Cosmological No-Hair Theorems

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We briefly dicuss the successes and failures of the old cosmological model and the resolvement of those failures in inflationary models. We prove the cosmological no-hair conjecture for Bianchi-type universes with a positive cosmological constant in the context of the general theory of relativity.

I. INTRODUCTION

The cosmological no-hair "theorems" are, more precisely, a class of conjectures dealing with the loss of information of certain initial conditions during the evolution of a given cosmological model. Conjectures of this class generally have the following structure: *'For a geometrically defined class of spacetimes and physically motivated properties of the energy-momentum tensor, all the solutions of the gravitational field equation asymptotically converge to a space of constant curvature'*. Our aim is to prove the conjecture in the context of general relativity for Bianchi-type universes with a positive cosmological constant.

First we recap the fundamentals of the (old) standard cosmology model and list a few key issues of the model. These issues eventually led to the idea of inflation which we briefly summarise. Then we turn ourselves to proving the no-hair conjecture in the context of GR. In the Appendix we derive the Raychaudhuri equation, a tool needed for proving the conjecture and briefly explain the energy conditions used in the proof. Finally, for completeness, we derive the classical Friedmann-Lemaître equations.

II. THE STANDARD COSMOLOGICAL MODEL

The (generalised) Ehlers-Geren-Sachs theorem states that if, in a given universe, all freely falling observers measure the cosmic background radiation to be (nearly) isotropic, then that universe is a (nearly) isotropic and homogeneous FLRW spacetime (see section V C). In the electromagnetic spectrum the contribution of the cosmic microwave background is by far larger than the other branches and constitutes, roughly speaking, 93% of the the whole emission. The temperature of CMB is measured to be highly isotropic - it varies only by a tenth of thousandth of a Kelvin with the direction of observations. Making the Copernican assumption that Earth does not occupy a privileged cosmic position, this constitutes one of the strongest available evidence for our own universe's homogeneity and isotropy, and hence for the foundation of current standard cosmological models.

The FLRW cosmology succesfully predicts the expansion of the universe, the large scale uniformity of the universe, the light-element abundances (such as ^{4}He) and possibly the age of the universe. In view of these successes the FLRW cosmology became accepted as the standard cosmological model (SCM). However, there are serious issues with the model which led to its eventual modification. In the following subsections, we list and briefly explain some of those issues and how they may be resolved.

A. Problems of the SCM

a. The horizon problem For analysis of the causal structure of cosmological models, we can define two important quantities; the proper distance of the event horizon:

$$d_e(t) = a(t)\int_t^{t_{max}} \frac{dt'}{a(t')}, \tag{1}$$

and the proper distance of the particle horizon

$$d_p(t) = a(t)\int_{t_{min}}^{t} \frac{dt'}{a(t')}. \tag{2}$$

The event horizon measures the distance over which we can admit a causal connection *even in the future*. The particle horizon measures instead the size of causally connected regions at time t.

Let us consider a flat Universe with an equation of state $p = (\frac{2}{3\alpha} - 1)\rho$, where $0 < \alpha < 1$. For instance, in a matter-dominated Universe $\alpha = \frac{2}{3}$ and in a radiation dominated Universe we have $\alpha = \frac{1}{2}$. It follows from the Friedmann-Lemaître Eqns. (49) and (51) that $a \propto t^\alpha$ and so we have $\dot{a} > 0$ and $\ddot{a} < 0$, i.e. the Universe expands in a deccelerating fashion and the Hubble parameter $H = \frac{\dot{a}}{a} \propto t^{-1}$. It can be shown that in the SCM we have

$$\begin{aligned} &\lim_{t_{min}\to 0} d_p(t) \to \frac{\alpha}{1-\alpha}H^{-1}(t) \\ &\lim_{t_{max}\to\infty} d_e(t) \to \infty, \end{aligned} \tag{3}$$

ie. in the SCM the event horizon does not exist but the particle horizon exists and is of order H^{-1}. At the very early stages of the universe (Planck time), the particle horizon is then approximately given by (restoring the units of c for the moment):

$$d_p(t_p) \approx ct_p = 10^{-33}cm. \tag{4}$$

The Hubble radius is defined as the distance from the observer at which the recession velocity of an object would equal the speed of light. Current observations ($H = 71$ km/s/Mpc) place the Hubble radius at around 13.7 billion l.y. In reference [2] it is calculated that the size of the Hubble radius blueshifted to Planck time would be of order μm. This gives 10^{87} causally disconnected regions inside a sphere of radius $\sim\mu m$. A question arises of how did the universe came to be nearly homogeneous at early times, if microwaves coming from regions seperated by more than a few degrees of the sky were causally disconnected by many horizon distances at last scattering? Therefore the SCM cannot explain the large scale homogeneity of the universe, but must rather take it as an assumption.

