Isometries of Spacetime

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Here we look at the isometries, the distance preserving transformations, of spacetime. This leads us to two important properties: isotropy and homogeneity which we explore through Lie groups and algebras resulting in the Bianchi cosmological models.

I. INTRODUCTION

Our goal here is to look at the isometries of spacetime and all the consequences they entail. These wonderful consequences are, namely, the isotropy and homogeneity of the spacetime. We will start off easy by looking at the maximally symmetric spacetimes, but will abandon them fairly quickly for they are 'not reasonable models of the real world'[1]; p.323. After that come the Robertson-Walker metric and the Friedmann equations where we will leave the universe homogeneous and isotropic in space, but let it evolve in time.

The bulk of the work here, however, will be in exloring the generalization of these concepts via Lie groups and algebras, moving over to the homogeneous spaces and Killing vectors and all the way over to the Bianchi cosmological models and its many types, finally wrapping it all up with the generalized Friedmann equations. So, let's get started.

II. MAXIMALLY SYMMETRIC UNIVERSES

Contemporary cosmological models are based on the Copernican principle; the idea that the universe is pretty much the same everywhere. The Copernican principle can be more formally understood through two mathemtically precise properties; isotropy and homogeneity. Their rigorous definitions are left for later in the text, for we do not need them here. Suffice to say isotropy states the space looks the same no matter in what direction you look and homogeneity is the statement the metric is the same throughout the manifold. The isotropy and homogeneity imply the space is maximally symmetric and we can write the Riemann tensor as[1]; Eq.8.1:

$$R_{\rho\sigma\mu\nu} = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}), \tag{1}$$

where κ is the measure of the Ricci curvature:

$$\kappa = \frac{R}{n(n-1)}, \tag{2}$$

and n is the dimension of the space in question. We have 3 spacetimes of maximal symmetry: Minkowski ($\kappa = 0$), de Sitter ($\kappa > 0$) and anti-de Sitter ($\kappa < 0$), the details of which are not as interesting here. We will however check if they are the solutions of the Einstein's equation. Start by taking the trace of Eq. (1):

$$R_{\mu\nu} = 3\kappa g_{\mu\nu},\ R = 12\kappa. \tag{3}$$

The Einstein tensor is then:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -3\kappa g_{\mu\nu}. \tag{4}$$

The Einstein's equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ then implies the energy-momentum tensor is proportional to the metric:

$$T_{\mu\nu} = -\frac{3\kappa}{8\pi G}g_{\mu\nu}. \tag{5}$$

The energy density and pressure are given by:

$$\rho = -p = \frac{3\kappa}{8\pi G}. \tag{6}$$

If ρ is positive, we get the de Sitter solution, if it is negative, the anti-de Sitter. But in our universe, we have ordinary matter and radiation, as well a possible vacuum energy. Our maximally symmetric spacetimes are not compatible with a dynamically interesting amount of matter and/or radiation.

The maximally symmetric spacetimes are, therefore, not reasonable models of the real world so why bother with them here? Simply; to show how fundamental isotropy and homogeneity are in considering cosmological models. Now we will discard these properties in time, but keep them in space.

III. ROBERTSON-WALKER METRICS

Consider our spacetime to be $\mathbf{R} \times \Sigma$, where $\mathbf{R}$ represents the time direction and Σ is a maximally symmetric 3-manifold. The spacetime metric then takes the form of:

$$ds^2 = -dt^2 + R^2(t)d\sigma^2, \tag{7}$$

where t is the timelike coordinate, $R(t)$ is a function called the scale factor and $d\sigma^2$ the metric on Σ. We now look at the maximally symmetric Euclidean 3-metrics γ_{ij} that obey:

$$^{(3)}R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \tag{8}$$

which is just Eq. (1) written with different leters,

$$k =^{(3)} R/6 \tag{9}$$

for convenience and the superscript $^{(3)}$ to remind us it is associated with the 3-metric. The Ricci tensor is then:

$$^{(3)}R_{jl} = 2k\gamma_{jl}. \tag{10}$$

If the space is to be maximally symmetric, then it will certainly be spherically symmetric and have the metric of the form:

$$d\sigma^2 = \gamma_{ij} du^i du^j = e^{2\beta(\tilde{r})} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2, \tag{11}$$

where $\tilde{r}$ is the radial coordinate and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The components of the Ricci tensor are then:

$$\begin{aligned} R_{11} &= \frac{2}{\tilde{r}}\partial_1\beta, \\ R_{22} &= e^{-2\beta}(\tilde{r}\partial\beta - 1) + 1, \\ R_{33} &= [e^{-2\beta}(\tilde{r}\partial\beta - 1) + 1]\sin^2\theta. \end{aligned} \tag{12}$$

Using (9) we can solve for β:

$$\beta = -\frac{1}{2}\ln(1 - k\tilde{r}^2). \tag{13}$$

And the 3-metric is:

$$d\sigma^2 = \frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2 d\Omega^2. \tag{14}$$

k sets the curvature and it is common to normalize this so $k \in \{-1, 0, 1\}$. The choice $k = -1$ corresponds to constant negative curvature on Σ and is called 'open', the $k = +1$ to positive and is called 'closed'. The $k = 0$ corresponds to zero curvature on Σ and is called flat. It is beneficial to introduce a new radial coordinate χ defined by:

$$d\chi = \frac{d\tilde{r}}{\sqrt{1 - k\tilde{r}^2}}, \tag{15}$$

which can be solved for $\tilde{r}$ and gives $sin(\chi)$ for $k = +1$, $sinh(\chi)$ for $k = -1$ and χ for $k = 0$. For the flat case, $k = 0$, the metric on Σ becomes:

$$d\sigma^2 = dx^2 + dy^2 + dz^2, \tag{16}$$

which is just the $\mathbf{R}^3$. For the closed case, $k = +1$, we get:

$$d\sigma^2 = d\chi^2 + \sin^2\chi d\Omega^2, \tag{17}$$

which is a metric of a 3-sphere. And for the open case, $k = -1$, we obtain:

$$d\sigma^2 = d\chi^2 + \sinh^2\chi d\Omega^2, \tag{18}$$

which is a three-dimensional space of negative curvature. When we put it all together, we can write the metric for our $\mathbf{R} \times \Sigma$ as follows:

$$ds^2 = -dt^2 + R^2(t)\left[\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + r^2 d\Omega^2\right] \tag{19}$$

This is the Robertson-Walker metric. We can work with the dimensionless factor $a(t) = R(t)/R_0$, a coordinate with dimensions of distance $r = R_0\tilde{r}$ and a curvature parameter with dimensions of $(length)^{-2}$ $\kappa = k/R_0^2$. The metric is then:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right] \tag{20}$$

Setting $\dot{a} \equiv da/dt$, we get the nonzero components of the Ricci tensor:

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a}, \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1 - \kappa r^2}, \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + \kappa), \\ R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + \kappa)\sin^2\theta. \end{aligned} \tag{21}$$

The Ricci scalar is:

$$R = 6\left[\frac{\ddot{a}}{a}\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]. \tag{22}$$

IV. THE FRIEDMANN EQUATIONS

We can derive the Friedmann equations from the Robertson-Walker metric and the Einstein equation. Let us model matter and energy by a perfect fluid. Let us set the four velocity to:

$$U^\mu = (1, 0, 0, 0), \tag{23}$$

and the energy-momentum tensor:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{24}$$

with one index raised takes the form:

$$T^\mu{}_\nu = diag(-\rho, p, p, p). \tag{25}$$

Then, the trace is:

$$T = T^\mu{}_\mu = -\rho + 3p. \tag{26}$$

The Einstein equation can be written in the form:

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T). \tag{27}$$

The $\mu\nu = 00$ equation is:

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p), \tag{28}$$

and the $\mu\nu = ij$ equations give:

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\kappa}{a^2} = 4\pi G(\rho - 3p). \tag{29}$$

Using Eq. (28) we can do away with the second derivatives in Eq. (29) and we finally obtain the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{2\pi G}{3}(\rho + 3p). \tag{30}$$

