Israel's theorem

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In this paper I give a short review of the black hole uniqueness theorem in static vacuum spacetimes, due to Werner Israel 1967. First I introduce the terminology and preliminary results from differential geometry and general relativity, followed by the theorem itself.

I. Preliminary Remarks and Notation

First I will introduce my notation and conventions.

I will be using natural units, in which $G_N = c = \hbar = 1$, where G_N is the Newton gravitational constant, and c is the speed of light.

Index convetions are as follows:

- Greek indices $\mu, \nu, \rho, ...$ - **tensor components, in a coordinate system**
- lower case Latin indices from the beginning of the alphabet $a, b, c, ...$ - **abstract index notation**
- lower case Latin indices starting with i, j, k, ... - **spatial indices x,y,z,...**
- upper case Latin indices A, B - **angular spatial indices, corresopnding to ϕ, θ in spherical coordinates**

The metric of the entire 4 dimensional spacetime is written as $g^{(4)}$ while the (induced) metric of a 3 dimensional hypersurface is g. Analogously, the metric of a 2 dimensional surface is denoted by $\tilde{g}$.

The Levi-Civita symbol that is the same in every coordinate system is written as $\in$, while the Levi-Civita tensor is ϵ

The Lie derivative of an arbitrary (p,q)-tensor T with respect to a vector field K is denoted by: $\pounds_K T^{a_1...a_p}_{b_1...b_q}$

I will occasionaly write the partial and covariant derivative in shorter notation as:

$$\partial_c T^{a_1...a_p}_{b_1...b_q} = T^{a_1...a_p}_{b_1...b_q,c}$$
$$\nabla_c T^{a_1...a_p}_{b_1...b_q} = T^{a_1...a_p}_{b_1...b_q;c}$$

(Note: the order of upper and lower incidices on tensor quantities usually matters. Only in this section will $T^{a_1...a_p}_{b_1...b_q}$ be written so ambiguously. This is done to save space and since the ordering of indices is irrelevant here and could be anything)

The Riemann tensor, Ricci tensor and Ricci scalar with respect to a space-time metric $g^{(4)}$ are denoted respectively as $R^{(4)}_{abcd}, R^{(4)}_{ab}$ and $R^{(4)}$. The same quantities with respect to a hypersurface metric g are written with an R instead of $R^{(4)}$.

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II. Manifolds, Spacetimes & Black Holes

This section will collect most of the differential geometry and general relativity results necessary for understanding the proof.

A spacetime is a smooth Lorentzian manifold - $(\mathcal{M}, g^{(4)}_{ab})$. Here, Lorentzian means g is non-degenerate and has signature (1, n), where (n+1) is the number of spacetime dimensions.

Some definitions necessary for the proof, relating to spacetimes and specifically black holes, are given here:

A. Hypersurfaces, normal and tangent vectors

Say we have a spacetime $(\mathcal{N}, g^{(4)})$ of dimension n+1. A hypersurface, call it Σ, is a n dimensional manifold embedded in $\mathcal{N}$.

1. Embeddings

This can be achieved one of two ways, the first one being more formal: One defines an embedding map $\Phi : \Sigma \to \mathcal{N}$, where Φ is a map between manifolds with certain nice properties (injective, structure-perserving, etc.) and Σ is an n dimensional manifold which exists independently of $\mathcal{N}$. Equipping $\mathcal{N}$ with coordinates x^α and Σ with coordinates $y^a(x)$, one can pull back the existing metric $g^{(4)}_{\alpha\beta}$ on $\mathcal{N}$ to Σ in the standard way, using:

$$g_{ab} = E^\alpha_a E^\beta_n g^{(4)}_{\alpha\beta} \tag{1}$$

where $E^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$ is the Jacobian.

The second approach, loosely speaking, identifies Σ with a subspace of $\mathcal{N}$ through a relation:

$$\Sigma = \{x \in \mathcal{N} \mid S(x) = 0\}$$

where $S(x)$ is a scalar function on $\mathcal{N}$. This is in direct analogy with the familiar case in $\mathbb{R}^3$, where the relation $f(x, y, z) = 0$ defines a 2 dimensional surface. The metric on Σ is then the induced metric:

$$g(x) = g^{(4)}(x) \mid_{S(x^\mu)=0}$$

These definitions become equivalent on spaces equipped with metrics, which is usually what we're working with anyway, so the second less abstract definition is what I'll be using from now on.

2. Normal and tangent vectors (fields)

A vector normal to the hypersurface $S(x) = 0$ is given by:

$$N_\mu = \partial_\mu S(x) \quad (2)$$

The logic here is that $S(x) = 0$ is a condition that holds on a specific hypersurface in $\mathcal{N}$. As soon as we move away from the hypersurface, the relation doesn't hold anymore, and therefore taking the derivative of $S(x)$ in a certain direction tells us "how much we're moving away from the hypersurface".

A tangent vector is defined as:

$$V^\mu \partial_\mu S(x) = 0 \quad (3)$$

which intuitively makes sense - a vector orthogonal to the normal vector is indeed a tangent vector. A hypersurface is called timelike if it's normal vector field is everywhere spacelike ($\partial_\mu S(x)\partial^\mu S(x) > 0$) and vice-versa it is called spacelike if it's normal vector field is everywhere timelike ($\partial_\mu S(x)\partial^\mu S(x) < 0$). This intuitively makes sense as the tangent vectors, i.e. the vectors that "live on the hypersurface", will be timelike/spacelike if the hypersurface is timelike/spacelike respectively.

A vector field:

$$N^\alpha = f(x) g^{(4)\alpha\beta} \partial_\beta S(x) \quad (4)$$

where f is a scalar (non-vanishing) function, satisfies condition (3) the same way (2) does and is therefore just as valid in the role of a vector field normal to the hypersurface $S(x) = 0$.

This can be used to normalize a normal vector field (4) by choosing $f(x)$ as $(\mid \partial_\mu S(x)\partial^\mu S(x) \mid)^{-1/2}$. Using this we have:

$$\begin{aligned} N_\mu N^\mu &= (\mid \partial_\nu S(x)\partial^\nu S(x) \mid)^{-1} \partial_\mu S(x)\partial^\mu S(x) \\ &= \mathrm{sgn}(\partial_\mu S(x)\partial^\mu S(x)) \equiv \epsilon \end{aligned}$$

where ϵ depends on if the vector field is timelike (-1), spacelike (+1) or null (0).

B. Foliations

A foliation is, informally, the decomposition of a manifold of dimension p into submanifolds of dimension q, satisfying some properties that guarantee it behaves nicely. This decomposition is then parametrized by (p-q) parameters. The submanifolds are called *leaves* of the foliation. Two simplet examples of foliations in $\mathbb{R}^3$ are:

- Choosing z as the parameter; z = const. gives us xy planes as leaves which cover the manifold as we vary z from $-\infty$ to $+\infty$.
- Choosing r, in spherical coordinates, as the parameter; r = const. gives us spheres of radius r as leaves which cover the manifold as we vary r from 0 to $+\infty$.

These two examples, in a slightly more complicated form, appear in Israel's proof later on and are therefore important to understand intuitively. While this all seems vague and hand-wavey, foliations are a fairly robust topic that would require a heap of definitions and it doesn't seem economical to introduce all of them since they would never be used - an intuitive notion will do for understanding the proof. A complete, formal definition can be found in the literature[3].

One important result required for the proof is that given a nowhere vanishing vector field dN on some manifold Σ, one can foliate the manifold with leaves $\Xi = \Sigma \mid_{N=\text{const.}}$, with parameter N. It is also the case that these leaves all have the same topology. These are some highly non-trivial claims and I won't directly support them here, but they can be found in the literature on foliations[3] and in the general relativity textbooks by Straumann[6] and Heusler[7].

C. Hypersurface orthogonality & Frobenius integrabilty

Say $S(x) = 0$ specifies a hypersurface in some spacetime. A feature of the vector field

$$N_\alpha = f(x)\partial_\alpha S(x) \quad (5)$$

is that it is everywhere orthogonal to the hypersurface $S(x) = 0$, by definition. We are now interested in finding a condition on an arbitrary vector field ξ_α that guarantees that it can be written as (5) - or in other words that there exists a hypersurface $S(x)$ such that ξ_α is orthogonal to it at every point. This proposition is reminiscent (and is in fact a generalization to manifolds) of the well known statement that a vector field with vanishing rotation can be written as the gradient of a scalar field.

