

DIFERENCIJALNE JEDNADŽBE

I

DINAMIČKI SUSTAVI

- nastavna skripta -

Danko Radić, 2014.

V1.2

LITERATURA:

- 1.) Steven H. Strogatz - Nonlinear Dynamics and Chaos
- 2.) Heinz G. Schuster & W. Just - Deterministic Chaos

Sadržaj:

§0	Dinamički sustavi - uvod	... 1
§1	Autonomni sustav I reda	... 3
1.1	Fiksne točke	... 5
1.2	Rješavanje dif. jedn. 1. reda - separacija varijabli	... 12
1.3	Tangentna polja	... 16
1.4	Numeričko rješavanje dif.-jednadžbi 1. reda	... 19
1.5	Bifurkacije	... 21
§2	Linearne dif. jednadžbe	... 26
2.1	Linearne dif. jednadžbe 1. reda	... 26
2.2	Linearne dif. jednadžbe višeg reda	... 31
2.3	Linearne dif. jednadžbe 2. reda (s konst. koeficijentima)	... 33
2.4	Tijekom harmonički oscilator	... 37
§3	Dinamički sustavi višeg reda	... 40
3.1	Fiksne točke sustava dif. jednadžbi	... 41
3.2	Liouvilleov teorema	... 43
3.3	Stabilnost trajektorija u fazovom prostoru	... 46
3.4	Metode istraživanja stabilnosti: Ly exponenat	... 47
§4	Autonomni sustavi II reda	... 51
4.1	Klasifikacija periodičkih točaka	... 51
4.2*	O svojstvima vektorijuma i vješnjima sustava	... 57
4.3*	Sustavi s isčravajućim svojstvom mijednošću	... 60
4.4	Numeričko rješavanje sustava diferencijalnih jednadžbi	... 65
4.5	Građani (harmonički i neharmonički) oscilator	... 74
4.6	Separatrica \rightarrow soliton	... 77
#	Value jednadžba i neilinearost	... 79
4.7	Granično kolo (limit cycle)	... 83
	Model sata - van der Polov oscilator	... 86
	Metode 2 vremenske skale	... 88
	LRC kružnik s negativnim diferencijalnim otporom	... 90
4.8	Hopfova bifurkacija	... 93
§5	Sustavi reda višeg od II - kaos	... 96
5.1	Duffingov oscilator	... 96
5.2	Poincaréovi presjeci	... 101
5.3	Hausdorff-Besicovitcheva dimenzija	... 106
5.4	1-dimenzionalna prstihareva	... 108
	Logistička mappa	... 110
	Trotentna mappa	... 113
5.5	Kicked rotator i Hénonova mappa	... 116
5.6	Lorenzov model	... 120



DINAMIČKI SUSTAVI

1

UVOD

- Dinamički sustav : sustav opisan određenim brojem varijabli koji se mijenja (evolucija) u vremenu povezane:

- fizikalni sustav : jednadžbe gibanja (generativne)
- biologija : populacijski modeli, oscilacije - mlad srce...
- kemija : dinamike kemijske reakcije
- tehnologija : oscilatori, hidrodinamika, turbulenca
- geologija : potoci
- meteorologija : modeli atmosfere i vremena
- kvantna mehanika
- termodinamika

MODELIRANJE SUSTAVA KOJI EVOLUIRA U VREMENU

1. korak: fizikalne pretpostavke modela (sadrže veze među veličinama koje promjenjuju)
2. korak: potpuno opisati sve varijable i parametre koje se koriste u modelu
 - varijable nezavisne (npr. vrijeme t)
 - ↳ zavisne (funkcije nezavisnih varijabli, npr. $x(t)$)
 - parametri - veličine koje se ne mijenjaju bez funkcije nezavisne varijable ali "s njom kontrolom" (pravljicom "iz neve") određuju svojstva modela (npr. masa u Newtonovu zakonu).
3. korak: Koristeći pretpostavke iz (1.) napisati jednadžke koje povezuju veličine iz (2.) i odrediti evoluciju zavisnih varijabli u vremenu.
pravilo: Korisiti nejednostavnije izraze koji zadovljave pretpostavke!

Uzimajući u obzir: 1) nevezane varijable - vrijeme (t).

$$\text{u zavisnosti od varijabli } \vec{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Može u nekom vremenskom intervalu t posljednju vrijednost po kojem se (sveke) veličine $x_i(t)$ mijenjaju $x_i(t+\Delta t)$, možemo reći da modelira evoluciju sustava u vremenu.

DINAMIČKI SOSTAV

1) kontinuirani model \Rightarrow diferencijalna jednadžba

$$\boxed{\frac{d\vec{x}(t)}{dt} = \vec{F}(\vec{x}(t); t)}$$

$$\begin{aligned} \frac{dx_1}{dt} &= F_1(x_1(t), \dots, x_n(t); t) \\ &\vdots \\ \frac{dx_n}{dt} &= F_n(x_1(t), \dots, x_n(t); t) \end{aligned}$$

(Sustav verovatno diferenzialnih jednadžbi 1. reda)

2) diskretni model \Rightarrow diferencijalna jednadžba (nepa)

$$\boxed{\vec{x}_{n+1} = \vec{F}(\vec{x}_n)}$$

diskretne vrijednosti

$$x_{n+1}^{(1)} = F_1(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(n)})$$

$$x_{n+1}^{(2)} = F_2(\quad \dots \quad)$$

\vdots

$$x_{n+1}^{(n)} = F_n(\quad \dots \quad)$$

Kontinuirani modeli - diferencijalne jednadžbe

a) Autonomni sustavi

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}(t))$$

\uparrow sans implicitna ovisnost o t !

Na svakoj okolini već djeluje (samo sebi određujuju budućost (evolucija)). (primjer: Hamiltonovi sustavi)

b) Neautonomni sustavi

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}(t); t)$$

\uparrow eksplicitna ovisnost o t !

Ujedno okolini dan je eksplicitnom ovisnošću o vremenu.

$$\frac{dx(t)}{dt} = f(x(t))$$

primjer (najjednostavniji)

Model neograničenog rasta populacije (biologija)

1. pretpostavke:

- Prvič populacije proporcionalan je ujedinj veličini.
(Motimo osim da u ovom modelu resursi, okređene i održive prostor ne igra ulogu, dokle odigledeno je da je model validan za veliku male populaciju u velikoj mjeri s izobiljem resursa.)

2. varijable:

t = vrijeme (vezana je varijable)

$P(t)$ = populacija (vezane varijable; $x(t)$)

k = koeficijent pristata (parametar) - konstante proporcionalnosti (tu se radi koga o jidinicama) između brzine rasta populacije [$\frac{1}{s}$] i same populacije [1] $\Rightarrow k[\frac{1}{s}]$

3. jednadžba:

$$\frac{dP(t)}{dt} = k \cdot P(t)$$

\rightarrow vrlo d. j. jednadžbe 1. reda, linearne

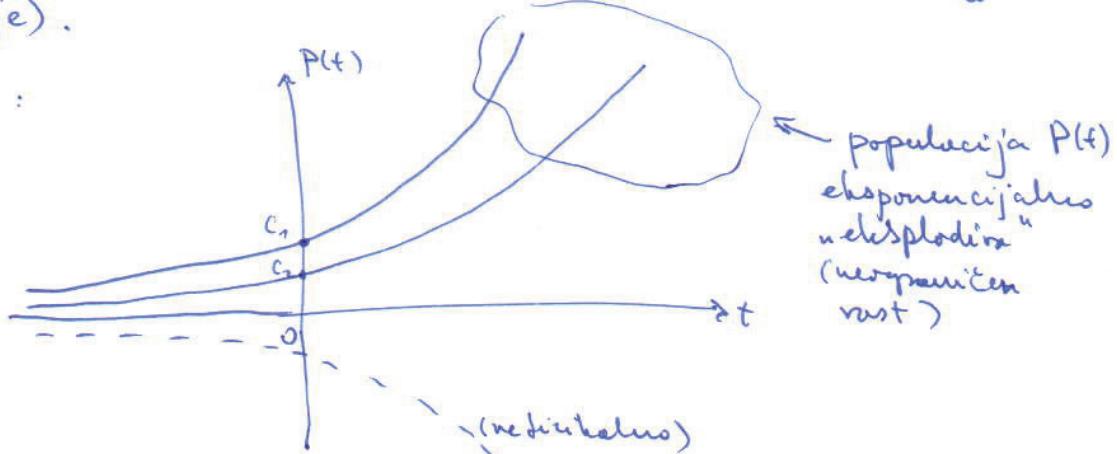
(*) napomena:

- obična d.j.: derivacija samo po 1 nezavisnoj varijabli
- parcijalna d.j.: derivacija po više nezavisnih varijabli
- red jednadžbe: najviši stupanj derivacije

- Pogledajmo funkciju $P(t) = 0$. Ova je vodilačna rješenja niti d.j. (kad ne mali, LHS = RHS). Brzina je konstantna $\neq t$, $P(t) = 0$ zove se i RAVNOTEŽNO RJEŠENJE.
- Pogledajmo funkciju $P(t) = C \cdot e^{kt}$. Važno je razlikovati u d.j. da li se to rješenje ako je $k = k$, itd. Dakle $P(t) = C \cdot e^{kt}$ je drugo, OPĆE rješenje d.j. $\neq C$. Konstanta C određena je POČETNIM UVJETOM
 $P(0) = C \cdot e^{k \cdot 0} = C$

(ili u nekom drugom vremenu t_0 u krajnju znamo iznos populacije).

grafiki:



rješenje: $P(t) = P_0 e^{kt}$

→ partikularno rješenje (tj. rješenje za specifične početne uvjetove $C = P_0 = P(t=0)$).

primjene modela na populaciju SAD:

god.	t	stavni / 10^6	model / 10^6
1790	0	3.9	3.9
1800	10	5.3	5.3
1810	20	7.2	7.2
1820	30	9.6	9.5
1830	40	12	13
1840	50	17	18
1850	60	23	25
1860	70	31	33
1870	80	38	45
1880	100	62	84
1890	110	77	114
1900	120	91	155
1910	130	105	210
1920	140	122	286
1930	150	131	388
1940	160	151	528
1950	170	179	717
1960	180	203	945
1970	190	226	1320
1980	200	?	11300
2000	220	?	11300
2050	260	?	11300

← prvo treba ukloniti "model ne podstavlja tj. vrednosti konstante k "

1.) $t = 0 \quad C \Rightarrow$ godine 1790.