Here we can introduce a few useful parameters. The Hubble parameter is defined as:

$$H = \frac{\dot{a}}{a}. \tag{31}$$

The value of the Hubble parameter at the present time is the Hubble constant $H_0 = 70 \pm 10\, km\, s^{-1} Mpc^{-1}$. The cosmological scales are set by the Hubble length $d_H = H_0^{-1}c = 3 \times 10^3 h^{-1} Mpc$ and the Hubble time $t_H = H_0^{-1} = 9.78 \times 10^9 h^{-1} yr$. Another useful parameter is the density parameter:

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{crit}}, \tag{32}$$

where the critical density is defined by $\rho_{crit} = 3H^2/(8\pi G)$. We have:

$$\begin{aligned} \rho < \rho_{crit} &\leftrightarrow \Omega < 1 \leftrightarrow \kappa < 0 \leftrightarrow open, \\ \rho = \rho_{crit} &\leftrightarrow \Omega = 1 \leftrightarrow \kappa = 0 \leftrightarrow flat, \\ \rho > \rho_{crit} &\leftrightarrow \Omega > 1 \leftrightarrow \kappa > 0 \leftrightarrow closed. \end{aligned} \tag{33}$$

The density parameter tells us which of the three Robertson-Walker geometries describes our universe.

Okay, by now we have forgotten all about the isotropy and homogeneity. So let's get back to them. We can start with a gentelman called Lie.

V. LIE GROUPS AND LIE ALGEBRAS

Definition: A Lie group, G, is a topological space that has the following properties:

1. G is a manifold.
2. The group multiplication $m : G \times G \mapsto G$ is smooth.
3. Inversion $i : G \mapsto G$ is smooth.

An example of one such group is the $SO(3)$ group. To show it has these three properties is not really that hard. The multiplication and inversion are continuous operations. Each element in $SO(3)$ corresponds to rotation and rotations are continous operations. $SO(3)$[2]; p.402 is equal to the manifold $\mathbb{P}^3$ and we have satisfied all 3 conditions for it being a Lie group. Moving on.

Definition: A real (or complex) Lie algebra, $\mathfrak{g}$ is a (finite) dimensional vector space equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ which satisfies the following properties:

1. $[\mathbf{X}, \mathbf{X}] = 0 \, \forall \, \mathbf{X} \in \mathfrak{g}$
2. Jacobi's identity:
   $[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0.$

An example of a Lie algebra is the space of all $n \times n$ matrices $\mathfrak{gl}(n)$ with the bilinear map defined by $[A, B] = AB - BA$. The Lie algebra of $SO(3)$, usually written as $\mathfrak{so}(3)$ consists of all skew-symmetric matrices (those with $[A, B] = -[B, A]$).

There is a connection between these two concepts; a Lie algebra is a vector space, while a Lie group is a group manifold. We have the following theorem.

Theorem: Let G be a Lie group. The tangent space of G at the identity element, $T_e G$, is a Lie algebra, i.e.

$$\mathfrak{g} = T_e G.$$

We can, by calculating the tangent space of a Lie group, find a corresponding Lie algebra.

Each element of a Lie algebra can be considered as a vector at the unit element of a manifold, so if $\mathbf{X}$ is a vector in the Lie algebra, we can define the local flow ϕ_t of the vector $\mathbf{X}$ by:

$$(\phi_t)^{-1}\frac{\partial \phi_t}{\partial t} = \mathbf{X}, \qquad \phi_0(e) = e, \tag{34}$$

where e is the unit element. The solution to these equations is an exponential map[2]; sec.6.9 :

$$\phi_t(e) = \exp(t\mathbf{X}). \tag{35}$$

We can then define the exponential map $\exp : \mathfrak{g} \mapsto G$ as:

$$\exp(\mathbf{X}) = \phi_1(e) \in G, \tag{36}$$

and use it to get from the Lie algebra to the Lie group. The inverse can also be defined:

$$\log : U \subset G \mapsto \mathfrak{g}, \qquad \log \equiv \exp^{-1}\big|_U. \tag{37}$$

Let $\{\mathbf{X}_i\}$ be the basis for the Lie algebra $\mathfrak{g}$. We define the structure constants C_{ij}^k by:

$$[\mathbf{X}_i, \mathbf{X}_j] = C_{ij}^k \mathbf{X}_k, \tag{38}$$

which are antisymmetric in the lower indices.