A useful quantity to consider for this will be the "rotation" of a vector field:

$$\nabla_a \xi_b - \nabla_b \xi_a = \partial_a \xi_b - \partial_b \xi_a \quad (6)$$

Looking at the identity for scalar functions:

$$\nabla_a(\partial_b S(x)) - \nabla_b(\partial_a S(x)) = \partial_a\partial_b S(x) - \partial_b\partial_a S(x) = 0$$

it is clear that a normal vector field written as a four-derivative of a scalar function has vanishing rotation. More generally a normal vector field of the form (5) would satisfy:

$$\begin{gathered} \nabla_a N_b - \nabla_b N_a = \\ \nabla_a(f(x)\partial_b S(x)) - \nabla_b(f(x)\partial_a S(x)) = \\ \cancel{f\partial_a(\partial_b S)} - \cancel{f\partial_b(\partial_a S)} + (\partial_a f)(\partial_b S) - (\partial_b f)(\partial_a S) = \\ (\partial_a f)(\partial_b S) - (\partial_b f)(\partial_a S) = \\ (\partial_a \ln f) N_b - (\partial_b \ln f) N_a \end{gathered}$$

Or in short:

$$\nabla_{[a} N_{b]} = (\partial_{[a} \ln f) N_{b]}$$

If we now suspect that an arbitrary vector field ξ_a could be written as (5), i.e. normal to some hypersurface, a necessary condition would be that it at least satisfy the same equation:

$$\nabla_{[a}\xi_{b]} = (\partial_{[a}\ln f)\xi_{b]} \tag{7}$$

The trouble with this is that we don't know $f(x)$ and therefore can't check this, given only ξ_a. Luckily, multiplying (7) by ξ_c and antisymmetrizing all three indices gives:

$$\xi_{[c}\nabla_a\xi_{b]} = 0 \tag{8}$$

where $\xi_{[c}(\partial_a \ln f)\xi_{b]}$ vanishes because we're antisymmetrizing two of the exact same vector fields, ξ_c and ξ_b. What we've obtained in (8) is the **Frobenius integrability condition**! Checking that this holds for a vector field is fairly straight forward and implies that, at least locally, it can be written as a hypersurface orthogonal vector field - (5).

(Note - in the language of differential forms, (8) takes the simple form:

$$\xi \wedge d\xi = 0$$

as it is often written in the literature.)

D. Extrinsic geometry: the 1st and 2nd fundamental forms & mean curvature

1. *Projected tensors and the 1st fundamental form*

Say we have a spacetime $(\mathcal{N}, g^{(4)}_{ab})$ and a globally timelike or spacelike hypersurface Σ with a normalized normal vector field N_a. The hypersurface Σ, with the induced metric $g = g^{(4)}\mid_\Sigma$, is a spacetime in it's own right and it is completely valid to forget about the surrounding higher-dimensional spacetime and calculate all the familiar curvature properties of Σ (say the Riemann tensor) using it's induced metric, covariant derivative etc. The "projected" or induced metric on Σ is given by:

$$g_{\mu\nu} = g^{(4)}_{\mu\nu} - \epsilon N_\mu N_\nu \tag{9}$$

which turns out to be equivalent to the pulled back metric (1). Some intuitive properties of this metric are that it is orthogonal to N^μ:

$$\begin{aligned} g_{\mu\nu}N^\mu &= g^{(4)}_{\mu\nu}N^\mu - \epsilon(N_\mu N^\mu)N_\nu \\ &= N_\nu - \epsilon^2 N_\nu = 0 \end{aligned} \tag{10}$$

and that, for a tangent vector V^μ:

$$g_{\mu\nu}V^\mu = g^{(4)}_{\mu\nu}V^\mu - \epsilon\cancelto{0}{(N_\mu V^\mu)}N_\nu = V_\nu \tag{11}$$

from which we see that, for tangent vectors, the induced metric g functions equivalently to the full spacetime metric $g^{(4)}$. Overall, we see that g is degenerate for N^μ and is therefore not a valid metric, unless we restrict ourselves to the tangent vectors of the hypersurface Σ. g is called the **1st fundamental form** of the hypersurface Σ. By construction, g has the form of a projector, in that it takes a tensor and subtracts any components normal to the hypersurface. This can be used to define projected tensors, denoted $\bar{T}^{a\dots}_{b\dots}$, as:

$$\bar{T}^{a_1\dots a_p}{}_{b_1\dots b_q} = g^{a_1}{}_{a'_1}\dots g^{a_p}{}_{a'_p}g^{b'_1}{}_{b_1}\dots g^{b'_q}{}_{b_q}T^{a'_1\dots a'_p}{}_{b'_1\dots b'_q}$$

One should note that the raising and lowering of indices should generally always be done using the full metric $g^{(4)}$, however on projected tensors $\bar{T}$ this becomes equivalent to using the projected metric g, because the normal vectors appearing in definition (9) vanish when contracted with g as can be seen from (10):

$$N_\mu \bar{T}^\mu{}_\nu = N_\mu g^\mu{}_\rho T^\rho{}_\nu = 0$$

Say we want to covariantly differentiate something on the surface Σ; there are two ways we could arrive at a covariant derivative operator:

- Treating Σ as an independent manifold - calculating Christoffel symbols using the induced metric g, denoted $\Gamma^{(g)}$, and writing:

$$\nabla^{(g)}_\alpha V_\beta = \partial_\alpha V_\beta - \Gamma^{(g)}{}^\rho_{\alpha\beta}V_\rho$$

- By projecting down the (p,q+1)-tensor $\nabla_a T$ using the induced induced metric g:

$$\bar{\nabla}_\alpha \bar{T}_{\dots} = g^{\alpha'}{}_\alpha g^{\dots'}{}_{\dots}\nabla_\alpha T_{\dots'}$$

The first approach gives a valid covariant derivative on the hypersurface, as long as V is a tensor with only components tangent to Σ, since no information about the ambient manifold goes into the definition. It would be nice if the second approach was equivalent, i.e. produced a valid covariant derivative as well. This can be checked by confirming the conditions that a given operator be a covariant derivative: (1) linearity in both variables, (2) the Leibniz rule, (3) commutation on scalars, (4) commutation with contractions and (5) compatibility with the induced metric g. Everything except (5) is trivial to check and follows directly from the properties of the full covariant derivative before projecting it down to Σ. To check (5), we do:

$$\begin{aligned} \bar{\nabla}_\alpha g_{\beta\gamma} &= g^{\alpha'}_\alpha g^{\beta'}_\beta g^{\gamma'}_\gamma \nabla_{\alpha'}g_{\beta'\gamma'} \\ &= g^{\alpha'}_\alpha g^{\beta'}_\beta g^{\gamma'}_\gamma \nabla_{\alpha'}(g^{(4)}_{\beta'\gamma'} - \epsilon N_{\beta'}N_{\gamma'}) \end{aligned}$$

which using the compatibility of $g^{(4)}$ with the full covariant derivative gives:

$$= -\epsilon\; g^{\alpha'}_\alpha g^{\beta'}_\beta g^{\gamma'}_\gamma \nabla_{\alpha'}(N_{\beta'}N_{\gamma'})$$

and now by Leibniz, for the full covariant derivative we have:

$$= -\epsilon\; g^{\alpha'}_\alpha g^{\beta'}_\beta g^{\gamma'}_\gamma [N_{\beta'}\nabla_{\alpha'}(N_{\gamma'}) + N_{\gamma'}\nabla_{\alpha'}(N_{\beta'})] = 0$$

Contracting g with the normal vectors not in the covariant derivative operator, we get 0 for both terms, thus proving metric compatibility for g. This should be reassuring - projecting using g is a valid way of calculating tensors on the hypersurface Σ.

2. *Extrinsic curvature and the 2nd fundamental form*

The curvature information that we can get by only using g and the vectors tangent to Σ is called the **intrinsic** curvature, since it makes no reference to the ambient space and therefore doesn't depend at all on how Σ sits in it.