b. The flatness problem The total energy density in critical units can be written as

$$\Omega_t(a) = 1 + \frac{k}{a^2H^2}. \tag{5}$$

According to experimental data, at present time we have:

$$\Omega_{t_0}(a) = 1.02 \pm 0.02 \tag{6}$$

It can be shown that, for the universe to be flat in the present, it must have been even flatter in the past, i.e. we must require enormous fine-tuning at Planck time:

$$\frac{|k|}{a_P^2H_P^2} \approx 10^{-60}. \tag{7}$$

Thus the SCM cannot explain the flatness of space, but must again take it as an assumption. Other problems of the SCM include the singularity problem, the small-scale inhomogeneity of the universe, the entropy problem etc. For more details see [2].

B. Inflationary models

Inflationary models are modifications of the SCM which attempt to resolve all the issues mentioned above. The basic idea of inflation is that the early universe underwent a short period during which matter was in a metastable false vacuum state driving the evolution of the universe into exponential expansion. During this period the scale factor increases by a tremendous factor. A small, subhorizon-sized volume of the Universe can grow large enough to encompass the entire observable Universe. The flatness and horizon problems are immediately solved. The tremendous expansion stretches quantum fluctuations on microscopic scales ($\leq 10^{-23}$ cm) to astrophysical scales ($\geq$ Mpc) which solves the problem of local inhomogeneites. The basic mechanism that drives inflation is a scalar field ϕ described by the Lagrangian density

$$L = -\frac{1}{2}\nabla_a\phi\nabla^a\phi - V(\phi). \tag{8}$$

Its energy-momentum tensor $T_{ab}^{(\phi)} = \nabla_a\phi\nabla_b\phi + g_{ab}L$ can be written as

$$T_{ab}^{(\phi)} = \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\left(\partial_c\phi\partial^c\phi + 2V(\phi)\right) \tag{9}$$

where we have taken into account that we are dealing with a scalar field.

Inflation begins with the scalar field displaced from the minimum of its potential; as it evolves toward the potential energy minimum the scalar field potential drives a nearly exponential expansion. In most models, the time required to evolve to the minimum is many hundreds or thousands of Hubble times, i.e. the potential must be very flat around the starting point $\phi = 0$. As the scalar field approaches its true-vacuum value, the energy density falls to zero, H decreases and inflation ends.

Now we assume that during the inflatory phase $V(\phi) = V$, where V is a constant, and that ϕ is a constant field across all spacetime. This represents a restructuring of the vacuum energy density in the sense that the vacuum energy density changes by a quantity proportional to V. In *GR* this affects the properties of spacetime in the following way:

$$\begin{aligned} G_{ab} &= 8\pi T_{ab} \\ &= 8\pi(T_{ab}^{(m)} + T_{ab}^{(\phi)}) \\ &= 8\pi(T_{ab}^{(m)} - g_{ab}V) \end{aligned} \tag{10}$$

where $T_{ab}^{(m)}$ is the energy-momentum tensor of ordinary matter. This is just the Einstein equation with a cosmological constant

$$\Lambda = 8\pi V. \tag{11}$$

We can view Eq. (10) in vacuum as describing a perfect fluid with $p = -\rho = -V$. This negative pressure has the effect of making a homogeneous and isotropic universe expand exponentially. The FLRW metric (44) and the Friedmann-Lemaître equations (51) and (52) imply for the scale factor:

$$a(t) = \left\{ \begin{array}{ll} H^{-1}\cosh Ht, & \text{if } \tilde{k} = +1 \\ H^{-1}e^{Ht}, & \text{if } \tilde{k} = 0 \\ H^{-1}\sinh Ht, & \text{if } \tilde{k} = -1 \end{array} \right\},$$

where we have set $H \equiv \sqrt{\frac{\Lambda}{3}}$. This solution is referred to as *de Sitter spacetime*.