VI. HOMOGENEOUS SPACES

Let us now introduce the concept of homogeneous spaces. Simply put, a homogeneous space is a space where you can get from one point to any other point using an isometry. Firstly, let us define an isometry.

Definition: Isometry. Let M and N be metric spaces with metrics d_M and d_N. A map $\phi : M \to N$ is called an isometry or distance preserving if for any $p, q \in M$ one has $d_N(\phi(p), \phi(q)) = d_M(p, q)$.

Now consider a space M with a metric g. We define the isometry group $Isom(M)$ by:

$$Isom(M) \equiv \{\phi : M \to M \,|\, \phi \,\text{is an isometry}\}. \tag{39}$$

The isometry group will, in general, be a Lie group (Myers–Steenrod theorem[3]). A Killing vector field generates an isometry[2];p.422, i.e. the Killing vector field forms a finite dimensional vector space isomorphic to the Lie algebra of $Isom(M)$.

Now we can define the isotropy subgroup of a point $p \in M$ by:

$$I_p(M) = \{\phi \in Isom(M) | \phi(p) = p\}. \tag{40}$$

I.e., the isotropy subgroup is the subgroup of the isometry group that leaves the point p fixed. Finally we can define a homogeneous space.

Definition: Homogeneous space. If for each pair of points $p, q \in M$ there exist $\phi \in Isom(M)$ so that $\phi(p) = q$, we say that M is a homogeneous (or transitive) space.

Let the dimension of $Isom(M)$ be n and $I_p(M)$ be m. A necessary condition for M to be homogeneous is that $n > dim(M)$. We call M **simply transitive** if it is homogeneous and $n = dim(M)$ and **multiply transitive** if it is homogeneous and $n > dim(M)$. E.g. the maximally symmetric spaces are multiply transitive.

Consider the subspace of M given by:

$$H_p = \{q \in M | q = \phi(p) \,\text{for a}\, \phi \in Isom(M)\}, \tag{41}$$

for a point $p \in M$. The subspace H_p is called the orbit of p under the isometry group. All the points we can reach by the action of an isometry on p, is in the orbit of p. If $H_p = M$, M is transitive.

By now we have seen what Lie algebra and groups are, how are they connected and what all of this has to do with homogeneous spaces. Let us now construct spaces that are simply transitive. If we consider simply transitive spaces, there exist a set of Killing vetor fields that obey:

$$[\xi_i, \xi_j] = D^k_{ij}\xi. \tag{42}$$

These can be taken to be the basis vectors. However, it is more convenient to define a basis $\mathbf{e}_i$ at a point p. We define a left invariant frame by Lie transporting this basis around the space:

$$\pounds_{\xi_j}\mathbf{e}_i = [\xi_j, \mathbf{e}_i] = 0. \tag{43}$$

Since:

$$\pounds_{\xi_j}[\mathbf{e}_i, \mathbf{e}_k] = 0, \tag{44}$$

for some constants C^K_{ij} we have:

$$[\mathbf{e}_i, \mathbf{e}_j] = C^k_{ij}\mathbf{e}_k. \tag{45}$$

One might wonder if these two Lie algebras are just different representations of the Lie algebra of the isometry group. She might be discouraged to try to prove it in the face of some tedious math (tedious to type it out anyhow), but here goes: Assume the vector fields ξ_i and $\mathbf{e}_j$ coincide at the point p. Then there exists an invertible matrix α^i_j such that:

$$\mathbf{e}_j = \alpha^i_j\xi_i, \; \alpha^i_j\big|_p = \delta^i_j. \tag{46}$$

In general, the matrix is dependant on the position, but the structure constants are not. We have:

$$\begin{aligned}\pounds_{\xi_j}\mathbf{e}_i = [\xi_j, \mathbf{e}_i] = \alpha^k_i[\xi_j, \xi_k] + \xi_j(\alpha^k_i)\xi_k = \\ = (\alpha^l_i D^k_{jl} + \xi_j(\alpha^k_i)\xi_k) = 0,\end{aligned} \tag{47}$$