However, if we are told that Σ does in fact live in a higher dimensional space, and that N_μ is it's normal vector, which extends into this extra dimension, then there is additional information one can extract about Σ. For example, the Riemann tensor on Σ, denoted $\bar{R}_{ijkl}$, is valid as a characterisation of it's intrinsic curvature for components tanget to Σ, denoted by Latin letters. It is, on the other hand, also true that the ambient spacetime $\mathcal{N}$ has it's own Riemann tensor, $R_{\mu\nu\rho\sigma}$, and that there certainly exist mixed components $N^\mu R_{\mu jkl}$, evaluated on the surface Σ, that tell us "how the hypersurface bends" in the surrounding space. These additional curvature components can be considered as **extrinsic** and are information that doesn't exist on the standalone manifolds with metric - they certainly cannot be expressed purely using the metric and tangent vectors.

This motivates us to try to define a quantity on Σ that captures how Σ is embedded in the ambient space, using the normal vector field N_μ. To this end, we define the **2nd fundamental form** or **extrinsic curvature**:

$$K_{\alpha\beta} = g^\gamma{}_\alpha g^\delta{}_\beta \nabla_\gamma N_\delta \tag{12}$$

which is a *projected* tensor, with components tangent to the hypersurface Σ. It contains information about how the normal vector field changes as we move along Σ. Intuitively, it makes sense that if the normal stays exactly the same, then our hypersurface is embedded in such a way that it's completely flat in the ambient space and doesn't turn at all. If the normal changes a lot, then that means that our hypersurface is also turning violently in the ambient space. One might argue that this quantity vanishes since we're projecting the normal vector onto the tangent plane, however $\nabla_\gamma N_\delta$ compares normal vectors at two close points and as such very well might have components tangent to the plane. We also can't just contract N_δ with g^δ_β since:

$$g^\gamma{}_\alpha g^\delta{}_\beta \nabla_\gamma N_\delta \neq g^\gamma{}_\alpha \nabla_\gamma (g^\delta{}_\beta N_\delta) = 0$$

because

$$\nabla_\gamma g_{\alpha\beta} = \nabla_\gamma (g^{(4)}_{\alpha\beta} - \epsilon N_\alpha N_\beta) = -\epsilon \nabla_\gamma (N_\alpha N_\beta) \neq 0$$

It can be shown that the extrinsic curvatuer tensor is symmetric. Since it is a projected tensor, we only need consider it's action on two arbitrary vectors tangent to Σ; X, Y

$$K_{ab}X^aY^b - K_{ba}X^aY^b =$$
$$X^aY^b\nabla_a N_b - X^aY^b\nabla_b N_a =$$
$$-X^a\nabla_a(Y^b)N_b + Y^b\nabla_b(X^a)N_a =$$
$$-\nabla_X(Y^b)N_b + \nabla_Y(X^a)N_a = -[X,Y]^cN_c$$

Now we just need to prove that for two tangent vectors X, Y, their commutator is also tangent. For $S(x) = 0$ defining the hypersurface in question, it is by definition of a tangent vector that

$$X(S(x)) = X^\mu \partial_\mu S(x) = X^\mu N_\mu = 0$$

Following from this, we have:

$$\begin{aligned} [X,Y]S(x) &= X^\mu\partial_\mu(Y^\nu\partial_\nu S(x)) - Y^\mu\partial_\mu(X^\nu\partial_\nu S(x)) \\ &= X(Y(S(x))) - Y(X(S(x))) \\ &= X(0) - Y(0) = 0 \end{aligned}$$

which is exactly the condition that $[X,Y]^a$ is tangent. Returning back to the symmetrization of K_{ab}, we have $[X,Y]^cN_c$ which is the contraction of a tangent vector with a normal vector, i.e. equal to 0:

$$K_{[ab]}X^aY^b = -[X,Y]^cN_c = 0$$

from which it follows that:

$$K_{ab}X^aY^b = K_{ba}X^aY^b$$

on arbitrary vectors X, Y and therefore in general.

An alternative and equivalent definition of extrinsic curvature can be given in terms of a Lie derivative:

$$K_{\alpha\beta} \stackrel{?}{=} \frac{1}{2} g_\alpha{}^\gamma g_\beta{}^\delta \pounds_N g^{(4)}_{\gamma\delta} \tag{13}$$

Writing this out and using metric compatibility, we get

$$\begin{aligned} \frac{1}{2} g_\alpha{}^\gamma g_\beta{}^\delta \pounds_N g^{(4)}_{\gamma\delta} &= \frac{1}{2} g_\alpha{}^\gamma g_\beta{}^\delta (\nabla_\gamma N_\delta + \nabla_\delta N_\gamma) \\ &= \frac{1}{2}(K_{\alpha\beta} + K_{\beta\alpha}) = K_{(\alpha\beta)} = K_{\alpha\beta} \end{aligned}$$

where in the last line, we recognized the first definition of the extrinsic curvature and the symmetrization of a tensor with 2 indices.

This definition is useful because it manifestly shows that if we have a surface whose only normal(s) is (are) Killing vector fields, then it's extrinsic curvature automatically vanishes, since Killing vectors satisfy $\pounds_N g^{(4)}_{\gamma\delta} = 0$.

3. *Mean curvature*

The mean curvature is defined as the trace of the extrinsic curvature (up to a factor), given by:

$$K \equiv K^{ij} g^{(4)ij} = K^i_i \tag{14}$$

E. Stationary metrics

A spacetime $(\mathcal{N}, g^{(4)}_{ab})$ is said to be **stationary** if there exists a **timelike Killing vector field** on $\mathcal{N}$. This is equivalent to the vanishing of the Lie derivative for a metric-compatible connection:

$$\begin{aligned} 0 = \pounds_K g^{(4)}_{ab} &= K^c \cancelto{0}{\nabla_c g^{(4)}_{ab}} + g^{(4)}_{ac}\nabla_b K^c + g^{(4)}_{cb}\nabla_a K^c \\ &= g^{(4)}_{ac}\nabla_b K^c + g^{(4)}_{cb}\nabla_a K^c \\ &= \nabla_b K_a + \nabla_a K_b = 0 \end{aligned} \tag{15}$$

along with the timelike condition: $K_cK^c < 0$.

F. Adapted coordinates for stationary spacetimes

The plan is now to give an intuitive interpretation of this somewhat formal Lie derivative vanishing condition and, to this end, to construct a special coordinate system that will be used throughout the proof.

To start, we should remember the property of the Lie derivative - it can be written using partial derivatives instead of covariant derivatives, since the Christoffel symbols exactly cancel. Using this, we could have equivalently written the first equation in (15) as:

$$\pounds_K g^{(4)}_{ab} = K^c \partial_c g^{(4)}_{ab} + g^{(4)}_{ac} \partial_b K^c + g^{(4)}_{cb} \partial_a K^c = 0 \tag{16}$$

in which case the derivative of the metric doesn't vanish on it's own as before.

Next we want to construct a *special coordinate system adapted to the Killing field K*. This amounts to choosing the time coordinate to point in the direction of the vector field K so that the above equation (16) takes on a simpler form.

We can start by considering a 4-dimensional Lorentzian manifold $(\mathcal{N}, g^{(4)}_{ab})$. Let us now say that we want to define some (smooth, etc.) 3-dimensional hypersurface Σ on $\mathcal{N}$. We could, for example, perscribe such a hypersurface by the condition $t = 0$, where t is the time coordinate in some specific coordinate system given on an open set equipped with a chart of $\mathcal{N}$. The assumed smoothness of the manifold $\mathcal{N}$ then guarantees that comparing the results of this procedure on overlapping charts would produce a smooth surface Σ across the entire manifold $\mathcal{N}$. More generally, the condition $t = 0$ could be replaced by: $S(x^\mu) = 0$, where S is some scalar function defined on $\mathcal{N}$, as discussed in the hypersurface section, earlier.

Either way, assume we have successfully specified a hypersurface by setting $S(x^\mu) = 0$ and that the hypersurface is spacelike. This is easy to check as the vector field normal to $S(x^\mu) = 0$ is given by:

$$N_\mu = \partial_\mu S(x) \tag{17}$$

The spacelike condition then simply reads: $\partial^\mu S(x) \partial_\mu S(x) < 0$ for every x on Σ.