$P(0) = 3.9$

2.) godine 1800 $\rightarrow t = 10$,

$P(10) = 5.3$

$5.3 = 3.9 \cdot e^{k \cdot 10}$

$k = \frac{1}{10} \ln \frac{5.3}{3.9} \approx 0.03067$

← $P(t) = 3.9 e^{0.03067 \cdot t}$

end

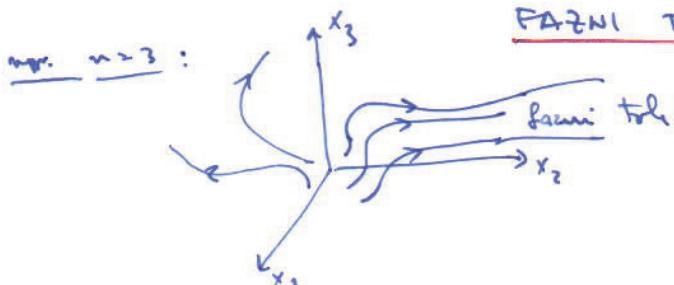
1.1FIKSNE TOČKE

(sarmi prostor, sarmi tok, fiksne točke)

Zadani sustav redovitib i funkcija $\vec{F}(\vec{x}(t))$ određuje sve moguće rješenja (za svaki početni uvjet) - one čine FАЗНИ TOK (sve trajektorije formog prostora).

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}(t), t)$$

$$\vec{x} = (x_1, x_2, \dots, x_n)$$

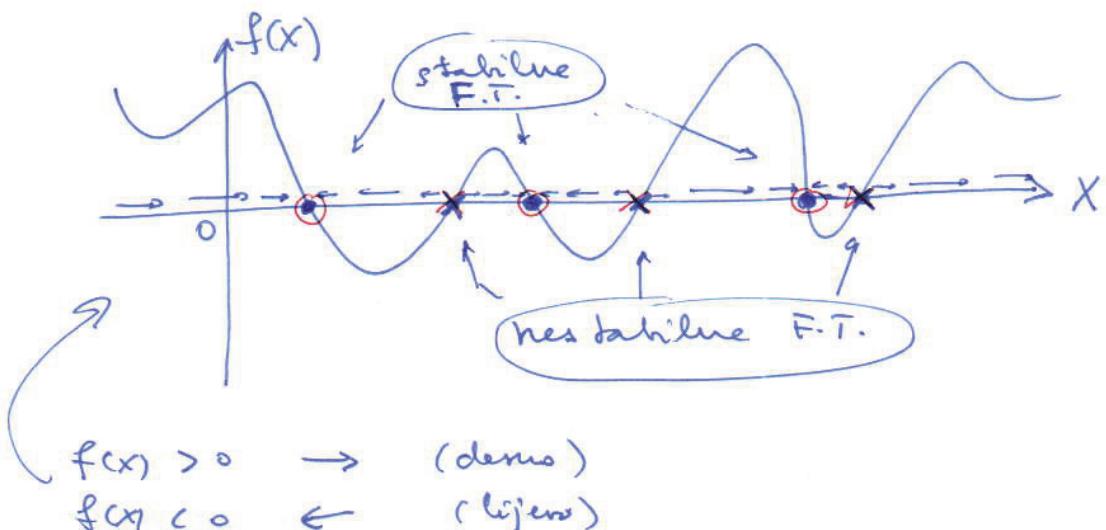


Pri tome se razlikuju vektorski sarmi x_i , $i=1\dots n$ koji narivaju u FАЗНИ PROSTOR (dimenzije n). Skup svih trajektorija u sarmom prostoru, tj. rješenja $\vec{x}(t)$ za sve moguće početne uvjete, nariva se FАЗNI TOK.

Cilj nam je analizirati taj sarmi tok. Promatrano 2 trajektorije koje su u početnom trenutku po vrijednosti bliske te promatrano vrijednost evoluciju u vremenu. Zašto? Na taj način određujemo stabilitet trajektorija čime se klasificira dionični dinamički sustav (npr. ako se utvrdi hiperosjetljivost na početne uvjete tj. ako 2 početno bliske trajektorije nakon dovoljno dugo vremena zavise na suprotnim krajevima formog prostora, sustav spada u klasi sustava s manifestiranim determinističkim (caosom)). O tome će biti riječi kasnije.

Početak te analize je metoda FIKSIH TOČAKA tj. RAVNOSTI NIH TOČAKA u sarmom prostoru. One određuju topologiju sarmog prostora i vrijednosti (točke) kojima sarmi tok asimptotski terji (za $t \rightarrow \pm\infty$). Fiksne točke su u skoru asimptotike (stacionarne) rješenje sustava d.j. $\dot{\vec{x}} = \vec{F}(\vec{x}, t)$.

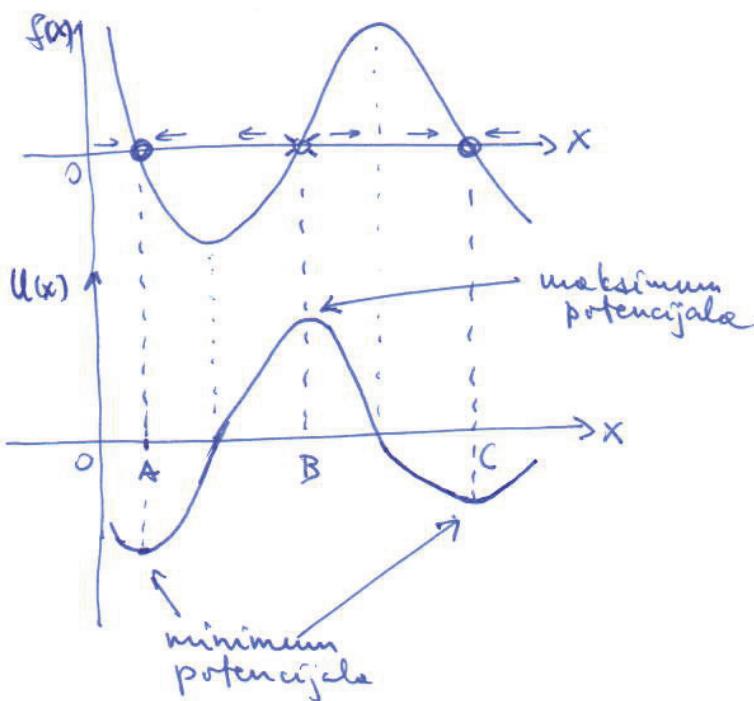
$$\frac{dx}{dt} = f(x) \quad ; \quad f(x_c) = 0 \Rightarrow \text{FIXNA TOČKA!}$$



Fixne točke je ravnostine, zaustavne točke kroz koje dinamički sustav evolucijom ne more proći (on jej se asimptotski približava u vremenu).

Zanishimo dinamički potencijal generiran "silom" $f(x)$

$$f(x) = -\frac{dU(x)}{dx} \Rightarrow U(x) = - \int f(x') dx'$$



Ako je sustav u fixnoj točki, ne more proći (evoluirati) kroz nju. Tu ostaje zadržati!

Fixne točke dijele fizički prostor (prostor raspeta s $\{x_1, x_2, \dots, x_n\}$) u INVARIJANTNE SKUPOVE.

(u našem primjeru imamo 4 inv. skupove:
 $(-\infty, A), (A, B), (B, C), (C, \infty)$)

U svakom inv. skupu neće biti jidre trajektorija.

(*) neponene

$$\frac{dU}{dt} = \frac{dU}{dx} \cdot \frac{dx}{dt} = f(x) \equiv -\frac{dU}{dx}$$

$$= -\left(\frac{dU}{dx}\right)^2 < 0 !$$

Kako t raste, tako U pada duž projektorije \Rightarrow sustav se giba prema minimum "potencijala" U dok u fix. točki ($\frac{dU}{dx} = 0$) konzervira stane.

L

(**)

Egistence rješenja dif. jednadžbe i njegova jedinstvenost:

Teorem o egistenceji i jedinstvosti (!)