so that:

$$D^k_{ij} = -\beta^l_j\xi_i(\alpha^k_l), \tag{48}$$

where $\beta^l_j = (\alpha^{-1})^l_j$, and, similarly:

$$C^k_{ij} = -\alpha^l_j\mathbf{e}_i(\beta^k_l). \tag{49}$$

Since the structure constants are not dependent on the position, we can evaluate them at the point p where $\beta^l_j = \alpha^l_j = \delta^l_j$ and $\xi_i = \mathbf{e}_i$. The derivative of β^k_l can be written in terms of the derivative α^k_l:

$$\xi_i(\beta^k_l) = -\beta^k_n\beta^m_l\xi_i(\alpha^n_m). \tag{50}$$

At the point p this reduces to:

$$\xi_i(\beta^k_l) = -\xi_i(\alpha^k_l), \tag{51}$$

giving us:

$$C^k_{ij} = -\alpha^l_j\mathbf{e}_i(\beta^k_l) = \xi_i(\alpha^k_j) = -D^k_{ij}. \tag{52}$$

The tedious proof is done and done.

We say the frame $\mathbf{e}_j$ defines a left invariant frame and the frame ξ_i a right invariant frame. Now we can construct a homogeneous space as follows. Take the structure constants of a Lie algebra, C^k_{ij}, and define a left invariant frame:

$$[\mathbf{e}_i, \mathbf{e}_j] = C^k_{ij}\mathbf{e}_k. \tag{53}$$

If ω^k is the dual basis to $\mathbf{e}_k$, then:

$$d\omega^k = -\frac{1}{2}C^k_{ij}\omega^i \wedge \omega^j. \tag{54}$$

These basis 1-forms will be left invariant $\pounds_{\xi_i}\omega^k = 0$ and using them we can equip the space with an invariant metric given by:

$$ds^2 = g_{ij}\omega^i \otimes \omega^j, \tag{55}$$

where the metric coefficients g_{ij} are constants. This metric is a homogeneous metric on the said space.

VII. THE BIANCHI MODELS

In cosmology, we are interested in three-dimensional spatial sections. The Bianchi models are cosmological models that have spatially homogeneous sections, invariant under the action of a three-dimensional Lie group. We assume the four-dimensional space to be:

$$M = \mathbb{R} \times \Sigma_t, \tag{56}$$

where $\mathbb{R}$ is the time variable and each Σ_t is labeled with the time variable. Σ_t is a homogeneous three-dimensional space by construction. We have three different possibilities:

1. $\dim Isom(M) = 6 : \Sigma_t$ is a multiply transitive space of maximal symmetry. These are the Friedmann-Robertson-Walker models.
2. $\dim Isom(M) = 4 : \Sigma_t$ is a multiply transitive space with an isotropy subgroup $I_p(M) = SO(2)$.
3. $\dim Isom(M) = 3 : \Sigma_t$ is a simply transitive space.

All of the spaces in the categories 1 and 2 bar one (the one where Σ_t has the covering space $\mathbb{R} \times S^2$) has a subgroup $H \subset Isom(M)$ such that H acts simply transitive on Σ_t. So we can just consider the simply transitive spaces. What possibilities do we then have for Σ_t? We need a classification of three-dimensional Lie algebras. The classification in question is called the Bianchi classification and each Lie algebra is labeled by a number I-IX. By using one of these algebras we can construct a spatially homogeneous cosmological model, i.e. a Bianchi model. The table listing the Bianchi models is given in Table 1.