Assume now, additionally, that a timelike vector field K^a is given, nonvanishing on the entire manifold $\mathcal{N}$. we can then use this vector field as the definition of the time coordinate through the following procedure - we consider a family of curves $u(s) : \mathbb{R} \to \mathcal{N}$, obeying the equation:

$$\frac{d}{ds} u(s) = K(u(s))$$

with the set of initial conditions $(u(s = 0) = p_0)$ for all points p_0 on the hypersurface Σ. This locally defines a family of ODEs with unique solutions, which can in principle be extended across the entire manifold $\mathcal{N}$ due to it being C^∞. Taking now any old coordinates $\{x^\mu\}$, our new coordinate system along a specific curve originating from the point $p_0 \in \Sigma$ is then given by:

$$\{x^0 = s, x^1(u(s)), x^2(u(s)), x^3(u(s))\}.$$

In this way, we've constructed a good coordinate system in which "forward in time" points along the vector field K and the spatial coordinates follow it around the manifold. This is particularly useful as, in these coordinates, the Killing vector field K takes the simple form: $K = \frac{\partial}{\partial x^0}$ since the time coordinate points exactly in it's direction, by construction.

Since $K = \frac{\partial}{\partial x^0}$ is now constant, the partial derivatives of K in (16) vanish, and we're left with:

$$\pounds_K g^{(4)}_{\mu\nu} = \left(\frac{\partial}{\partial x^0}\right)^\rho \partial_\rho g^{(4)}_{\mu\nu} = 0 \tag{18}$$

$$= \partial_0 g^{(4)}_{\mu\nu} = 0 \tag{19}$$

We now conclude that in adapted coordinates, the stationary spacetime condition (15) reduces to the simple statement that *none of the metric components depend explicitly on time*, as one would expect from the name "stationary". This might seem like it was a lot of work, just to arrive at the standard interpretation of the Lie derivative, but this *adapted coordinate system* construction will appear in the theorem itself, twice, and was therefore instructive to go through properly. For reference, this construction goes by the name "Flow box theorem".

G. Static metrics

Static metrics are a special subset of stationary metrics, as introduced in the previous section. To recapitulate, formally: a stationary spacetime is one that admits a timelike Killing vector field.

The specialization to a static metric is then straightforward - it is additionally required that the Killing vector field be also hypersurface orthogonal, or in other words that it satisfies the Frobenius integrability condition (8).

1. *The block diagonal form*

I will now show that stationarity together with hypersurface orthogonality together imply that the metric can be written in block diagonal form, separating the time block from the spatial block, or in other words, such that $g^{(4)}_{0i} = 0$ for i $= x, y, z$.

Say we have a Killing vector field $K^a = (\partial_t)^0$ in adapted coordinates, with components δ^α_0, that satisfies the Killing equation: $\nabla_a K_b = \nabla_b K_a$. The components of the corresponding covector field are then:

$$K_\alpha = g^{(4)}_{\alpha\beta} \delta^\beta_0 = g^{(4)}_{\alpha 0} \tag{20}$$

Referring to the hypersurface orthogonality condition and the Killing equation for K, we have:

$$K^\gamma K_{[\alpha} \nabla_\beta K_{\gamma]} = K_{[\alpha} \nabla_{\beta]} K_\gamma + \frac{1}{2} K_\gamma \nabla_{[\alpha} K_{\beta]} = 0$$

and now contracting with K^γ and rearraging we get:

$$K_{[\alpha} \nabla_{\beta]} (K_\gamma K^\gamma) + K^\gamma K_\gamma \nabla_{[\alpha} K_{\beta]} = 0$$

writing $K_\gamma K^\gamma = K^2$ we can rewrite this as:

$$\nabla_\alpha(K_\beta/K^2) - \nabla_\beta(K_\alpha/K^2) = 0$$

which is precisely equation (6). This implies that K_μ/K^2 can be written as $\partial_\mu S(x)$, or in other words - that $K_\mu = K^2\partial_\mu S(x)$. Comparing this with (20), we conclude:

$$g^{(4)}_{\mu 0} = K^2\partial_\mu S(x)$$

The stationary metric condition, in adapted coordaintes then implies:

$$\partial_0(g^{(4)}_{\alpha\beta}) = 0 \longrightarrow \partial_0\left(\frac{g^{(4)}_{i0}}{g^{(4)}_{00}}\right) = \partial_0\left(\frac{\partial_i S(x)}{\partial_0 S(x)}\right) = 0 \quad (21)$$

$$\longrightarrow \partial_0(K^2\partial_0 S(x)) = 0 \quad (22)$$

Using these, we can conclude two things:

- While $S(x)$ has to depend on t so that $\partial_0 S(x) \neq 0$, $\partial_0 S(x)$ as well as $\partial_i S(x)$ shouldn't because of (21). This implies that S is at most linear in time and a general solution to this is then
$$S(x) = At + B + f(x^i) \sim t + \tilde{f}(x^i)$$
The A can be dropped since it is just a constant and amouts to rescaling the time coordinate.

- Using (22) we can conclude that K^2 shouldn't depend on time.

Writing out the one form of a scalar function:

$$dS = (\partial_\mu S(x))dx^\mu = K^{-2}g^{(4)}_{0\mu}dx^\mu$$
$$= K^{-2}g^{(4)}_{00}(dt + g^{(4)}_{0i}/g^{(4)}_{00}dx^i)$$

The metric can then be rewritten as:

$$g^{(4)}_{\alpha\beta}dx^\alpha dx^\beta = g^{(4)}_{00}dt^2 + 2g^{(4)}_{0i}dx^i dt + g^{(4)}_{ij}dx^i dx^j$$

completing the square

$$= g^{(4)}_{00}\left(dt^2 + 2(g^{(4)}_{0i}/g^{(4)}_{00})dx^i dt + (g^{(4)}_{0i}g^{(4)}_{0j}/(g^{(4)}_{00})^2)dx^i dx^j\right)$$
$$- (g^{(4)}_{0i}g^{(4)}_{0j}/(g^{(4)}_{00})^2)dx^i dx^j + g^{(4)}_{ij}dx^i dx^j$$

and grouping some terms

$$= g^{(4)}_{00}\left(dt + (g^{(4)}_{0i}/g^{(4)}_{00})dx^i\right)^2$$
$$+ \left(g^{(4)}_{ij} - (g^{(4)}_{0i}g^{(4)}_{0j}/(g^{(4)}_{00})^2)\right)dx^i dx^j$$

Now using $g^{(4)}_{\mu 0} = K^2\partial_\mu S(x)$ and $S(x) = t + \tilde{f}(x^k)$, we can recognize dS^2 in the first bracket:

$$g^{(4)}_{\alpha\beta}dx^\alpha dx^\beta = K^2(dS)^2 + \left(g^{(4)}_{ij} - (g^{(4)}_{0i}g^{(4)}_{0j}/(g^{(4)}_{00})^2)\right)dx^i dx^j$$

Thus, using $T = S(x)$ as a new time coordinate, we arrive at a metric block diagonal in time and space! This choice is equivalent to the old time coordinate - we're still in the Killing field adapted coordinate system (as we saw, $T = S(x)$ is linear in t so $\partial_t = \partial_T$ are the same vector fields and hence both Killing vector fields. In these coordinates, a generic spatial hypersurface orthogonal to the Killing field is given by the condition $T =$ const.

2. *Recap & Static vs Standard-Static*

From what we've seen up until now, it is possible to (locally) write any static space-time metric in the form:

$$g^{(4)}_{\alpha\beta}dx^\alpha dx^\beta = -(N(\vec{x}))^2(dt)^2 + g_{ij}(\vec{x})dx^i dx^j \quad (23)$$

Due to clash of notation with the mean curvature, N^2 will from now on denote the norm of the Killing vector field, sometimes referred to as the *lapse function*. It should be noted that N does not depend on t, and as such only varies across Σ. The time coordinate is here renamed back to t, but is in fact the dS from the previous section. $g_{ij} := g^{(4)}_{ij} - g^{(4)}_{0i}g^{(4)}_{0j}/(g^{(4)}_{00})^2$ was appropriately redefined in these new coordinates in the spacial block, which is still completely time independent.