Za $\dot{x} = f(x)$ i $x(t_0) = x_0$, velje da $f(x)$ i $f'(x)$ kontinuirane na intervalu $R(x)$ i velje je $x_0 \in R$. Tada problem s početnim uvjetom ima rješenje $x(t)$ na intervalu $t \in (-\tau, \tau)$ da $t=0$ i to rješenje je jedinstveno!

formulas matematičkih

$$\frac{dx}{dt} = f(x)$$

separacija
varijabli:

$$\frac{dx}{f(x)} = dt \quad | \int \Rightarrow$$

$$t - t_0 = \int_{x_0}^x \frac{dx}{f(x)}$$

Ako $f(x) = 0$, sustav je nointegrabilen.

VRSTE PLESNIH TOČAKA

1.) $\begin{cases} f(x_0) = 0 \\ f'(x_0) \neq 0 \end{cases} \Rightarrow$ JEDNOSTAVNA F.T.

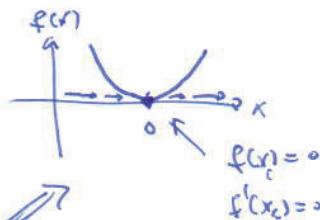
STABILNA ($f'(x_0) < 0$)

NESTABILNA ($f'(x_0) > 0$)

2.) $f(x_0) = 0 \Rightarrow$ SLOŽENA F.T.

$\Rightarrow f'(x_0) = 0$

upr. $\dot{x} = cx^2$
 $f(x) = 0$
 $x_0 = 0$



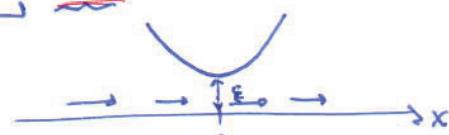
Ako približimo s lijeva, f. točka je stabilna.

Ako približimo s desno, f. točka je nestabilna.

→ Dodamo malu perturbaciju ε u sustav:

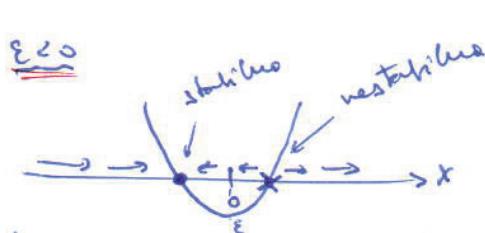
$$\dot{x} = cx^2 + \varepsilon$$

1.) $\varepsilon > 0$



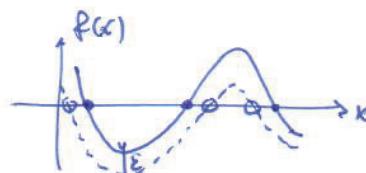
F.T. nestaje!

2.) $\varepsilon < 0$



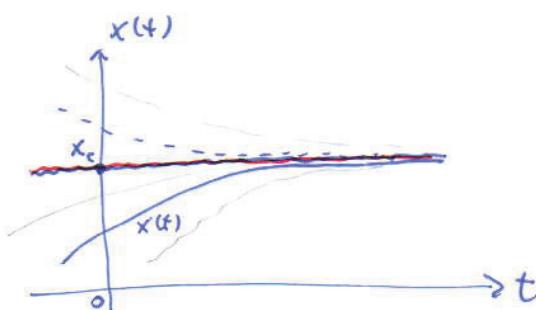
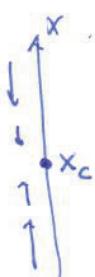
Od jedne struk. nest. F.T. dolihimo druge F.T., jednu stabilnu, drugu nestabilnu.

- (*) Kad jednokrakih F.T. toga nema.
 Dodatak perturbacije u sustav F.T. je samo mala promjena, ali neće pojaviti da nastane/nestane.



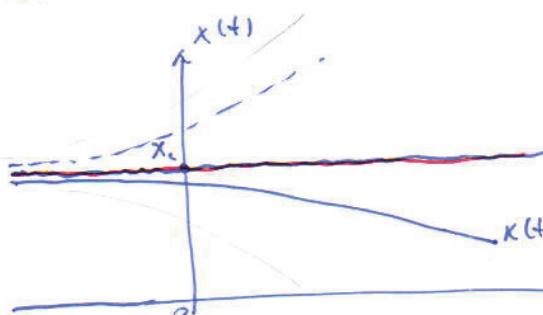
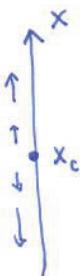
povezani rješenja - kvalitativna metoda

a) stabilna F.T.



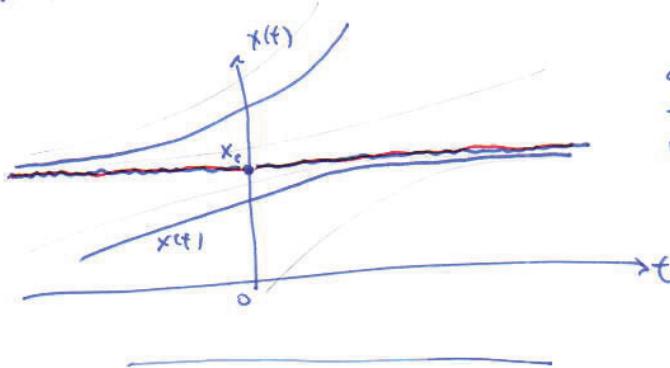
Prijenosi $x(t)$ asimptotski se približava x_c kada $t \rightarrow \infty$. Tačka je **PONOR** (faznog toka).

b) nestabilna F.T.



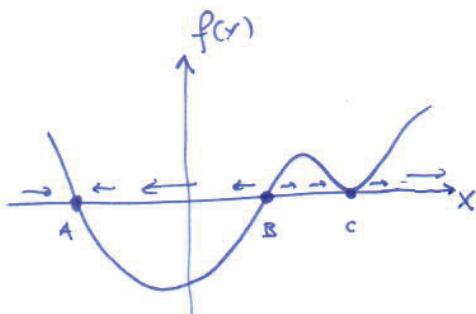
Prijenosi $x(t)$ asimptotski se udaljavaju od x_c kada se vrijeme povećava od $t = -\infty$. Tačka je **IZVOR** (faznog toka).

c) strukturno nestabilna F.T.

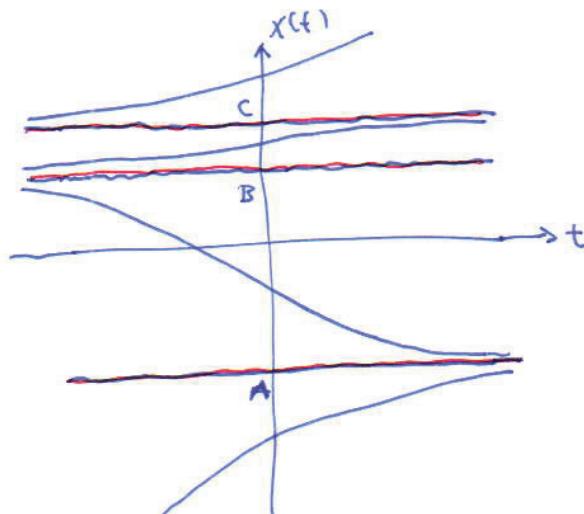


Prijenosi se s jedne strane asimptotski približavaju, a s druge strane asimptotski udaljuju se od x_c za vremena $\pm \infty$. Tačka je **ČVOR** (faznog toka).

↳ upr.



\Rightarrow



primjerLogistički populacijski model

1. pretpostavke:

- Ako je populacija mala, stopa rasta populacije proporcionalna je njenoj veličini.
- Ako je populacija prevelika da bi se okolice i resursi mogli podržati, one će se manjirati proporcionalno veličini (stopa rasta je negativna).

2. varijable i parametri

 t - vijeme (nerav. var.) $P(t)$ - populacija (zav. var.) k - koeficijent rasta za mala populacija (parametar) N - kapacitet područje okolice (parametar)

Modificirajući prethodni model

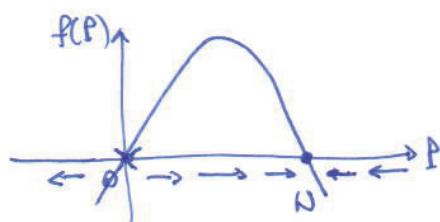
$$\frac{dP}{dt} = k \cdot P \cdot ?$$

mogućnost:
 $\begin{cases} x_1 & \text{za } P \ll N \\ < 0 & \text{za } P > N \end{cases} \Rightarrow ? = (1 - \frac{P}{N})$

$$\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{N}\right) \quad - \text{logistički populacijski model (kontinuirani!)}$$

Analiza (kvalitativna)

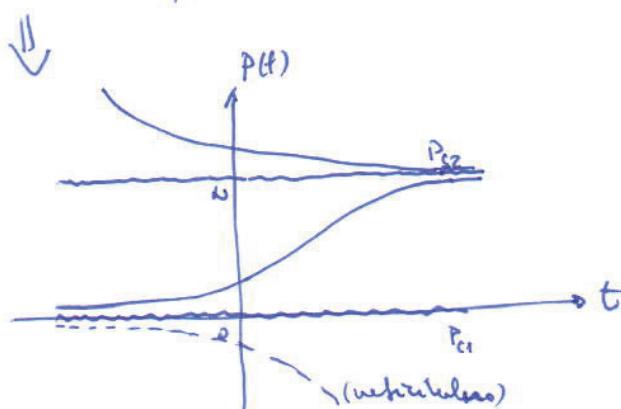
$$f(P) = k \cdot P \cdot \left(1 - \frac{P}{N}\right) \rightarrow \text{F.I. } f(P_c) = 0$$



$$\begin{cases} P_{c_1} = 0 \\ P_{c_2} = N \end{cases}$$

- nestabilna
- stabilna!

