates the dynamics of both the "electric" field A_p and the "magnetic" field B_{D-p-2}.

4 EM duality in linearized gravity

It is possible to extend the above arguments to fields that are not differential forms, i.e. fully antisymmetric matrices. The most simple and physically relevant example is that of the linearized graviton. Suppose that we perturb the metric tensor g (graviton) around the Minkowski metric η by a small perturbation h (linearized graviton). The linearized graviton h is a symmetric 2-tensor and not a 2-form. It can be expanded as

$$h_{1,1} = h_{(ij)}\, dx^i \otimes dx^j\,. \tag{4.1}$$

Thus, the linearized graviton can be thought of as being a composite object constructed out of two 1-forms. In group-theoretical terminology, it corresponds to an irreducible representation of $SO(D-1,1)$ having the symmetries of a (1,1) Young tableau.

Another important notion in gravity is the one of the Riemann curvature. At the linearized level, it is defined with respect to the linearized graviton as

$$R_{2,2} = \frac{1}{4} R_{ijkl}\, dx^i \wedge dx^j \otimes dx^k \wedge dx^l\,, \qquad R_{2,2} := d\, \widetilde{d}\, h_{1,1}\,, \tag{4.2}$$

where $d, \widetilde{d}$ are exterior derivatives acting on the first and second slot of $h_{1,1}$. In component form, the above definition reads as $R_{ijkl} := 4h_{[j,i][l,k]}$. The above definition implies the index symmetries

$$R_{ijkl} = R_{[ij][kl]}\,, \qquad R_{ijkl} = R_{klij}\,, \qquad R_{[ijk]l} = 0 = R_{i[jkl]} \tag{4.3}$$

and, thus, $R_{2,2}$ is thus in an irreducible representation related to a (2,2) Young tableau. More specifically, the last two symmetries above indicate that

$$tr \star R_{2,2} = 0\,. \tag{4.4}$$

Moreover, the Riemann tensor satisfies the two Bianchi identities

$$d\, R_{2,2} = 0 = \widetilde{d}\, R_{2,2} \tag{4.5}$$

and the linearized Einstein equations correspond to the vanishing of its trace, namely

$$tr R_{2,2} = 0\,. \tag{4.6}$$

Note that $\widetilde{d} R_{2,2} = 0$ also implies $\widetilde{d} \star R_{2,2} = 0$, since $\star$ commutes with $\widetilde{d}$. Finally, the Riemann tensor can also be seen to be divergenceless

$$d \star R_{2,2} = 0\,, \tag{4.7}$$

which is due to the tracelessness condition (4.6) and the second Bianchi identity in (4.5). What we have just shown should be quite obvious by now. The system of Bianchi identities (4.5) and Einstein equations (4.6) is invariant under the duality transformation

$$R_{2,2} \;\mapsto\; \star R_{2,2}\,, \tag{4.8}$$

since this transformation maps them to (4.7) and (4.4) respectively. This is in full accordance with the Maxwell equations, which were invariant under $F \mapsto \star F$. Here, an alternative tensor can be defined as

$$G_{D-2,2} := \star R_{2,2} \tag{4.9}$$

obeying the same Bianchi identities and Einstein field equations as $R_{2,2}$. This tensor has D indices and it is antisymmetric in the first $D-2$ and in the last 2. Moreover, its index symmetries, inherited by the index symmetries of $R_{2,2}$, relate it to a $(D-2,2)$ Young tableau representation. This time, one should use a generalized version of the Poincaré lemma and locally identify $G_{D-2,2}$ as the Riemann tensor of a $(D-3,1)$ Young tableau field $k_{D-3,1}$:

$$G_{D-2,2} := d\,\widetilde{d}\,k_{D-3,1}\,. \tag{4.10}$$

The newly introduced field $k_{D-3,1}$ is the magnetic representation of the linearized graviton, with $h_{1,1}$ being the electric one. We observe again that in $D=4$ self-duality is present, a trait shared with the $D=4$ Maxwell theory. However, self-duality is broken in other number of dimensions. For example, in $D=5$ the electric representation $h_{1,1}$ is dual to a magnetic $k_{2,1}$ field. This 3-tensor field is also known in the literature as the Curtright field [5].

Finally, this case (and many more involving objects which are not differential forms) can also be studied in the context of parent Lagrangians using tools from Graded Geometry [6].

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