In the literature there is mention of a distinction between a static space-time and a standard static space-time. A static space-time is defined exactly as I had written at the beginning of this section: stationarity + hypersurface orthogonality of the Killing field. This is however only true locally, as the Frobenius integrability condition (8) is a local notion. It could happen that different portions of the manifold admit different functions $S(x)$ which do not patch together smoothly; we then don't have a global choice of time coordiante $T = S(x)$ that is hypersurface orthogonal to the Killing field and (23) only holds locally as well.
A standard static space-time is then, as one could guess, defined such that there in fact **does** exist a global choice of a smooth time coordinate $T = S(x)$ that is hypersurface orthogonal, or in other words that (23) holds everywhere on the space-time. This is the definition that everyone usually works with, including Israel in his theorem, and is the one we'll be using from now on.

Standard static spacetimes can be denoted globally by the set: (Σ, g, N). This contains all the same information as $(\mathcal{N}, g^{(4)})$, where we used the standard static property to write $\mathcal{N}$ as the direct product $(\mathbb{R}\times\Sigma)$ and, correspondingly the space-time metric $g^{(4)}$ as $(N \otimes g)$, where g is a Riemannian 3-metric.

3. *Curvature and the Einstein equations for a standard static metric*

Given a (standard) static metric:

$$g^{(4)} = -N(x^k)^2dt^2 + g(x^k)$$

it amounts to a straightforward but time-consuming calculation to check that the Riemann tensor, with latin indices $dx^{ijk\dots}$ the tangent, and $dx^0 = Ndt$ the orthogonal coordinates to Σ, is given as:

$$R^{(4)i}{}_{jkl} = R^i_{jkl} \qquad R^{(4)0}{}_{jk0} = \frac{1}{N}\nabla^{(g)}_j\nabla^{(g)}_k N \qquad R^{(4)0}{}_{jkl} = 0$$

and after contracting,

$$R^{(4)}_{00} = \frac{1}{N} g^{jk} (\nabla^{(g)}_j \nabla^{(g)}_k) N \equiv \frac{1}{N} \Delta^{(g)} N$$

$$R^{(4)}_{jk} = -\frac{1}{N} \nabla^{(g)}_j \nabla^{(g)}_k N + R_{jk} \qquad R^{(4)}_{i0} = 0$$

and once again, for the Ricci scalar:

$$R^{(4)} = R - \frac{2}{N} \Delta^{(g)} N$$

(Note that here, the $dx^0 = Ndt$ choice of coordinates introduces extra factors of $\frac{1}{N}$ in some of the tensor components, and also makes $g^{(4)}_{00} = -1$)

The Einstein equations in vacuum $R^{(4)}_{\mu\nu} - \frac{1}{2} g^{(4)}_{\mu\nu} R^{(4)} = 0$ then simplify to:

$$G^{(4)}_{00} = R^{(4)}_{00} - \frac{1}{2} g^{(4)}_{00} R^{(4)} = \frac{1}{2} R = 0 \tag{24}$$

(00-component, the intrinsic curvature of Σ vanishes)

$$G^{(4)}_{0i} = \cancelto{0}{R^{(4)}_{i0}} - \frac{1}{2} \cancelto{0}{g^{(4)}_{i0}} R^{(4)} = 0 \tag{25}$$

(0i-component)

$$G^{(4)}_{jk} = \nabla^{(g)}_j \nabla^{(g)}_k N = N R_{jk} \tag{26}$$

(ij-component) which after tracing and using the $R = 0$ equation gives:

$$\Delta^{(g)} N = 0 \tag{27}$$

or in other words - the lapse function is harmonic on Σ. These will be useful in the proof. A complete derivation can be found in the literature[6].

H. The Gauss-Codazzi equations

The Gauss-Codazzi equations express different components of the Riemann curvature tensor on the hypersurface Σ entirely using Σ's intrinsic curvature quantities - the projected Riemann tensor $\bar{R}_{ijkl}$ and the extrinsic curvature $K_{\alpha\beta}$. Deriving them is fairly straightforward and can be found in great detail in the literature[4]. The form we're going to be using in the proof are expressions for the Einstein tensor, where the index 0 represents the normal component to the hypersurface while all the other indices are exclusively tangent components.

$$G^{(4)}_{00} = -\frac{1}{2} R + \frac{1}{2} (K^2 - K_{ij} K^{ij}) \tag{28}$$

$$G^{(4)}_{0i} = \bar{\nabla}_i K - \bar{\nabla}_j K^j_i \tag{29}$$

$\bar{\nabla}$ is the covariant derivative operator on the hypersurface Σ. $K = (K^i_i)$ denotes the trace of the extrinsic curvature tensor.

I. The Schwarzschild Metric

The static, spherically symmetric solution to the vacuum Einstein equations is the Schwarzschild metric, given here:

$$ds^2 = -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2 \tag{30}$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the standard angular metric element in spherical coordinates. This solution has a curvature singularity at $r = 0$ and a coordinate singularity at $r = 2M$. We will concern ourselves with it's validity on $r > 2M$, where we assume the energy momentum tensor vanishes, as required for the Schwarzschild metric to be a solution, and where the spacetime is standard static.

J. The black hole horizon

1. *Black hole horizons for static metrics*

Let (Σ, g, N) be a (standard) static space-time. Our manifold is the region $N > 0$, and has a boundary at $N = 0$ where the norm of the timelike Killing field vanishes. The reasoning for this $N > 0$ restriction is that in the region$N \leq 0$, ∂_t is no longer a timelike Killing field, and therefore doesn't fit the description of a stationary or static spacetime.
If the boundary surface ($\partial\Sigma$), for a constant time slice of the the space-time, is closed (i.e. compat and has no boundary) as well as having $N = 0$ on it, we call this surface a (spacelike slice of a) **Killing horizon**.
This implies that the timelike Killing vector field becomes null on such a surface, which intuitively means that the lightcone has tilted so much that it's future oriented half points entirely inward on the surface - no timelike or lightlike particle can escape this region once inside.
An event horizon is usually defined somewhat differently, as the "boundary of the past of future null infinity" or more loosely speaking: "the boundary from inside which a lightray following a geodesic would never manage to arrive at spatial infinity in infinite time". This seems far removed from the idea of a timelike Killing field vanishing on a surface, which is a local geometric notion. Nevertheless, thanks to certain *rigidity theorems*[5], due to Hawking and Ellis[2], which relate the definition of a Killing horizon to that of an event horizon for stationary black holes, we can consider them equivalent and proceed to use the more practical Killing horizon definition.

2. *Static black hole horizons*

If the null vector field on a Killing horizon $\partial\Sigma \times \mathbb{R}$ with $N \mid_{\partial\Sigma}$, as defined above, is also hypersurface orthogonal *on the horizon surface itself*, we call it a *static horizon.* This can be shown to be equivalent to a vanishing mean curvature $K = K^i{}_i = 0$ on the horizon. The name "static" follows from the condition that $N = 0$ - the Killing vector field is null on the horizon and this implies that it's also normal to the horizon. A normal vector field that is

simultaniously Killing has vanishing extrinsic curvature, following from the second definition (13).
The horizon in Israel's proof is assumed to be a static horizon, and hence has mean curvature and lapse function vanishing on it.

3. Boundary conditions on a static black hole horizon

On a single static black hole horizon $\partial\Sigma$, in a (standard) static space-time, combining the Gauss-Codazzi equations (28) with the vacuum Einstein equations (24) we get the results:

- The extrinsic curvature tensor $K_{\alpha\beta}$ vanishes completely, which along with asssumed vanishing mean curvature implies that the trace-free part of the extrinsic curvature also vanishes:
$$\mathring{K}_{ij} \equiv K_{ij} - \frac{1}{2}\tilde{g}_{ij}K = 0$$
To elaborate, the extrinsic curvature vanishing follows directly from (24) and Gauss-Codazzi which give: $K_{ij}K^{ij} = 0$. This is quadratic in all components of K_{ij} and as the metric on $\partial\Sigma$ is positive definite this implies all the components vanish independantly.