The inflationary models are succesful in resolving the issues of the SCM. However, it is not obvious that cosmological models with non-FRW initial conditions ever enter an inflationary epoch nor is it obvious that, if inflation occurs, intial inhomogeneities and anisotropies will be smoothed out eventually. Therefore a question arises of the naturalness of the inflationary scenario: Does the inflationary phase in the evolution of the universe proceed from very general intial conditions? This is a question that is answered via the cosmological no-hair conjectures.

III. PROOF OF THE COSMOLOGICAL NO-HAIR CONJECTURE FOR HOMOGENEOUS COSMOLOGIES

In this chapter we prove the cosmological no-hair conjecture solely in the context of general relativity, i.e. we attribute the vacuum energy which drives inflation to a large cosmological constant. As there is no way to drive the cosmological constant to 0, we have no mechanism to finish inflation. We follow Wald to prove that *'all initially expanding Bianchi cosmologies with a positive cosmological constant Λ, except type-IX, evolve towards the de Sitter solution exponentially fast. The behaviour of type-IX models is similar provided that Λ is greater than a certain bound.'*

We consider a Bianchi universe of either type (I-IX), i.e. a spatially homogeneous, but not necessarily isotropic, spacetime (M, g). Such a spacetime can be foliated by a one-parameter family of spacelike hypersurfaces Σ_t orthogonal to a congruence of timelike geodesics parametrized with proper time t. Let $n = \frac{\partial}{\partial t}$ be the unit tangent vector field to the geodesics. Using Eq. (34) we can decompose the covariant derivative evaluated on Σ_t as

$$K_{ab} \equiv \nabla_a n_b = \frac{1}{3} K h_{ab} + \sigma_{ab}, \tag{12}$$

where $K = K^a{}_a$ and $\omega_{ab} = 0$ for a hypersurface orthogonal congruence. Since K_{ab} is symmetric, we notice that it can be written in terms of the Lie derivative of the metric tensor with respect to n:

$$\begin{aligned} L_n g_{ab} &= n^c \nabla_c g_{ab} + g_{cb} \nabla_a n^c + g_{ac} \nabla_b n^c \\ &= \nabla_a n_b + \nabla_b n_a \\ &= K_{ab} + K_{ba} \\ &= 2K_{ab}, \end{aligned} \tag{13}$$

so we can write:

$$\begin{aligned} K_{ab} &= \frac{1}{2} L_n g_{ab} \\ &= \frac{1}{2} L_n (h_{ab} - n_a n_b) \\ &= \frac{1}{2} L_n h_{ab}. \end{aligned} \tag{14}$$

It can be shown that $L_n n_a n_b = n_a L_n n_b + n_b L_n n_a = 0$ in the following way:

$$\begin{aligned} L_n n_a &= n_c \nabla^c n_a + n_c \nabla_a n^c \\ &= n_c \nabla_a n^c \\ &= \nabla_a (n_c n^c) - n^c \nabla_a n_c \\ &= -n_c \nabla_a n^c \\ &= 0, \end{aligned} \tag{15}$$

where we have used the geodesic equation and the fact that $n_a n^a = -1$. If we use a coordinate system adapted to n, we can write Eq. (14) as

$$K_{\mu\nu} = \frac{1}{2} \frac{\partial h_{\mu\nu}}{\partial t}. \tag{16}$$

In other words, we have obtained a differential equation for the spatial metric h_{ab} in terms of K_{ab}. The Raychaudhuri equation (37) for K is:

$$\frac{dK}{dt} = -\frac{1}{3} K^2 - \sigma^{ab} \sigma_{ab} - R_{ab} n^a n^b. \tag{17}$$

We can think of the spacelike hypersurfaces Σ_t as Riemannian 3-manifolds with the induced metric tensor h_{ab} and scalar curvature ${}^{(3)}R$. ${}^{(3)}R$ is nonpositive in all Bianchi models except type IX and satisfies the Gauss-Codacci equation:

$${}^{(3)}R = R + 2R_{ab} n^a n^b - K^2 + K_{ab} K^{ab}. \tag{18}$$

The last term may be further simplified:

$$\begin{aligned} K_{ab} K^{ab} &= \frac{1}{9} K^2 h_{ab} h^{ab} + \frac{2}{3} K h_{ab} \sigma^{ab} + \sigma_{ab} \sigma^{ab} \\ &= \frac{1}{3} K^2 + \sigma_{ab} \sigma^{ab}. \end{aligned} \tag{19}$$

where we used $h^{ab} h_{ab} = 3$. It can be shown that $h_{ab} \sigma^{ab} = 0$ by using the geodesic equation and the fact that σ^{ab} is traceless by assumption. We now write the Einstein equations:

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}. \tag{20}$$

We only need the time-time component:

$$G_{ab} n^a n^b - \Lambda - 8\pi T_{ab} n^a n^b = 0 \tag{21}$$

with the usual definition of the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$. We can use this definition and Eq. (21) to rewrite the last term of the Raychaudhuri equation (17) as follows:

$$\begin{aligned} R_{ab} n^a n^b &= G_{ab} n^a n^b - \frac{1}{2} R \\ &= 8\pi (T_{ab} - \Lambda g_{ab}) n^a n^b + 4\pi T - 2\Lambda \\ &= 8\pi (T_{ab} - \frac{1}{2} g_{ab} T) n^a n^b - \Lambda \end{aligned} \tag{22}$$

where $T = T^a{}_a$. Using equations (19) and (22) we obtain from the Gauss-Codacci equation (18):

$$K^2 = 3\Lambda - \frac{3}{2}\,{}^{(3)}R + \frac{3}{2}\sigma^{ab}\sigma_{ab} + 24\pi T_{ab}n^a n^b, \qquad (23)$$

and from the Raychaudhuri equation (17):

$$\frac{dK}{dt} = \Lambda - \frac{1}{3}K^2 - \sigma^{ab}\sigma_{ab} - 8\pi(T_{ab} - \frac{1}{2}g_{ab}T)n^a n^b. \quad (24)$$

Our strategy is to use equations (23) and (24) to find the asymptotic behaviour of K_{ab}. With that information we solve Eq. (16) for the spatial metric h_{ab} in the asymptotic regime. The first step is to show that K tends to a limit. We assume that the matter stress-energy tensor satisfies the strong and dominant energy conditions (see section V B) and that $K > 0$ for some arbitrary time $t_0 = 0$ (initially expanding Universe). First we consider cosmologies that aren't Bianchi type-IX. For such spacetimes it can be shown that ${}^{(3)}R \leq 0$. In that case all the terms on the right-hand side of Eq. (23) are positive and we can infer that:

$$K \geq \sqrt{3\Lambda}. \qquad (25)$$

Then it follows from the previous inequality that K is a strictly positive function of t. By similar reasoning we can infer from Eq. (24) that:

$$\frac{\dot{K}}{K^2 - 3\Lambda} \leq -\frac{1}{3}. \qquad (26)$$

Knowing that $K^2 \geq 3\Lambda$ we can integrate the previous inequality to obtain the upper limit on K:

$$\sqrt{3\Lambda} \leq K \leq \frac{\sqrt{3\Lambda}}{\tanh(\alpha t)}. \qquad (27)$$

Thus the upper limit on K exponentially approaches $\sqrt{3\Lambda}$ on a time scale $1/\alpha = \sqrt{\frac{3}{\Lambda}}$, i.e. K converges to $\sqrt{3\Lambda}$. Now it easily follows from Eq. (23) that:

$$0 \leq \sigma^{ab}\sigma_{ab} \leq \frac{2\Lambda}{\sinh^2(\alpha t)}, \qquad (28)$$

i.e. the shear σ_{ab} rapidly approaches zero and the universe isotropises. From the same equation we find that the energy density also approaches zero in a few time-constants α^{-1}:

$$T_{ab}n^a n^b \leq \frac{\Lambda}{8\pi}\frac{1}{\sinh^2(\alpha t)}. \qquad (29)$$