Bianchi Type	a_i	n	Structure constants
I	0	0	$C^i_{jk} = 0$
II	0	diag(1,0,0)	$C^1_{23}{=}-C^1_{32} = 1$, rest of $C^i_{jk} = 0$
III	$\frac{1}{2}\delta^3_i$	$-\frac{1}{2}A$	$C^1_{13}{=}-C^1_{31} = 1$, rest of $C^i_{jk} = 0$
IV	δ^3_i	diag(1,0,0)	$C^1_{13}{=}-C^1_{31} = 1$, $C^1_{23}{=}-C^1_{32} = 1$, $C^2_{23}{=}-C^2_{32} = 1$
V	δ^3_i	0	$C^1_{13}{=}-C^1_{31} = 1$, $C^2_{23}{=}-C^2_{32} = 1$, rest of $C^i_{jk} = 0$
VI$_h$	$\frac{\tilde{h}}{2}\delta^3_i$	$\frac{1}{2}(\tilde{h} - 2)A$	$C^1_{13}{=}-C^1_{31} = 1$, $C^2_{23}{=}-C^2_{32} = (\tilde{h}{-}1)$, rest of $C^i_{jk} = 0$
VII$_h$	$\frac{\tilde{h}}{2}\delta^3_i$	diag(-1,-1,0) $+\frac{\tilde{h}}{2}A$	$C^2_{13}{=}-C^2_{31} = 1$, $C^1_{23}{=}-C^1_{32} = 1$, $C^2_{23}{=}-C^2_{32} = \tilde{h}$, rest of $C^i_{jk} = 0$
VIII	0	diag(-1,1,1)	$C^1_{23}{=}-C^1_{32} = 1$, $C^2_{31}{=}-C^2_{13} = 1$, $C^3_{12}{=}-C^3_{21} = 1$, rest of $C^i_{jk} = 0$
IX	0	1	$C^i_{jk} = \epsilon_{ijk}$

Table 1: The Bianchi models in terms of their structure constants. A is just the first Gell-Mann matrix[2],t.15.1 The group parameter $h = -\tilde{h}^2/(\tilde{h}-2)^2$ for the Bianchi type VI$_h$ model and $h = \tilde{h}^2/(4-\tilde{h}^2)$ for the Bianchi type VII$_h$ model.

In column 2 and 3 the Bianchi types are written in terms of the Behr decomposition in which the structure constants are decomposed in terms of the trace-free part and the trace part:

$$C^k_{ij} = \epsilon_{ijl}n^{lk} + a_l(\delta^k_i\delta^l_j - \delta^k_j\delta^l_i), \tag{57}$$

where a_i is the 'vector' part of the Lie algebra. The trace of C^k_{ij} is:

$$C^j_{ij} = -2a_i. \tag{58}$$

We can always choose a basis such that $a_i = a\delta^3_i$. We usually call the models with $C^j_{ij} = 0$ the class A models. Those with $C^j_{ij} \neq 0$ are called class B models.

There are a couple of things to note. Bianchi type I corresponds to flat spatial section. Thus, it generalizes the Friedmann-Robertson-Walker model. Bianchi type IX corresponds to the Lie algebra $\mathfrak{so}(3)$. The class A models are: I, II, VI$_0$, VII$_0$ and IX. We have VI$_{-1}$=III.

The Bianchi models are then constructed as follows. For the specific Bianchi type, we choose an invariant basis $\{\omega_i\}$ that satisfies:

$$d\omega^k = -\frac{1}{2} C^k_{ij} \omega^i \wedge \omega^j. \quad (59)$$

The Bianchi model of the corresponding type can now be written as:

$$ds^2 = -dt^2 + g_{ij}(t) \omega^i \otimes \omega^j. \quad (60)$$

VIII. THE ORTHONORMAL FRAME APPROACH TO THE BIANCHI MODELS

Assume the energy-momentum tensor has the form:

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + \pi_{\mu\nu}, \quad (61)$$

where $u_{\mu\nu}$ is the 4-velocity of the fluid flow. We will also assume the 4-velocity is orthogonal to the hypersurfaces Σ_t spanned by the action of the isometry group. If this is the case we say the fluid in non-tilted, else it is called tilted. This assumption implies that the vorticity and the 4-acceleration of the fluid are 0, i.e. $\omega_{\mu\nu} = 0$, $u^\nu_{\mu;\nu} = 0$. We split the expansion tensor into trace and trace-free parts:

$$\theta_{\mu\nu} = U_{\mu;\nu} = \frac{1}{3} \theta h_{\mu\nu} + \sigma_{\mu\nu}. \quad (62)$$