- The normal derivative of the lapse function N on the static black hole horizon is constant along the black hole horizon:
$$\nabla_{t^a}(\nabla_{n^b}N(x)) = 0 \tag{31}$$
where t^a is a tangent vector and n^b is a normal vector on the black hole horizon.

- The scalar curvature $R^{(\tilde{g})}$ vanishes on the horizon.

Some of these are necessary for Israel's proof, and originally they were introduced as assumptions. Today, we know that they follow independently.

(Note: In all of these considerations of the lapse function N and the metric g on the boundary of Σ, it is important that these quantities do indeed extend to the boundary of the manifold, in a nice way, so that we can actually calculate things on $\partial\Sigma$. For the Schwarzschild metric for example, we obviously can't use the metric in the form (30) to calculate any quantities at $r = 2M$, as it diverges there. This, however, doesn't imply that any of the intrinsic or extrinsic curvature qunatities truly blow up, but in fact that we've only chosen a bad coordinate system. A suitable change of coordinates of the spacelike hypersurface metric $g = g^{(4)} \mid_\Sigma$ (say to a conformally flat coordinate $r = (1 + \frac{m}{2s})^2$ allows us do the calculations. A discussion on this coordinate singularity business can be found in most general relativity textbooks.)

K. Integration on Manifolds

1. Generally covariant Integration

Say we wanted to integrate a scalar function f defined everywhere on the manifold Σ equipped with metric g. This begs the question of how to extend the "flat" integral to curved spaces with metric?
One problem we immediately notice is that the "infinitesimal volume element" d^4x transforms as:
$$d^4x \longrightarrow d^4y = \left|\det\left(\frac{\partial y}{\partial x}\right)\right| d^4x = J\, d^4x \tag{32}$$
so not as a scalar but as a **tensor density of weight -1**. In general, tensor densities of weight w are defined by their transformation property:
$$\mathcal{T}'_{a'b'\ldots} = J^{-w} \times \left(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\ldots\mathcal{T}_{ab\ldots}\right)$$
so just as normal tensors, but with an extra Jacobian-to-$(-w)$-power term.

Assuming we're integrating a scalar function for which it holds that $f'(y) = f(x)$, under a change of coordinates $y \to x$, we get:
$$\int f(x)\, d^4x \neq \int f'(y)\, d^4y = \int f(x)\, J\, d^4x \tag{33}$$
for a general coordinate transformation with Jacobian $J = \left|\det\left(\frac{\partial y}{\partial x}\right)\right| \neq 1$. It seems that the integral of a scalar function doesn't transform as a scalar when integrated on a curved space, because the volume element doesn't transform as a scalar. This is to be intuitively expected since our notion of distance differes as we move around our space, in such a way that is encoded in the metric g, which is a (0,2)-tensor and hence isn't invariant to general transformations either.
If we want to have a concept of an integral that **is** invariant however, we'll have to do something to d^4x, such that when we integrate and change coordinates, this extra part in the integral cancels the $J = \left|\det\left(\frac{\partial y}{\partial x}\right)\right|$ that the d^4x introduces.

To this end we look at the determinant of the metric $|\det(g)|$ written $|g|$ for short, and it's transformation properties. We know the metric is a (0,2)-tensor, the components of whcih transforms as:
$$g'_{\mu'\nu'} = \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}g_{\mu\nu}$$
Taking the determinant of both sides then gives:
$$|g'| = \left|\det\left(\frac{\partial x}{\partial y}\right)\right|\left|\det\left(\frac{\partial x}{\partial y}\right)\right| \times |g| = J^{-2}|g|$$
and now taking the root:
$$\sqrt{|g'|} = J^{-1}\sqrt{|g|}$$
or in other words, $\sqrt{|g|}$ is a tensor density of weight 1.

Comparing this with (33), we see that if we instead perscribe a new infinitesimal volume element $\sqrt{g}\ d^4x$, this object is now truly scalar:

$$\sqrt{|g'|}\ d^4y = J^{-1}\sqrt{|g|}\ J\ d^4x = \sqrt{|g|}\ d^4x$$

With this notion of integration, the integral of a scalar is coordinate (or "observer") independent:

$$\int f(y)\sqrt{|g'|}\ d^4y = \int f(x)\sqrt{|g|}\ d^4x$$

as it should be.

We're now done - integrating functions on manifolds is settled! To convert this into modern notation however, we need to consider one more thing.

The Levi-Civita symbol $\in_{abcd}$ is defined as either 0 or 1 by the standard rules for Levi-Civita symbols **in every coordinate system**.
Let's now, for the sake of argument, pretend $\in$ are components of a tensor and try to transform them as perscribed by tensor transformation properties:

$$\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial x^\rho}{\partial y^{\rho'}}\frac{\partial x^\sigma}{\partial y^{\sigma'}} \in_{\mu\nu\rho\sigma} \equiv \det\left(\frac{\partial x}{\partial y}\right) \in'_{\mu'\nu'\rho'\sigma'}$$

From this we see that it holds true that ($\in' = J \in$) or in other words, that it is a tensor density of weight $+1$. This can again be remedied by multiplying with $\sqrt{|g|}$, which gives us now a true tensor:

$$\epsilon'_{abcd} = \sqrt{|g'|} \in'_{abcd} = J^{-1}\sqrt{|g|}J \in_{abcd} = \sqrt{|g|} \in_{abcd} = \epsilon_{abcd}$$

The reason we did this is because such a tensor already contains a $\sqrt{(|g|)}$ term, which we need when integrating on manifolds. Say we "integrate" ϵ_{abcd} with respect to the "naive" measure d^4x; we obtain:

$$\int \epsilon_{\mu\nu\rho\sigma} d^4x = \int \in_{\mu\nu\rho\sigma}\ \sqrt{|g|} d^4x$$

where now we recognize the coordinate-independent volume element $\sqrt{|g|}d^4x$ and the (by definition) coordinate independent $\in_{\mu\nu\rho\sigma}$.

It is somewhat unclear what to do with the indices on $\in_{\mu\nu\rho\sigma}$, but the general idea of "integrating" ϵ seems to be on the right track.

This essentially motivates the modern notation for integrating functions using differential forms:

$$\int \epsilon_{abcd}\ f = \int f(x)\epsilon_{\mu\nu\rho\sigma} dx^\mu \otimes dx^\nu \otimes dx^\rho \otimes dx^\sigma =$$

$$= \int f(x)\sqrt{|g|}\ dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \equiv \int f(x)\sqrt{|g|} d^4x$$

This is now manifestly completely covariant and is the notation used in most literature when cocerning integration.

(Note: The procedure given here - of multiplying with $\sqrt{|g|}$ - is in fact a general procedure for converting a tensor density of weight w into a tensor; we just multiply $\mathcal{T}_w$ by $|g|^{-w/2}$.)

2. *Green's Theorem*

Take Σ to be a compact oriented Riemannian manifold of dimension n, with boundary $\partial\Sigma$. If f is a smooth scalar function and X^a is a vector field, both on Σ, it then holds that:

$$\int_\Sigma \nabla_a(f)X^a + \int_\Sigma f\nabla_a X^a = \int_{\partial\Sigma} f\ n_a X^a \qquad (34)$$

with the appropriate volume forms on Σ and $\partial\Sigma$ respectively. n_a is the unit vector normal to the surface $\partial\Sigma$

If we take in particular $f = 1 = \text{const.}$, it then immedaitely follows that:

$$\int_\Sigma \cancelto{0}{\nabla_a(1)\nabla^a} N + \int_\Sigma 1\cdot\nabla_a\nabla^a N = \int_{\partial\Sigma} 1\cdot n_a\nabla^a N$$

Finally, for $\nabla^a N = X^a$, where N is some sufficiently smooth (lapse in this context) function on Σ; and for a harmonic $N \to \nabla_a\nabla^a N = 0$, we get:

$$\int_\Sigma \cancelto{0}{\nabla_a\nabla^a N} = \int_{\partial\Sigma} n_a\nabla^a N$$

I will call upon some of these results later on in the proof.