Due to the dominant energy condition $|T_{\mu\nu}| \leq T_{00}$ we deduce that all components of the energy-momentum tensor rapidly approach zero, as does the scalar curvature of spacelike hypersurfaces ${}^{(3)}R$ (also obtained from Eq. 23). Using the fact that $K \to \sqrt{3\Lambda}$ as $\sigma_{ab} \to 0$ From Eqns. (16) and (12) we obtain the asymptotic form of the spatial metric h_{ab}:

$$h_{ab}(t) = h_{ab}(t_0)e^{[2\alpha(t-t_0)]}. \qquad (30)$$

In conclusion, for $t \gg \alpha^{-1}$, any initially expanding Bianchi spacetime not of type-IX becomes isotropic ($\sigma_{ab} \to 0$), flat (${}^{(3)}R \to 0$), devoid of matter and expands at a constant rate $K = \sqrt{3\Lambda}$. Now we turn our attention to type-IX spacetimes. For such a spacetime, it can be shown that the spatial curvature ${}^{(3)}R$ has a maximum value given by:

$${}^{(3)}R_{\max} = \frac{3}{2}\frac{(\det M)^{2/3}}{h^{1/3}}, \qquad (31)$$

where $h = \det(h_{ab})$. From the Jacobi's formula for the derivative of a determinant we obtain:

$$\dot{h} = 2hK. \qquad (32)$$

Now we assume, as before, that $K > 0$ at some time $t_0 = 0$, but we also introduce an additional assumption for the value of the cosmological constant Λ. We suposse that initially the following inequality holds:

$$\Lambda < \frac{1}{2}\,{}^{(3)}R_{\max}(t_0), \qquad (33)$$

Given this assumption, $\dot{h}$ is intially positive and remains so if $K > 0$, which in turn remains positive for all $t > t_0$, so long as the previous assumption (33) holds for $t > t_0$. The assumption may only fail if h becomes smaller, but $\dot{h} > 0$. Therefore K is always positive, and the remainder of the proof proceeds similarly as for the case of Bianchi cosmologies not of type IX. So we have shown that the no-hair conjecture holds for type IX cosmologies, given an additional assumption for the cosmological constant Λ.

IV. CONCLUSION

We have summarised the standard cosmological model (FLRW universe) and its issues. Then we briefly showed how inflationary models may correct these issues and motivated the need for the cosmological no-hair theorem. Finally we proved the cosmological no-hair conjecture in the context of general relativity for Bianchi type universes with a positive cosmological constant. As we have no means to end inflation in pure GR, the next step would be to research cosmological no-hair conjectures in the context of inflationary models.

V. APPENDIX

A. The Raychaudhuri equation

Let O be an open region in spacetime. A *congruence* in O is a family of curves such that through each point in O there passes one and only one curve from this family. In analogy to fluid mechanics, these congruences are often called *flows*. The Raychaudhuri equations are a set of evolution equations for quantities describing the *kinematic characteristics of a flow*.

Consider a flow in spacetime (M, g) and let v be the corresponding normalized tangent vector field, i.e. $v_a v^a = \mp 1$. We define the *projection tensor* $h_{ab} = g_{ab} \pm v_a v_b$ (the plus sign is for timelike curves whereas the minus sign is for spacelike curves). Notice that ${h_a}^b v_b = {h^a}_b v^b = 0$, so ${h_a}^b$ can be regarded as the projection operator onto the subspace of the tangent space perpendicular to v. The covariant derivative of v is a second rank tensor and therefore it can be decomposed into its symmetric and anti-symmetric parts. The symmetric part can be further decomposed into its isotropic part involving the trace of the tensor and the symmetric traceless part. By performing such a decomposition, we have:

$$\nabla_b v_a = \sigma_{ab} + \omega_{ab} + \frac{1}{n-1} h_{ab} \Theta, \tag{34}$$

where n is the dimension of spacetime and Θ, σ_{ab} and ω_{ab} are the trace, the symmetric (traceless) part and the antisymmetric (traceless) part respectively. They are defined by:

$$\Theta = \nabla_a v^a \qquad \text{(expansion);}$$

$$\sigma_{ab} = \frac{1}{2}\Big(\nabla_b v_a + \nabla_a v_b\Big) - \frac{1}{n-1} h_{ab}\Theta \qquad \text{(shear);}$$

$$\omega_{ab} = \frac{1}{2}\Big(\nabla_b v_a - \nabla_a v_b\Big) \qquad \text{(rotation).}$$

The expansion, rotation and shear are related to the geometry of the cross sectional area (enclosing a fixed number of curves) orthogonal to the flow lines. The analogy with fluid flow is, usually, a good visual aid for understanding the change in the geometry of this area. If v represents the velocity field of a fluid we can interpret Θ as the expansion/contraction of the volume of the fluid, σ_{ab} as the distortion in shape without change in volume and ω_{ab} as the rotation without change in shape. The Raychaudhuri equations are the evolution equations for the expansion, shear and rotation along the flow.