Kako vijeme može $t \rightarrow \infty$, populacija se asimptotski približava vrijednosti N .



- \Rightarrow
- 1) Za $P \in (0, N)$ ona se asimptotski odvaja od $P_{c_1} = 0$ i približava asimptotski ka $P_{c_2} = N$
 - 2.) Za $P > N$, rast će stopiti i dobit će ka vrijednosti N

Modificirani logistički model

Želimo modificirati logistički model da opisuje situaciju da, ako je populacija premala, stopa rastuće bude negativna tj. populacija počne opadati (npr. zbog teškoće pri ponosaštenju partnera za parove ako je tvoj jedinik jake male - to opisuje već sljedeće organizme od ljudi npr. vježnice).

- Ako je populacija prevelika, stopa rastuće je negativna.
- Ako je populacija premala, stopa rastuće je negativna.
- Ako je populacija jednaka nuli, stopa rastuće je nula.

$$t = \text{vrijeme}$$

$$P(t) = \text{populacija}$$

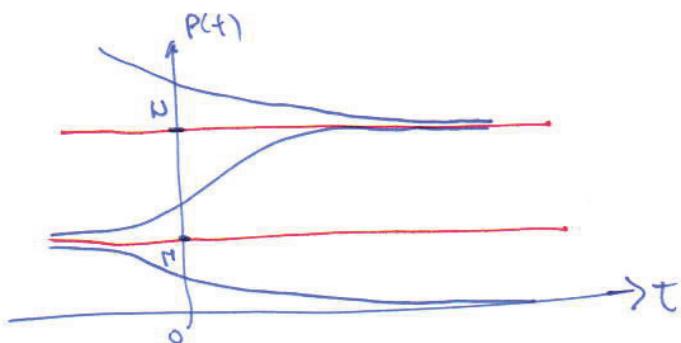
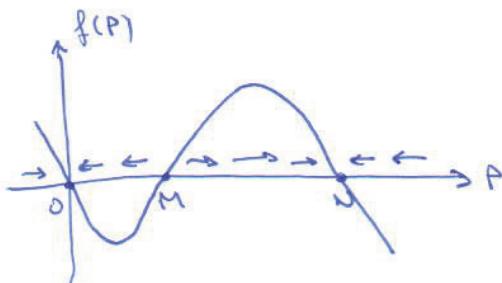
parametri: $k = \text{kof. rastuće za male populacije}$
 $N = \text{kapacitet područja}$
 $M = \text{konstanta „varijacije“}$

$$\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{N}\right) \cdot \left(\frac{P}{M} - 1\right) ; M < N$$

Modifikacija log. modela: ako iznos populacije pada ispod M , rast postane negativna & ako populacija doseže nulu, ostaje nula

$$f(P) = kP \left(1 - \frac{P}{N}\right) \left(\frac{P}{M} - 1\right) ; P.T.$$

- | | |
|--------------|--------------|
| $P_{01} = 0$ | - stabilna |
| $P_{02} = N$ | - stabilna |
| $P_{03} = M$ | - nestabilna |



1.2. RJEŠAVANJE DIFERENCIJALNIH JEDNADŽBI 1. REDA:

SEPARACIJA VARIJABLJI

[obične dif. jed. 1. reda]!

D.j. 1. reda (autonomna i neautonomna) separacija je
dajuće može napisati u obliku:

rješavanje:

$$\frac{dx}{dt} = g(t) \cdot h(x) \quad | : h(x) \quad | \cdot dt$$

$$\begin{aligned} \frac{dx}{h(x)} &= g(t) dt \\ \int_{x_0}^x \frac{dx'}{h(x')} &= \int_{t_0}^t g(t') dt' \end{aligned}$$

(*) F.T. $h(x_0) = 0$
nije integrabilna

(*) primjer 1
Model rastućeg
članova vrste
populacije

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k dt \quad ||$$

$$\ln x = kt + C$$

$$x = e^{kt+C}$$

$$x = e^{kt} \cdot e^C \quad \Rightarrow \tilde{C} \quad \text{opće rješenje: } x(t) = \tilde{C} e^{kt}$$

$$\Rightarrow x(t) = x_0 e^{kt}$$

partikularno rješenje s rubnim uvjetom u $x(t=0) = x_0$

(*) primjer 2
Logistički model

$$\frac{dx}{dt} = kx \left(1 - \frac{x}{N}\right)$$

$$\frac{dx}{x(1-\frac{x}{N})} = k dt \quad ||$$

$$\ln \frac{x}{x-N} = kt + C$$

$$\ln \frac{1}{1-\frac{N}{x}} = kt + C$$

$$\ln \left(1 - \frac{N}{x}\right) = -kt - C$$

$$1 - \frac{N}{X} = e^{-kt-c}$$

$$\frac{N}{X} = 1 - e^{-kt-c} = 1 - e^{-kt} \cdot \boxed{e^{-c}}$$

$$\underline{X(t) = \frac{N}{1 - \tilde{C} e^{-kt}}} \quad - \text{opeč vjeruje}$$

pričini ujet:

$$x_0 = x(t=0)$$

$$x_0 = \frac{N}{1 - \tilde{C}} \Rightarrow 1 - \tilde{C} = \frac{N}{x_0}$$

$$\tilde{C} = 1 - \frac{N}{x_0}$$

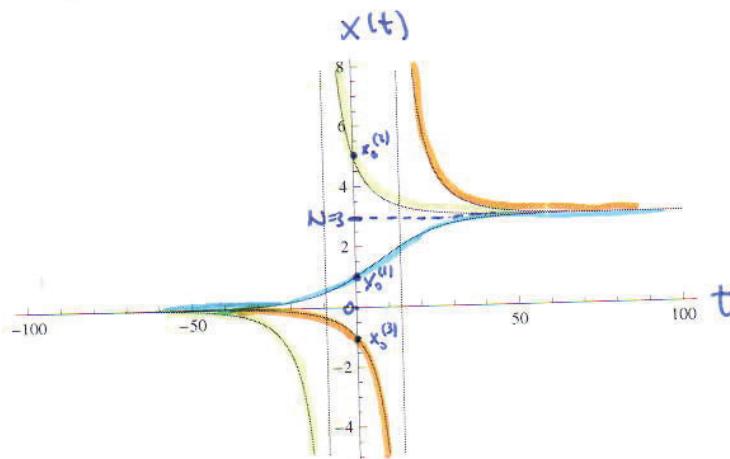
$$\boxed{x(t) = \frac{N}{1 - (1 - \frac{N}{x_0}) e^{-kt}}}$$

granice ponašanje (projekcijske):

$$t=0 : x(0) = \frac{N}{1 - (1 - \frac{N}{x_0})} = x_0 \checkmark$$

$$t=-\infty : x(-\infty) = \frac{N}{1 - (1 - \frac{N}{x_0}) e^{\infty}} = 0 \checkmark$$

$$t=+\infty : x(+\infty) = \frac{N}{1 - (1 - \frac{N}{x_0}) e^{-\infty}} = N \checkmark$$



Zadatak 1

neautonomi sustav

$$\frac{dx}{dt} = t^3 - 2 \sin t \quad \& \quad x(0) = 3$$

$$dx = (t^3 - 2 \sin t) dt \quad | \int$$

$$x(t) = \frac{1}{4}t^4 + 2 \cos t + C \rightarrow \text{opće rješenje}$$

pocetni uvjet: $x(0) = 3$:

$$3 = \phi + 2 \underset{=1}{\cos 0} + C \Rightarrow C = 1$$

$$x(t) = 1 + \frac{t^4}{4} + 2 \cos t$$

Zadatak 2

neautonomi jednadžba

$$\frac{dx}{dt} = -2tx^2, \quad x(0) = 1$$

$$\frac{dx}{x^2} = -2t dt \quad | \int$$

$$-\frac{1}{x} = -t^2 + C$$

$$x(t) = \frac{1}{t^2 - C} \sim \text{opće rješenje}$$

pocetni uvjet: $x(0) = 1$

$$1 = \frac{1}{-C} \Rightarrow C = -1$$

$$\Rightarrow x(t) = \frac{1}{1+t^2}$$

(*) napomena: Ispušteno rješenje (ravnateljne rješenja kada fix. točkom)

upr.

$$\frac{dx}{dt} = x^2$$

$$\frac{dx}{x^2} = dt \quad | \int$$

$$-\frac{1}{x} = t + C$$

$$x(t) = -\frac{1}{t+C}$$

Ovo opće rješenje ne opisuje svako partikularno rješenje!

$$\text{upr. za } t=0 : x(0) = -\frac{1}{C}, \text{ a}$$

to ne opisuje rješenje $x(0) = 0$.

Kad je da je rješenje $x(t) = 0$ upravo ravnateljne rješenje dobiveno iz fixne točke: $f(x) = 0$

$$x^2 = 0 \Rightarrow x_0 = 0$$

Dakle u rješenju separabilnih dif. jednadžbi teku pariti de se izmou-
notom skupu rješenja dodaju i ravnateljne rješenja.