3. *Gauss-Bonet theorem*

The Gauss-Bonnet theorem states that given a compact two-dimensional Riemannian manifold Ξ *without* boundary, and R is the Ricci scalar on Ξ, then:

$$\int_\Xi R = 4\pi\chi(\Xi) \qquad (35)$$

with the appropriate volume form on Ξ. χ denotes the Euler characteristic of Ξ, a definition of which can be found in the literature[8].

(There exists also a version with a boundary, but it includes additional definitions we won't ever use.)

4. *The Komar mass integral*

From the earlier mentioned harmonicity of N we can conclude, using the Green's theorem on Riemannian manifolds, that the integral over some subsection of Σ spanning from N_1 to N_2 be:

$$0 = \int_{\Sigma|_{N_1}^{N_2}} \Delta N\sqrt{g}(dn \wedge dt_1 \wedge dt_2) =$$

$$= \int_{\Xi|_{N_2}} \partial_n N\sqrt{\tilde{g}}(dt_1 \wedge dt_2) - \int_{\Xi|_{N_1}} \partial_n N\sqrt{\tilde{g}}(dt_1 \wedge dt_2)$$

where n and t are some normal and tangent vectors with respect to Ξ. In other words, the integral of the normal derivative of the lapse function is the same on any two $N = \text{const.}$ surfaces.
In particular, if we choose $N_2 = \infty$ using the perscribed asymptotics and Gauss-Bonnet we arrive at:

$$\int_{\Xi|_{N_1}} \partial_n N\sqrt{\tilde{g}}(dt_1 \wedge dt_2) = 4\pi M$$

Choosing also $N_1 = 0$, on the horizon, in conjuction with (31) we get that:

$$\int_{\Xi|_{N=0}} \partial_n N \sqrt{\tilde{g}}(dt_1 \wedge dt_2) =$$

$$\partial_n N \mid_{N=0} \times \int_{\Xi|_{N=0}} \sqrt{\tilde{g}}(dt_1 \wedge dt_2) =$$

$$= \partial_n N \times \mathcal{A} = 4\pi M \tag{36}$$

where $\mathcal{A}$ is the area of the surface $N = 0$, i.e. the black hole horizon.

III. Israel's Theorem

A. Israel's Statement of the Theorem

Let $(\mathcal{N}, g^{(4)})$ be a static spacetime with some spatial hypersurface Σ given by t =const. $\xi^a = \partial_t$ is a timelike Killing vector field on Σ. We consider possible (standard-)static metrics of the form

$$g^{(4)} = -N(x^i)^2 dt^2 + g(x^i)(dx^i)^2 \tag{37}$$

where $N^2 = \xi^a \xi_a$ is the norm of the timelike Killing field such that:

- **(A)** Σ is regular, empty (of matter), noncompact and asymptotically Euclidean. Asymptotically Euclidean precisely means that the metric behaves in the following way; when $r \equiv (\delta_{\alpha\beta} x^\alpha x^\beta)^{1/2} \to \infty$, the metric goes as:
$$g_{\alpha\beta} = \delta_{\alpha\beta} + O(r^{-1}) \tag{38}$$
$$g_{\alpha\beta,\gamma} = O(r^{-2}) \tag{39}$$
$$N = (-g^{(4)}_{00})^{1/2} = 1 - m/r + O(r^{-2}) \tag{40}$$
and analogously for higher derivatives. The condition on N is more analogous to the metric being "asymptotically Schwarzschildian". Intuitively, the metric (and it's derivatives) have to go quickly enough to a flat space; in particular as quickly as Schwarzschild does.

- **(B)** The surfaces N = const. > 0, t = const. are regular, simply connected and closed.

- **(C)** The invariant $R_{abcd}R^{abcd}$ is bounded on Σ

- **(D)** If N has a vanishing lower bound on Σ, the intrinsic curvature, given by the Riemann scalar on the lower-bound-surface, approaches a limit of a closed regular 2-space of finite area.

The theorem then states that **the only static space-time satisfying all four of thes econditions is Schwarzschild's spherically symmetric vacuum solution.**

B. Alternative modern statement

The theorem, as stated above, is the way Israel stated it in his original paper. The version I'm going to prove is stated a little bit differently, with less technical assumptions which have been shown to be true independently.

Theorem, Israel 1967. Let (Σ, g, N) be a standard static space-time, that is asymptotically Schwarzschild in the same sense Israel defines it in (40), with only one connected static black hole horizon $\partial\Sigma$. Assume that $dN \neq 0$ everywhere on Σ. Then, the space-time $g^{(4)} = -N^2 dt^2 + g$ is (isometric to) Schwarzschild (30).

C. Proof of the Israel theorem

In the proof we will be considering a spacelike slice of the space-time metric t =const., so we are working with g as a metric on Σ.

The structure of the proof is to write out the Einstein equations in special coordinates that foliate Σ in terms of N = const., turn them into inequalities by neglecting some terms. After some mathematical manipulations, we get that N = const. are in fact spheres as embedded in Σ - i.e. Σ is completely spherically symmetric for the radial coordinate N
At this point, we will have an array of equations that hold for various components of $g^{(4)}$ and by combining them all and picking the right coordinate transformations we will arrive at the Schwarzschild metric as given in (30).

Our first order of business is to introduce good coordinates on Σ that will be useful for the calculations to follow.

1. *Foliating Σ by N*

The non-vanishing condition $dN \neq 0$ is a technical assumption that cannot be dropped from Israel's proof, and with good reason as it is the basis of the entire coordinate-dependant construction that is to follow. It is important because, as dN is a nowhere vanishing vector field on Σ, it foliates the manifold with leaves $\Xi_N \mid_{N=\text{const.}}$ and parameter N. Notice that $N = 0$ corresponds to the static black hole horizon as one of the leaves in the foliation. It is also in this case true that all the leaves have the same topology, as mentioned in the earlier section on foliations.

Now, we can use the assumed asymptotic behaviour (40) for N to state that as $r \to \infty$, $N = (1 - m/r + \mathrm{O}(r^{-2})) \to 1 - \epsilon$, this implies $r \approx \frac{m}{\epsilon}$ =const., or in other words, the $\Xi_{m/\epsilon}$ leaf has the topology approaching a 2-sphere in a Riemannian 3-manifold as $r \to \infty$. This then implies that all the leaves share this topology and are all topological 2-spheres, including $\Xi_{N=0}$ which is the static black hole horizon. Since spheres are connected, we automatically get connectedness of the black hole horizon - we didn't have to assume we had only one horizon in the

statement of the theorem.
As we know the scalar curvature of the static black hole horizon is constant (and positive by Gauss-Bonnet theorem), we can conclude that our black hole horizon is indeed not only a topological sphere but an actual round sphere, as embedded in Σ.

2. *The radial coordinate N*

This foliation approach allows us to define now a radial coordinate N on Σ. As dN is nowhere vanishing on Σ and trivially sastisfies the frobenius integrability condition ($dN \wedge d^2N = 0$), we can use it to introduce adapted, surface ($N =$ const) orthogonal coordinates on Σ so that the metric now takes on the form:

$$g = \rho^2 dN^2 + \tilde{g}(x^A, x^B)$$

where x^A, x^B are tangent to the surfaces $N =$const., and $\rho^2 = \left((dN)^i (dN)_i\right)^{-1}$. The entire spacetime metric is now block diagonal in the time and 1st spatial coordinate, and the only part with potentially off-diagonal terms is AB, which is analogous to the angular part $d\Omega$ of a spherically symmetric metric.