Now we consider the special case of a congruence of timelike geodesics parametrized by proper time t in four-dimensional spacetime. The corresponding normalised tangent vector field v satisfies the geodesic equation $v^a \nabla_a v^b = 0$. We consider the quantity $v^c \nabla_c \nabla_b v^a$. From the definition of the Riemann curvature tensor we have

$$\nabla_c \nabla_b v_a = R_{adcb} v^d + \nabla_b \nabla_c v_a. \tag{35}$$

Using Eq. (35) and the geodesic equation we have

$$\begin{aligned} v^c \nabla_c \nabla_b v_a &= v^c R_{adcb} v^d + v^c \nabla_b \nabla_c v_a \\ &= R_{adcb} v^c v^d + \nabla_b (v^c \nabla_c v_a) - \nabla_b v^c \nabla_c v_a \\ &= R_{adcb} v^c v^d - (\nabla_b v^c)(\nabla_c v_a) \end{aligned} \tag{36}$$

We obtain the three Raychaudhuri equations from the trace, the symmetric and the antisymmetric parts of equation (36):

$$\frac{d\Theta}{dt} + \frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 = -R_{ab} v^a v^b \tag{37}$$

$$\begin{aligned} \frac{d\sigma_{ab}}{dt} = &-\frac{2}{3}\Theta\sigma_{ab} - \sigma_{ac} - \omega_{ac}{\omega^c}_b + \frac{1}{3} h_{ab}\Big(\sigma^2 - \omega^2\Big) \\ &+ C_{cbad} v^c v^d + \frac{1}{2}\tilde{R}_{ab} \end{aligned} \tag{38}$$

$$\frac{d\omega_{ab}}{dt} = -\frac{2}{3}\Theta\omega_{ab} - 2{\sigma^c}_{[b}\omega_{a]c} \tag{39}$$

where $\sigma^2 = \sigma_{ab}\sigma^{ab}$, $\omega^2 = \omega_{ab}\omega^{ab}$, C_{cbad} is the Weyl tensor and the quantity $\tilde{R}_{ab} = h_{ac} h_{bd} R^{cd} - \frac{1}{3} h_{ab} h_{cd} R^{cd}$.

These evolution equations are essentially *geometric statements*, independent of any reference to the Einstein field equations and, as such, are *not* equations but, essentially, identities. Once we use the Einstein equations or any other geometric property (e.g. Einstein space, or vacuum, etc.) as an extra input, the identities become equations. Eq. (37) for the expansion Θ is of interest to us and is usually refered to as the Raychaudhuri equation.

B. Energy conditions

The actual form of the energy-momentum tensor of the Universe is obviously too complicated to write, and so we confine ourselves to finding physically viable inequalities that T^m_{ab} should satisfy. The first energy condition we use is the *weak energy condition* (WEC):

$$T_{ab} u^a u^b \geq 0 \tag{40}$$

for all timelike vectors u. It means that the energy density as measured by an observer whose 4-velocity is u is non-negative. Continuing, we write the *strong energy condition* (SEC) which states:

$$T_{ab} n^a n^b \geq -\frac{1}{2} T \tag{41}$$

i.e. for every future-pointing timelike vector field n, the trace of the tidal tensor measured by the corresponding observers is always non-negative. The final energy condition we use is the *dominant energy condition* (DEC) which stipulates that

$$T_{ab}u^a u^b \geq 0 \tag{42}$$

and $T^a{}_b u^b$ is non-spacelike for all timelike vectors u. It implies that $|T_{\mu\nu}| \leq T_{00}$, where $T_{\mu\nu}$ are the components of T_{ab} in any orthonormal basis with n^a as the timelike element of this basis. For a perfect barotropic fluid the WEC implies that the energy density of the fluid is positive semi-definite, i.e. $\rho \geq 0$, the DEC implies that the enthalpy of the fluid is positive semi-definite $(\rho + p \geq 0)$ and the SEC demands that $\rho + 3p \geq 0$.