Zadatak 3

rijenje u
implicitnom
obliku

$$\frac{dx}{dt} = \frac{x}{1+x^2}$$

$$\frac{1+x^2}{x} dx = dt \quad | \int$$

$$\ln x + \frac{x^2}{2} = t + C$$

transcedentna forma (x se ne može iznijedriti direktno u obliku $x(t) = \text{RMS}$) - rijenje je u implicitnom obliku

Zadatak 4: Model štednje

1. pretpostavke: u banku ulazimo \$5000 s kontinuiranim interesom davanjem po stopi 5% godišnje. Takođe, želimo brojiti ušteterinu; to tako da nakon 10 godina posnuemo džizati \$1000 godišnje s račune.

Pitanje: do kada će nam novac trajati?

Dvojedne su dvije suprotnе tendencije: konzervativni (koji nema puni račun) i optimistički (koji nema posnući račun).

2. varijable: t - vremje (vrijeme u godinama)
 $X(t)$ - iznos na računu
 k - konstantna stopa ($k = 0.05$)
 P - iznos koji godišnje posnujemo s računa ($P = 1000$)

3. jednadžba:

$$\frac{dx}{dt} = \begin{cases} k \cdot x, & t < 10 \\ k \cdot x - P, & t \geq 10 \end{cases}$$

Jednadžba se sastoji od 2 dijela. Prvo treba riješiti prvi dio uz pretpostavku $X(0) = 5000$; onda to rješenje iskoristiti kroz početku ujet za rješavanje drugog dijela u trenutku $t=10$.

$$1^{\circ}) \quad \frac{dx}{dt} = kx \rightarrow x(t) = X_0 e^{kt}$$

$$X(10) = 5000 e^{0.05 \cdot 10} = 8244$$

$$2^{\circ}) \quad \frac{dx}{dt} = kx - P \quad \left. \begin{aligned} x(10) &= \frac{1}{k} (P + \tilde{C} e^{k \cdot 10}) \\ \Rightarrow \tilde{C} &= \frac{k \cdot x(10) - P}{e^{10k}} = \frac{0.05 \cdot 8244 - 1000}{e^{0.5}} \approx -356.52 \end{aligned} \right\}$$

supst.

$$u = kx - P : \quad \frac{1}{k} \frac{du}{u} = dt \quad | \cdot k \quad | \int$$

$$\ln u = kt + C$$

$$x(t) = \frac{1}{k} (P + \tilde{C} e^{kt})$$

$$\Rightarrow x(t) = \frac{1}{0.05} (1000 - 356.52 e^{0.05t})$$

$$e^{0.05t} = 2.805$$

$$t = 20.63 \text{ god} \Rightarrow \underline{\text{We're out of money!}}$$

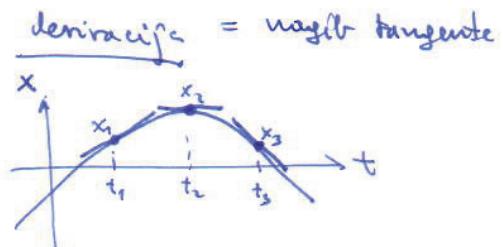
1.3

TANGENTNA POLJA

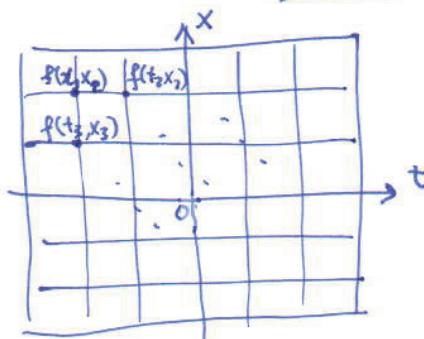
Mnoge (veline) d.j., čak i nekompleksne, ne mogu se analitički riješiti. No, preuzeti njihovu funkciju "dinamičkog potencijala" $f(x(t), t)$, možemo štovati reči o svojstima vježenja čak i bez rješavanja. Uz metode finih točaka, metode tangentnih polja još je jedna kvalitativna metoda analize fizičkog toka dinamičkih sustava.

$$\frac{dx}{dt} = \underbrace{f(t, x)}_{\text{derivacija}}$$

Ova funkcija određuje derivaciju (prirost funkcije / nagnutost tangente) u svakoj točki domene (t, x) tj. za sve početne uvjete moguće



! Derivacija je određen prirost funkcije t.j. kada će se funkcija promjeniti od t do $t+dt$.



Metode:

1. definiramo (t, x) domen za funkciju $f(t, x)$
2. pokrijemo domen mrežom (pravkutnikom) i definiramo parove (t_i, x_j)
3. na svakom ovoru (t_i, x_j) učrtamo mali vektor (tangentu) po pravilu da mu je t -komponenta jednaka 1, a x -komponenta jednaka vrijednosti funkcije $f(t_i, x_j)$ na tom ovoru

primjer 1.

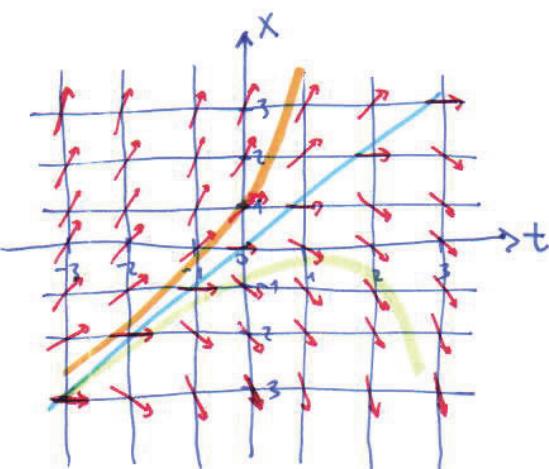
$$\frac{dx}{dt} = \underbrace{x-t}_{f(t,x)}$$

(one d.j. nije separabilne)

$$f(t,x) = x-t$$

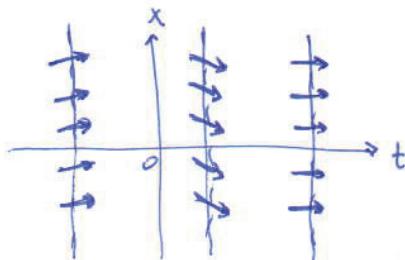
x/t	-3	-2	-1	0	1	2	3
-3	0	-1	-2	-3	-4	-5	-6
-2	1	0	-1	-2	-3	-4	-5
-1	2	1	0	-1	-2	-3	-4
0	3	2	1	0	-1	-2	-3
1	4	3	2	1	0	-1	-2
2	5	4	3	2	1	0	-1
3	6	5	4	3	2	1	0

\Rightarrow



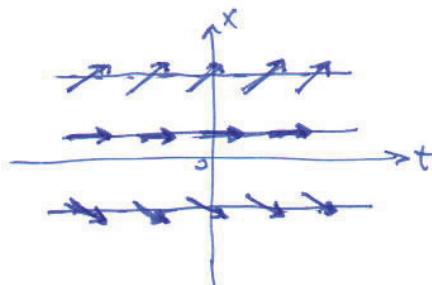
(*) poslovni smisajevi:

$$(1) \frac{dx}{dt} = f(t)$$



(2)

$$\frac{dx}{dt} = f(x) \quad (\text{autonomni sustav})$$



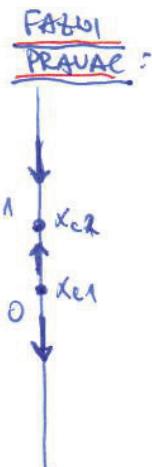
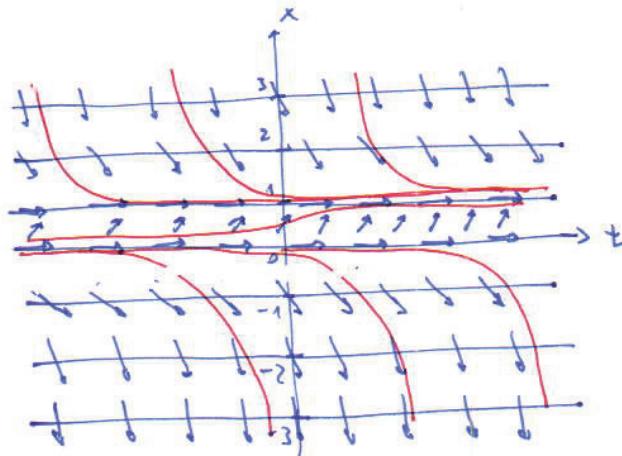
primjer 2.