The commutator functions $c^\alpha_{\ \mu\nu}$ are given by:

$$[\mathbf{e}_\mu, \mathbf{e}_\nu] = c^\alpha_{\ \mu\nu} \mathbf{e}_\alpha. \quad (63)$$

The functions are related to the connection coefficients via $c^\alpha_{\ \mu\nu} = \Gamma^\alpha_{\ \nu\mu} - \Gamma^\alpha_{\ \mu\nu}$. In an orthonormal frame, the rotation forms possess the antisymmetry, which makes it possible to write the connection coefficients in terms of the structure coefficients:

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}(g_{\alpha\beta} c^\beta_{\ \nu\mu} + g_{\mu\beta} c^\beta_{\ \alpha\nu} - g_{\nu\beta} c^\beta_{\ \mu\alpha}). \quad (64)$$

Since the vector $u_{\mu\nu}$ is orthogonal to the hypersurfaces Σ_t we have $\theta_{\mu\nu} = \Gamma^t_{\ \mu\nu}$ and hence $c^a_{\ tb} = -(\Gamma^a_{\ tb} - \Gamma^a_{\ bt})$. The antisymmetry of the rotation form implies:

$$\Gamma_{abt} = -\Gamma_{bat} \equiv \epsilon_{abc} \Omega^c, \quad (65)$$

where we have defined a rotation vector Ω^c by:

$$\Omega^a = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} u_\beta \mathbf{e}_\gamma \cdot \dot{\mathbf{e}}_\delta. \quad (66)$$

The structure coefficients can therefore be written as:

$$c^a_{\ tb} = -\theta^a_{\ b} + \epsilon^a_{\ bc} \Omega^c. \quad (67)$$

The vector Ω^c can be interpreted as the local angular velocity, in the rest frame of the observer with 4-velocity u^μ, of a set of Fermi-propagated axes with respect to the spatial fram $\{\mathbf{e}_a\}$. The remaining structure constants are all purely spatial and they must correspond to one of the Bianchi algebras. We write them as:

$$c^k_{\ ij} = \epsilon_{ijl} n^{lk} + a_l (\delta^k_i \delta^l_j - \delta^k_j \delta^l_i), \quad (68)$$

where n^{lk} is a symmetric matrix. n^{lk} and a_i are only functions of time and we can find the evolution of these function by applying the Jacobi identity. The Jacobi identity holds, in particular, for the set of vectors $(\mathbf{u}, \mathbf{e}_a, \mathbf{e}_b)$ which leads to the identity:

$$\mathbf{u}(c^k_{\ ab}) + c^k_{\ td} c^d_{\ ab} + c^k_{\ ad} c^d_{\ bt} + c^k_{\ bd} c^d_{\ ta} = 0. \quad (69)$$

Contracting we get:

$$n^{ij} a_i = 0, \quad (70)$$

the propagation equation for a_i:

$$\mathbf{u}(a_i) + \frac{1}{3} \theta a_i + \sigma_{ij} a^j + \epsilon_{ijk} a^j \Omega^k = 0, \quad (71)$$

and the trace-free part:

$$\mathbf{u}(a_i) + \frac{1}{3} \theta n_{ab} + 2 n^k_{\ (a} \epsilon_{b)kl} \Omega^l - 2 n_{\ k(a} \sigma^k_{\ b)} = 0. \quad (72)$$

Out of these, of particular interest is the Eq. (69) which implies a_i is in the kernel of the matrix n^{ij}. Since n^{ij} is a symmetric matrix, we can diagonalize it and, without any loss of generality, assume that:

$$n_{ij} = \mathrm{diag}(n_1, n_2, n_3)\, a_i = (0, 0, a). \quad (73)$$

The Bianchi models can now be characterised by the relative signs of the eigenvalues n_1, n_2, n_3 and a as in Table 2.