C. Friedmann-Lemaître equations

The general form of the Friedmann - Lemaître - Robertson - Walker (abbreviated as FLRW) metric is derived from purely geometric considerations. The assumption of homogeneity and isotropy implies that the geometry of the Universe is invariant under spatial roto-translations. In four-dimensional spacetime the metric tensor will have 10 independent components. Using homogeneity and isotropy the number of independent components can be reduced from 10 to 4 (having taken into account the 3 rotational and 3 translational degrees of freedom). The most general form of a line element exhibiting spatial spherical symmetry can be written as:

$$\begin{aligned} ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + e^{\mu}(r^2 d\theta^2 + r^2 sin^2\theta d\phi^2) \\ - 2e^{\sigma}drdt \end{aligned} \tag{43}$$

where ν, λ and μ are functions of r. By exploiting the freedom of choosing a gauge the metric can be reduced to its canonical FLRW form:

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\Big] \end{aligned} \tag{44}$$

where $g_{\mu\nu}$ is the metric tensor of the FLRW geometry, $a(t)$ is the scale factor and t is the *cosmic* time coordinate. The FLRW metric describes a homogeneous, isotropic expanding or contracting universe that is path connected, but not necessarily simply connected. $k = 0$ corresponds to a spatially flat Universe; if $k > 1$ the Universe is spatially closed and, finally, $k < 1$ corresponds to a spatially open Universe. The line element (44) is invariant under the following transformation:

$$\begin{aligned} r &\to \tilde{r} = \frac{r}{r_0} \\ a(t) &\to \tilde{a}(t) = a(t) r_0 \\ k &\to \tilde{k} = k r_0^2 \end{aligned} \tag{45}$$

where r_0 is a dimensionful constant. In the parametrization of Eq. (45), $\tilde{k}$ is 0, +1 or −1 depending on the spatial curvature of the internal space. We employ the parametrization where the scale factor is dimensionless.

The time evolution of the scale factor $a(t)$ is governed by the Friedmann-Lemaître equations which are just the Einstein equations written in an FLRW metric and supplemented by the covariant conservation of the total stress-energy tensor. The explicit form of the stress energy tensor depends on the physical model of the matter content of the Universe being used. We consider a perfect barotropic fluid (i.e. a non-viscous fluid with a definite relation between pressure and energy density). Denoting the pressure of such a fluid with p and its density with ρ, the stress-energy tensor can be written in the following general form:

$$T^{\mu\nu} = (p+\rho)u^\mu u^\nu + p g^{\mu\nu}, \tag{46}$$

where u^μ is the velocity field of the (total) fluid satisfying $g_{\mu\nu}u^\mu u^\nu = -1$. By definition of the covariant derivative, covariant conservation of the stress-energy tensor can be written as:

$$0 = \nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\mu\sigma}T^{\sigma\nu} + \Gamma^\nu{}_{\mu\sigma}T^{\mu\sigma}. \tag{47}$$

The Christoffel symbols can be calculated from the metric tensor $g_{\mu\nu}$:

$$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}\Big(\partial_\nu g_{\sigma\lambda} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\nu\sigma}\Big). \tag{48}$$

In the FLRW metric Eqns. (46) and (47) then imply:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{49}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and the over-dot denotes differentiation with respect to the cosmic time coordinate t.

The components of the Ricci tensor and the Ricci scalar computed from the metric (44) are:

$$\begin{aligned} R^0_0 &= -3(H^2 + \dot{H}) \\ R^j_i &= -\Big(\dot{H} + 3H^2 + \frac{2k}{a^2}\Big)\delta^j_i \\ R &= 6\Big(\dot{H} + 2H^2 + \frac{k}{a^2}\Big). \end{aligned} \tag{50}$$

By using Eqns. (50) and (46), the following equations are obtained from the (00) component and the linear combination of the (00) component and the (ij) components of the Einstein equation respectively:

$$H^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{51}$$

$$\dot{H} = -4\pi(\rho + p) + \frac{k}{a^2}. \tag{52}$$

Eqns. (49), (51) and (52) are refered to as the Friedmann-Lemaître equations. Applied to a fluid with a given equation of state $f(\rho, p) = 0$, the Friedmann-Lemaître equations yield the time evolution and geometry of the universe as a function of the fluid density.

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