Logističke jednadžbe

$$\frac{dx}{dt} = 4x(1-x)$$

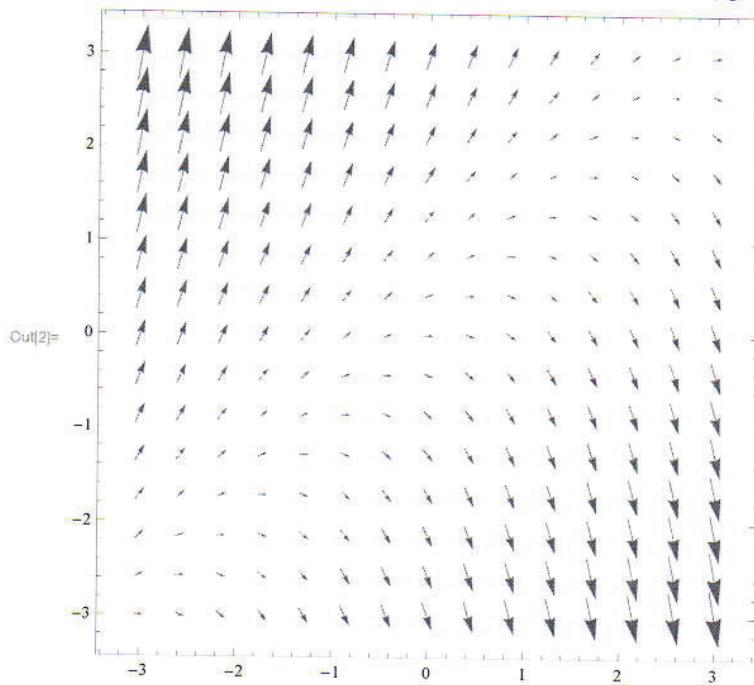
$$x_{c1} = 0 \\ x_{c2} = 1$$

x	$f(x)$
-3	-48
-2	-24
-1	-8
0	0
0.5	1
1	0
2	-8
3	-24



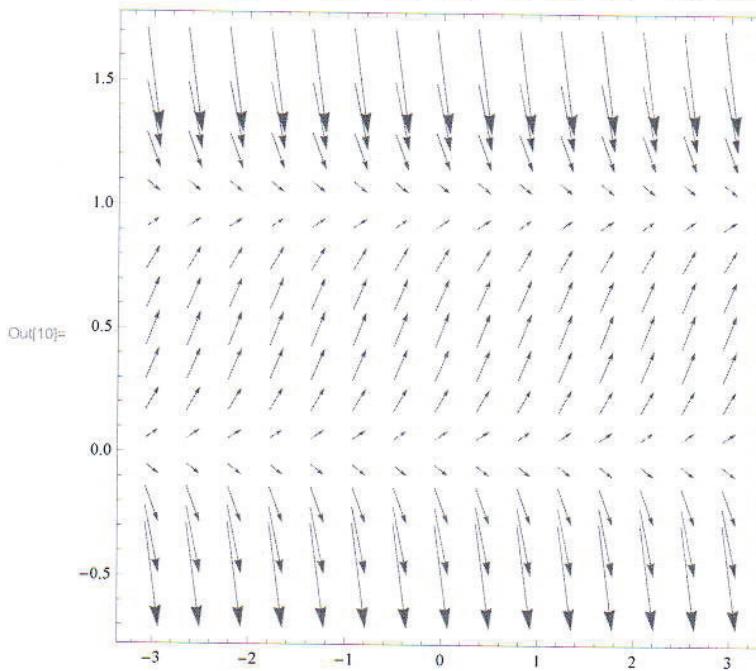
- (*) Za autonomne sustave formi pravac je isti t.
- Za neautonomne sustave postoji drugačiji formi pravac za svaki trenutak t.

In[1]:= $f[t_, x_] := x - t$
 $\text{VectorPlot}[\{1, f[t, x]\}, \{t, -3, 3\}, \{x, -3, 3\}]$



neutrale
Spirale

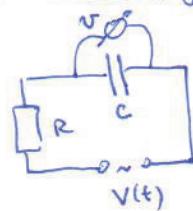
In[9]:= $f[t_, x_] := 4x(1-x)$
 $\text{VectorPlot}[\{1, f[t, x]\}, \{t, -3, 3\}, \{x, -.5, 1.5\}]$



attractiv
Spirale

Zadanie

RC - kmit -



$$V(t) = I \cdot R + v$$

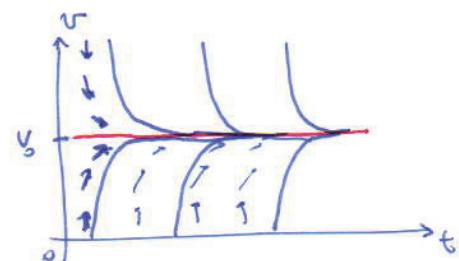
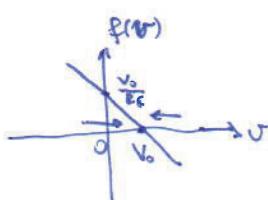
$$I = \frac{dv}{dt} = C \frac{dv}{dt}$$

$$V(t) = RC \frac{dv}{dt} + v \rightarrow \frac{dv}{dt} = \frac{1}{RC} (V(t) - v)$$

1°) $V(t) = V_0$ - konstantni napon

$$\frac{dv}{dt} = \frac{1}{RC} (V_0 - v)$$

$$\text{F.T. } v_c = V_0$$



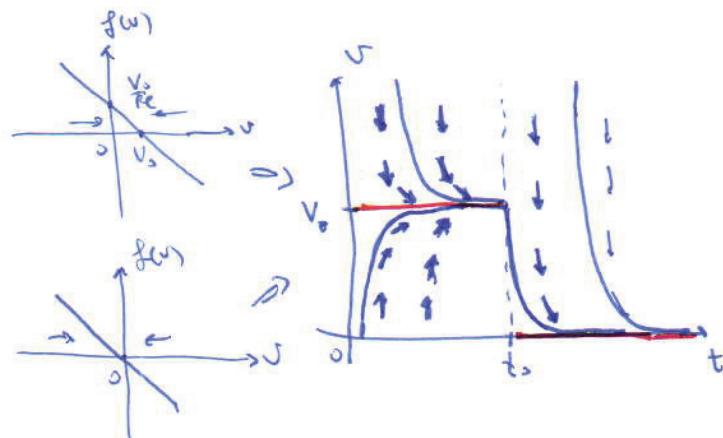
2°) $V(t) = \begin{cases} V_0, & 0 \leq t \leq t_0 \\ 0, & t > t_0 \end{cases}$ - pulz-gari napon

$$0 \leq t \leq t_0 : \frac{dv}{dt} = \frac{1}{RC} (V_0 - v)$$

$$\text{F.T. } v_c = V_0$$

$$t > t_0 : \frac{dv}{dt} = -\frac{1}{RC} v$$

$$\text{F.T. } = 0$$



3°) $V(t)$ = periodički pulz-gari napon s periodom $2t_0$

1.4.

NUMERIČKO RJEŠAVANJE DIFFERENCIJALNIH JEDNADŽBI (1. reda)

19

- za $\frac{dx}{dt} = f(t, x)$ funkcija $f(t, x)$ predstavlja izvor koji u svakoj točki (t, x) definira prirost funkcije. Dakle, ako izaberemo početni vrijednost (t_0, x_0) , možemo izračunati (t_1, x_1) gdje je $t_1 = t_0 + \Delta t$ itd. To je analogno rješavanju po tangentnom polinomu pr. vrednosti (t_0, x_0) pa dolje u smjeru vektora - tangenti.

①

EULEROVA METODA

jednačina: $\frac{dx}{dt} = f(t, x)$

$$\downarrow$$

approx: $\frac{x(t+\Delta t) - x(t)}{\Delta t} \approx f(t, x(t)) \quad | \Delta t$

$$x(t+\Delta t) = x(t) + f(t, x(t)) \cdot \Delta t$$

algoritam:

def. $f(t, x)$

input: t_0, t, x_0, n

$$\Delta t = \frac{t-t_0}{n}$$

(petlja)

\rightarrow
 $\Delta x_i = f(t_i, x_i) \Delta t, i = 1, 2, \dots, n$
 $t_i = t_0 + i \cdot \Delta t$
 $x_{i+1} = x_i + \Delta x_i$

Eulerova metoda

19-a

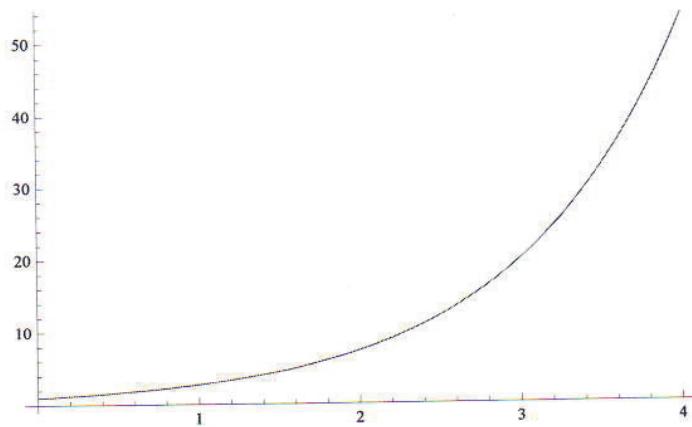
$$\frac{dx}{dt} = k \cdot x \quad ; \quad x(0) = 1 \\ k = 1$$

```
f[x_] := k * x  
  
k = 1;  
t0 = 0;  
t = 4;  
n = 4;  
dt = (t - t0) / n;  
  
s = 1;  
For[i = 1, i ≤ n, i++,  
  ds = f[s] * dt;  
  t = t0 + i * dt;  
  s = s + ds  
]  
s // N  
Exp[k * t] // N  
16.  
54.5982
```