3. *The Israel inequalities*

This metric can be plugged into the Einstein vacuum equations (25) - (26) and again split up into the normal vector part $x^1 = \rho dN$ and the tangent parts $x^{A,B}$. Doing this is a lot of work, but not particularly instructive, so I'll skip it here[7], and claim that only the following three equations are relevant to the rest of the proof:

$$\begin{aligned} 0 = G_{00} + G_{11} &= N^{-2} G_{tt} + \rho^{-2} G_{NN} \\ &= \frac{1}{\rho}\left(\frac{K}{N} - K_{,N} - \frac{\rho}{2}K^2\right) \\ &\quad - \frac{1}{2}\left\{ \frac{\nabla_A^{(\tilde{g})}\rho \nabla^{(\tilde{g})A}\rho}{\rho^2} + 2\mathring{K}_{AB}\mathring{K}^{AB} \right\} \end{aligned} \tag{41}$$

$$\begin{aligned} 0 = \; G_{00} + 3G_{11} &= N^{-2} G_{tt} + 3\rho^{-2} G_{SS} \\ &= \frac{1}{\rho}\left(\frac{3K}{N} - K_{,N}\right) - R^{(\tilde{g})} - \Delta^{(\tilde{g})} \ln \rho \\ &\quad - \left\{ \frac{\nabla_A^{(\tilde{g})}\rho \nabla^{(\tilde{g})A}\rho}{\rho^2} + 2\mathring{K}_{AB}\mathring{K}^{AB} \right\} \end{aligned} \tag{42}$$

$$0 = \rho_{,N} - \rho^2 K \tag{43}$$

where objects with A, B indices are projected tensors onto the 2-surfaces Ξ_N. Note that the curly brackets in the above two equations are positive definite - they entirely consist of squared quantities on a space with Riemannian (positive-definite) metric. As such, they can be dropped, writing the equations instead as inequalities $0 \leq ...$ ((42) and (43) without the curly brackers).

Using this, the "Einstein inequalities" and the Lie-derivative definition of the extrinsic curvature:

$$\begin{aligned} K_{AB} &= \frac{1}{2}\pounds_N \tilde{g}_{AB} \\ &= \frac{1}{2}\left(\partial_1 \tilde{g}_{AB} + \tilde{g}_{A1}\cancelto{0}{\partial_B(\partial_1)} + \tilde{g}_{1B}\cancelto{0}{\partial_A(\partial_1)}\right) \\ &= \frac{1}{2}\rho^{-1}(\partial_N \tilde{g}_{AB}) \end{aligned}$$

where ∂_1 is the surface ($N =$ const.) orthogonal vector field dN in adapted coordinates, gives us two inequalities:

$$\partial_N \left(\frac{|\tilde{g}| K}{\sqrt{\rho} N} \right) \leq -2\frac{\sqrt{|\tilde{g}|}}{N} \Delta^{(\tilde{g})}\sqrt{\rho} \tag{44}$$

$$\partial_N \left(\frac{|\tilde{g}|}{\sqrt{\rho}} (KN + \frac{4}{\rho}) \right) \leq -N\sqrt{|\tilde{g}|}\left(\Delta^{(\tilde{g})} \ln \rho + R^{(\tilde{g})}\right) \tag{45}$$

These two inequalities are crucial to the proof

4. *Integrating the inequalities*

Now we integrate both inequalities wrt. $dx^A dx^B dN$, over the entirety of Σ.
Note the right hand sides contain the determianant of the metric on Ξ_N and as such give immediately the area element:

$$\int_{\Xi_N} dA_N \Delta^{(\tilde{g})}\sqrt{\rho} = 0$$

$$\int_{\Xi_N} dA_N \Delta^{(\tilde{g})} \ln \rho = 0$$

which both vanish due to Green's theorem. The only surviving part on the right hand sides is the $R^{(\tilde{g})}$, which by Gauss-Bonnet gives:

$$\int_0^1 N dN \int_{\Xi_N} dA_N \left(R^{(\tilde{g})}\right) = \frac{1}{2} 8\pi$$

On the left hand signs, we use stokes theorem and evaluate the expressions on the horizon and at infinity. For this, we use the derived boundary conditions on static black hole horizons and asymptotics at infinity, as perscribed by the proof. This leads us to:

$$\left[\int dA_N \frac{K}{\sqrt{\rho} N} \right]_0^1 \leq 0$$

$$\left[\int dA_N \frac{KN + 4\rho^{-1}}{\rho} \right]_0^1 \leq -4\pi$$

i.e. the angular integrals as evaluated on the horizon and at spatial infinity. Using the asymptotics $\rho^{-1} \to \frac{M}{r^2}, K \to \frac{2}{r}$ & $N \to 1 - \frac{M}{r}$ and the earlier derived boundary conditions on the horizon, we arrive at:

$$M \leq \frac{1}{4}\rho_{\Xi_0} \tag{46}$$

$$\frac{1}{4}\rho_{\Xi_0} \leq \frac{1}{4\pi}\mathcal{A}\rho_{\Xi_0}^{-1} \tag{47}$$

where $\mathcal{A}$ is the surface area of the static black hole horizon at $N=0$ and ρ_{Ξ_0} is ρ evaluated on the blackhole horizon (we know this is a constant on the entire black hole horizon from equation (31) as well as the fact that the tangential derivatives vanish since $N=0$ on Ξ_0). The mass term comes from the asymptotic condition that the metric approach Schwarzschild with mass M.

The proof at this point calls upon the Komar mass integral (36), which relates the mass M and surface area of a horizon in vacuum as:

$$M = \frac{\rho_{\Xi_0}^{-1}}{4\pi}\mathcal{A}$$

which then implies

$$\frac{1}{4}\rho_{\Xi_0} \leq \frac{\rho_{\Xi_0}^{-1}}{4\pi}\mathcal{A} = M \leq \frac{1}{4}\rho_{\Xi_0}$$

which implies that these inequalities are actually equalities!

The great news here is that this also implies that the inequalities (44) & (45), created by neglecting the big curly brackets in equations (41) & (42), are **also equalities** - the brackets are in fact 0:

$$\left\{\frac{\nabla_A^{(\tilde{g})}\rho\nabla^{(\tilde{g})A}\rho}{\rho^2} + 2\mathring{K}_{AB}\mathring{K}^{AB}\right\} = 0 \tag{48}$$

and as both parts of appearing inside the bracket are positive definite on their own, they must both vanish on their own.

That is:

$$\mathring{K}_{AB}\mathring{K}^{AB} = 0 \longrightarrow \mathring{K}_{AB} = 0 \tag{49}$$

$$\frac{\nabla_A^{(\tilde{g})}\rho\nabla^{(\tilde{g})A}\rho}{\rho^2} = 0 \longrightarrow \nabla^{(\tilde{g})}\rho = 0 \longrightarrow \rho = \text{const.} \tag{50}$$

Since equations (41) & (42) hold for every N, everything derived from them also holds for every N; so this is true on the entirety of Σ.

Using now equation (43), we can also conclude from constancy of ρ (50) that $K=0$ for all N, and from this that every surface $N=$ const. has constant mean curvature. Additionally, from (42), we see that every quantitiy except $R^{(\tilde{g})}$ is now constant on the leaves $N=$ const. and therefore that $R^{(\tilde{g})}$ is as well. We already know from the theory of foliations that each $N=$ const. are topological 2-spheres. Further, using these two facts, it follows that they are all also embedded 2-sphere - i.e. the spacetime is completely spherically symmetric.

5. *Schwarzschild Obtained*

We now have full 2-spherical symmetry for arbitrary N which turns out to be enough to explicitly reconstruct the Schwarzschild spacetime metric. Plugging (49) into (41), (42) and (43), and solving we get:

$$\rho = \frac{4c}{(1-N^2)^2} \qquad K = \frac{N}{c}(1-N^2) \qquad R^{(\tilde{g})} = \frac{(1-N^2)^2}{2c^2}$$

where c is an integration constant from integrating the $\nabla^{(\tilde{g})}\rho$ to get ρ.

All that's left is to choose the correct coordinates for the function $N(r)$, where r is the radial coordinate in Schwarzschild, to reproduce the exact form of (30).

Taking:

$$\frac{(1-N^2)^2}{2c^2} = \frac{2}{r^2} \qquad\qquad c = M$$

we get:

$$N^2 = 1 - \frac{2M}{r} \qquad \rho^2 dN^2 = (1-\frac{2M}{r})^{-1}dr^2 \qquad \tilde{g} = r^2 d\Omega^2$$

where the $\tilde{g}$ condition follows from the constant-$N \approx$ constant-r spherical symmetry condition described earlier.

Writing the full spacetime metric in these new coordiantes then yields exactly Schwarzschild (30):

$$g^{(4)} = -N^2dt^2 + \rho^2 dN^2 + r^2 d\Omega^2$$

□

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