Class	Type	a	n_1	n_2	n_1
A	I	0	0	0	0
	II	0	+	0	0
	VI_0	0	+	−	0
	VII_0	0	+	+	0
	IX	0	+	+	−
	VIII	0	+	+	+
B	V	+	0	0	0
	IV	+	+	0	0
	VI_h	+	+	−	0
	VII_h	+	+	+	0

Table 2: The Bianchi types in terms of the algebraic properties of the structure coefficients.

IX. EINSTEIN'S FIELD EQUATIONS FOR BIANCHI TYPE UNIVERSES

The results of the previous section can be used to obtain the Einstein equations for Bianchi type universes. The Ricci tensor can be found by contracting the Riemann tensor. The three-dimensional Ricci tensor is given by[2];eq.15.71:

$$^{(3)}R_{ab} = \Gamma^{d}_{\ ab}\Gamma^{c}_{\ dc} - \Gamma^{d}_{\ bc}\Gamma^{c}_{\ ad}. \tag{74}$$

Using Eq. (64) and Eq. (68) we get:

$$\begin{aligned} R_{ab} = &-2\epsilon^{cd}_{\ \ (a}n_{b)c}a_d + 2n_{ad}n^{d}_{\ b} - nn_{ab} - \\ &- h_{ab}\left(2a^2 + n_{cd}n^{cd} - \frac{1}{2}n^2\right), \end{aligned} \tag{75}$$

where $n = n^{d}_{\ d}$. We have[2],eq.15.73:

$$\dot{\sigma}_{ab} = \mathbf{u}(\sigma_{ab}) - \Gamma^{\mu}_{\ a\nu}\sigma_{\mu b}u^{\nu} - \Gamma^{\mu}_{\ b\nu}\sigma_{a\mu}u^{\nu}, \tag{76}$$

which can be written, using Eq. (65), as:

$$\dot{\sigma}_{ab} = \mathbf{u}(\sigma_{ab}) - 2\sigma^{d}_{\ (a}\epsilon_{b)cd}\Omega^{c}. \tag{77}$$

Einstein field equations then give us the shear propagation equations:

$$\begin{aligned} &\mathbf{u}(\sigma_{ab}) + \theta\sigma_{ab} - 2\sigma^{d}_{\ (a}\epsilon_{b)cd}\Omega^{c} +^{(3)} R_{ab} - \\ &- \frac{1}{3}h^{(3)}_{ab}R = \kappa\pi_{ab}, \end{aligned} \tag{78}$$

the Raychauduri's equation:

$$\dot{\theta} + \frac{1}{3}\theta^2 + \sigma_{ab}\sigma^{ab} + \frac{\kappa}{2}(\rho + 3p) - \Lambda = 0, \tag{79}$$

and the Friedman equation:

$$\frac{1}{3}\theta^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} - \frac{1}{2}{}^{(3)}R + \kappa\rho + \Lambda, \tag{80}$$

where:

$$^{(3)}R =^{(3)} R^{a}_{a} = -\left(6a^2 + n_{cd}n^{cd} - \frac{1}{2}n^2\right). \tag{81}$$

These are the equations for Bianchi type universes. Fin.

X. SUMMARY

In this text I have gone over some properties of spacetime, namely, the isotropy and homogeneity which trace their roots to little something we call isometries; the distance preserving transformations.

Over the course of this work I have touched upon things like maximally symmetric universes; they don't work (well, they do, but they don't correspond to reality), then the maximally-symmetric-in-space-only-universes which gave us the Friedmann equations. All of this could be, and was, made more mathematically rigorous via the Lie groups and algebras which led to, again, homogeneous spaces, whose mostly exciting, sometimes tedious, mathematical framework was developed here. This culminated in the Bianchi cosmological models.

The last part of this report dealt with the orthonormal approach to the Bianchi models and the field equations. I ommited some stuff, just simply cited many more, and could have certainly included such things as model 8 geometries, compact quotients etc. which would make this work truely gargantuan and at which point one might as well just read the book[2].

XI. CITATIONS

[1] S. Carroll: Spacetime and Geometry, 2004.
[2] Ø. Grøn, S. Hervik: Einstein's General theory of Relativity, 2004.
[3] S. B. Myers, N.E. Steenrod, The group of isometries of a Riemannian manifold, 1939.