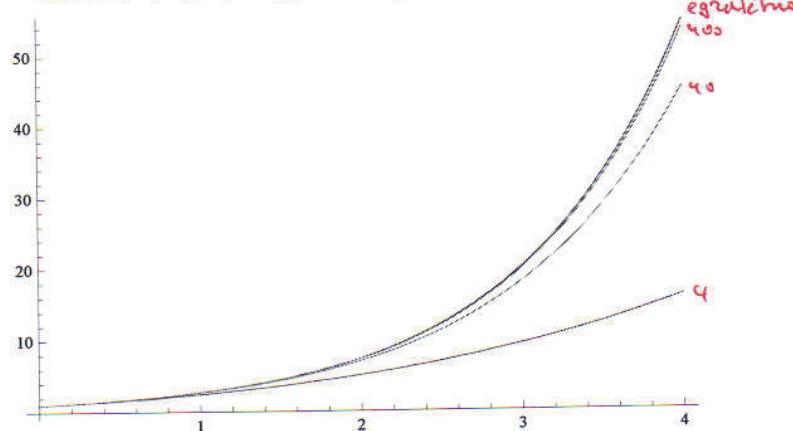
```
f[x_, t_] := k * x
```

```
Euler[t_, t0_, x0_, n_] := Module[{i, xx, dx, tt, dt},  
  dt = (t - t0) / n;  
  xx = x0;  
  For[i = 1, i ≤ n, i++,  
    dx = f[xx, tt] * dt;  
    tt = t0 + i * dt;  
    xx = xx + dx  
  ];  
  xx // N  
]  
  
k = 1;  
  
Euler[4, 0, 1, 400]  
53.5241
```

```
Plot[Euler[t, 0, 1, 400], {t, 0, 4}]
```



```
Plot[{Exp[k*t], Euler[t, 0, 1, 4],  
Euler[t, 0, 1, 40], Euler[t, 0, 1, 400]}, {t, 0, 4}]
```



RUNGE - KUTTA IV

20

def. $f(t, x)$

input: t_0, t, x_0, n

$$\Delta t = \frac{t - t_0}{n}$$

(petlja)

$$\begin{aligned}
 k_i^{(1)} &= f(t_i, x_i) \cdot \Delta t \\
 k_i^{(2)} &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{k_i^{(1)}}{2}\right) \cdot \Delta t \\
 k_i^{(3)} &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{k_i^{(2)}}{2}\right) \cdot \Delta t \\
 k_i^{(4)} &= f\left(t_i + \Delta t, x_i + k_i^{(3)}\right) \cdot \Delta t \\
 \Delta x_i &= \frac{1}{6} (k_i^{(1)} + 2k_i^{(2)} + 2k_i^{(3)} + k_i^{(4)}) \\
 t_i &= t_0 + i \cdot \Delta t \\
 x_{i+1} &= x_i + \Delta x_i
 \end{aligned}$$

RUNGE-KUTTA IV

$$\frac{dx}{dt} = kx, \quad x(0) = 1, \quad k=1$$

$f[x_, t_] := k * x$

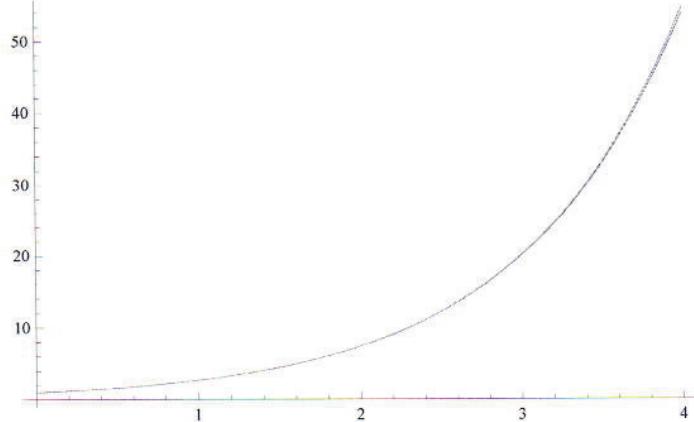
```
RK[t_, t0_, x0_, n_] := Module[{i, xx, dx, tt, dt, k1, k2, k3, k4},
  dt = (t - t0) / n;
  xx = x0;
  For[i = 1, i <= n, i++,
    k1 = f[xx, tt] * dt;
    k2 = f[xx + k1/2, tt + dt/2] * dt;
    k3 = f[xx + k2/2, tt + dt/2] * dt;
    k4 = f[xx + k3, tt + dt] * dt;
    dx = (k1 + 2*k2 + 2*k3 + k4) / 6;
    tt = t0 + i * dt;
    xx = xx + dx
  ];
  xx // N
]
```

$k = 1;$

$RK[4, 0, 1, 4]$

53.8032

$\text{Plot}[\{\text{Exp}[k*t], \text{RK}[t, 0, 1, 4], \text{RK}[t, 0, 1, 40], \text{RK}[t, 0, 1, 400]\}, \{t, 0, 4\}]$



wspomnienia Euler vs. Runge-Kutta IV :

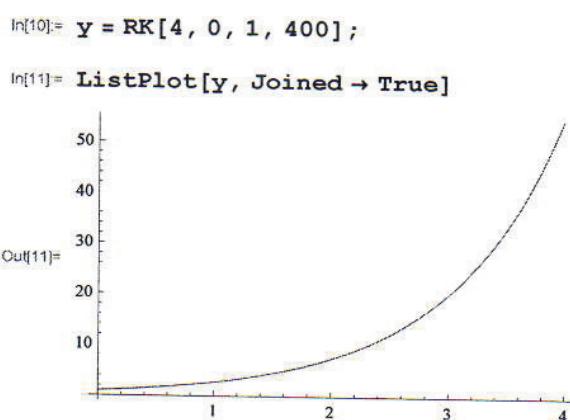
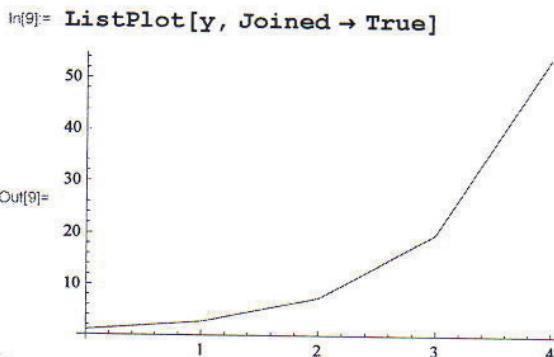
$$\underline{x(t=4)} \quad \left[\begin{array}{l} \text{egz. } x(4) = e^4 \\ = 54.5982 \end{array} \right]$$

n	4	40	400	4000
Euler	1e	45.2593	53.5241	54.4891
R-K	53.8032	54.5980	54.5982	-

```
In[6]:= f[x_, t_] := x

RK[t_, t0_, x0_, n_] := Module[{i, xx, dx, tt = t0, dt, k1, k2, k3, k4, T},
  dt = (t - t0) / n;
  xx = x0;
  T = Table[{0, 0}, {n + 1}];
  T[[1]] = {t0, x0};
  For[i = 1, i <= n, i++,
    k1 = f[xx, tt] * dt;
    k2 = f[xx + k1 / 2, tt + dt / 2] * dt;
    k3 = f[xx + k2 / 2, tt + dt / 2] * dt;
    k4 = f[xx + k3, tt + dt] * dt;
    dx = (k1 + 2 * k2 + 2 * k3 + k4) / 6;
    tt = t0 + i * dt;
    xx = xx + dx;
    T[[i + 1]] = {tt, xx}
  ];
  T // N
]

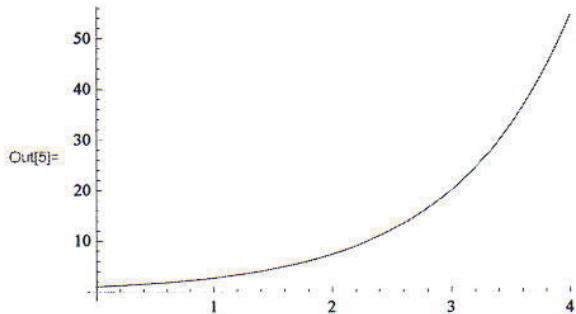
In[8]:= y = RK[4, 0, 1, 4]
Out[8]= {{0., 1.}, {1., 2.70833}, {2., 7.33507}, {3., 19.8658}, {4., 53.8032}}
```



```
In[3]:= g[y_, t_] := y
s = NDSolve[{y'[t] == g[y[t], t], y[0] == 1}, y, {t, 0, 4}]
Out[4]= {{y → InterpolatingFunction[{{0., 4.}}, <>]}}
```

2 | RungeKutta improved.nb

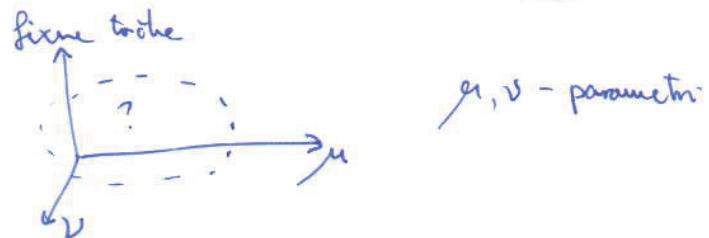
```
In[5]= Plot[Evaluate[y[t] /. s], {t, 0, 4}, PlotRange -> All]
```



Promjenom odlike farnog toka sustava ovise o vrijednosti parametra (parametara).

BIFURKACIJA je pojava pri kojoj mala promjena vrijednosti parametra izazove topolosku promjenu u farnom prostoru dinamičkog sustava (odnosno u asimptotskom ponašanju rješenja za $t \rightarrow \infty$).

Svi parametri dinamičkog sustava (mora ih biti više) razapinju k.v. PARAMETRSKI PROSTOR. U njemu, ovise o parametrima, možemo topoloski varirati odlike sustava f_j : fixne točke i dogadaje s/medu njima ovise o tome kako se parametri mijenjaju. Fixne točke su te koje određuju asimptotsko ponašanje rješenja f_j topologiju farnog prostora.



JEDNOPARAMETARSKA ORBITA (autonomnih dinamičkih sustava)s
JEDNOM BIFURKACIJOM

$$\frac{dx}{dt} = f_{\mu}(x)$$

parametar : za svaku vrijednost parametra je možno dobiti jednadžbu s drugačijim rješenjem koji moramo poslušati

■ Kada se javlja bifurkacija?

Da li se pojavlja bifurkacija, tj. kvalitativne topoloske promjene u farnom prostoru, treba imati svjeda (strukturno ustabilna) F.T. za koju je $f_{\mu}(x_0) = 0 : f'_{\mu}(x_0) = 0$. Već smo pokazali da mala perturbacija sistema s jednokrovnom F.T. ($f_{\mu}(x_0) = 0, f'_{\mu}(x_0) \neq 0$) samo mala ponešte F.T., ali ne vodi na topoloske kvalitativne promjene farnog prostora.

Vrste bifurkacija

#1 TANGENCIJALNA BIFURKACIJA

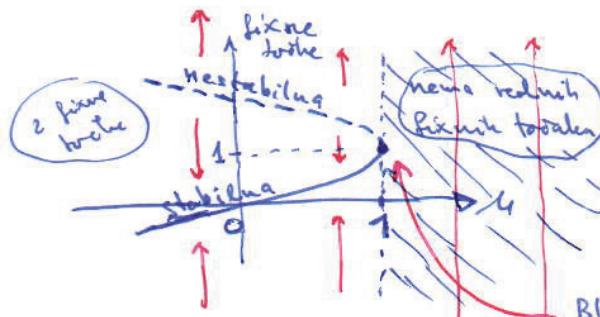
$$\frac{dx}{dt} = x^2 - 2x + \mu$$

parameter

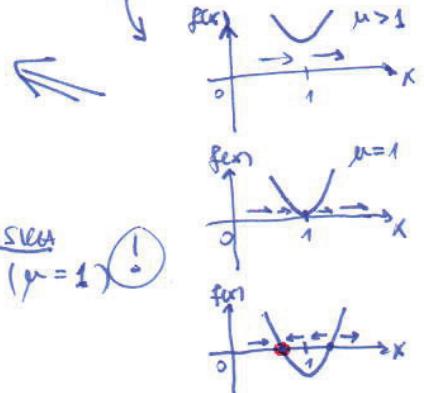
$$F.T.: f(x) = x^2 - 2x + \mu = 0$$

$$x_{1,2} = \frac{1}{2} (2 \pm \sqrt{4-4\mu}) = 1 \pm \sqrt{1-\mu} \Rightarrow$$

(*) Uočimo da je globalni formi tako još dobro od bifurkacijske točke ($\mu=1$) ostao nepromjenjiv u bifurkacijom!



$$\begin{cases} \mu < 1 ; 2 \text{ reale F.T.} \\ \mu = 1 ; 1 \text{ reale F.T.} \\ \mu > 1 ; \text{nemreale F.T.} \end{cases}$$



rijecije:

$$\frac{dx}{x^2 - 2x + \mu} = dt \quad | \int$$

$$\frac{1}{\sqrt{\mu-1}} \operatorname{arc tg} \frac{x-1}{\sqrt{\mu-1}} = t + C$$

$$\frac{x-1}{\sqrt{\mu-1}} = \operatorname{tg} (\sqrt{\mu-1}(t+C))$$

$$x(t) = 1 + \sqrt{\mu-1} \operatorname{tg} (\sqrt{\mu-1}(t+C)) \quad - \text{opće rješenje}$$

$$x(0) = x_0 = 1 + \sqrt{\mu-1} \operatorname{tg} (\sqrt{\mu-1} \cdot C)$$

$$\operatorname{arc tg} \frac{x_0-1}{\sqrt{\mu-1}} = \sqrt{\mu-1} \cdot C$$

$$C = \frac{\operatorname{arc tg} \frac{x_0-1}{\sqrt{\mu-1}}}{\sqrt{\mu-1}}$$

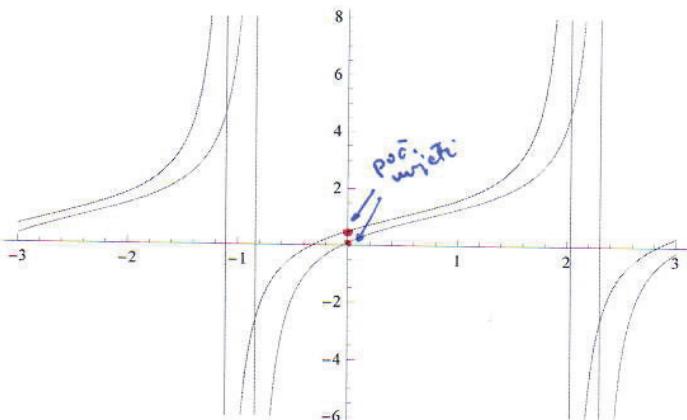
$$x(t) = 1 + \sqrt{\mu-1} \operatorname{tg} \left(\sqrt{\mu-1} t + \sqrt{\mu-1} \frac{\operatorname{arc tg} \frac{x_0-1}{\sqrt{\mu-1}}}{\sqrt{\mu-1}} \right)$$

$$x(t) = 1 + \sqrt{\mu-1} \operatorname{tg} \left(\sqrt{\mu-1} t + \operatorname{arc tg} \frac{x_0-1}{\sqrt{\mu-1}} \right)$$

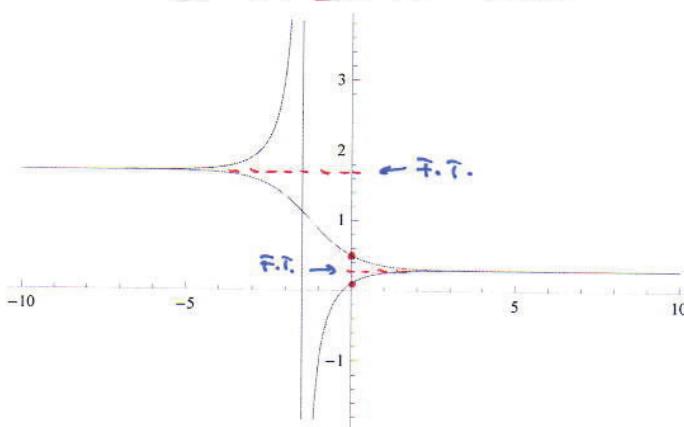
u bifurkacijskoj točki:

$$\lim_{\mu \rightarrow 1} x(t) = 1 \quad \checkmark, \quad \forall t, \forall x_0$$

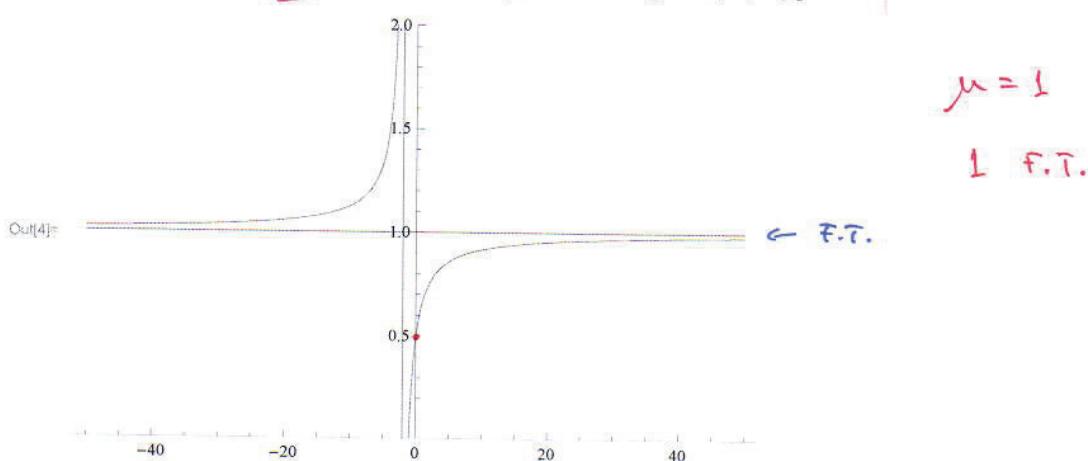
```
x[t_, x0_] := 1 + Sqrt[μ - 1] * Tan[Sqrt[μ - 1] * t + ArcTan[(x0 - 1) / Sqrt[μ - 1]]]
μ = 2;
Plot[{x[t, .5], x[t, .1]}, {t, -3, 3}]
```



```
μ = .5;
Plot[{x[t, .1], x[t, .5]}, {t, -10, 10}]
```



```
In[2]:= g[t_, x0_] :=
  Limit[1 + Sqrt[μ - 1] * Tan[Sqrt[μ - 1] * t + ArcTan[(x0 - 1) / Sqrt[μ - 1]]], μ → 1]
In[4]:= Plot[{1, g[t, .5]}, {t, -50, 50}, PlotRange → {0, 2}]
```



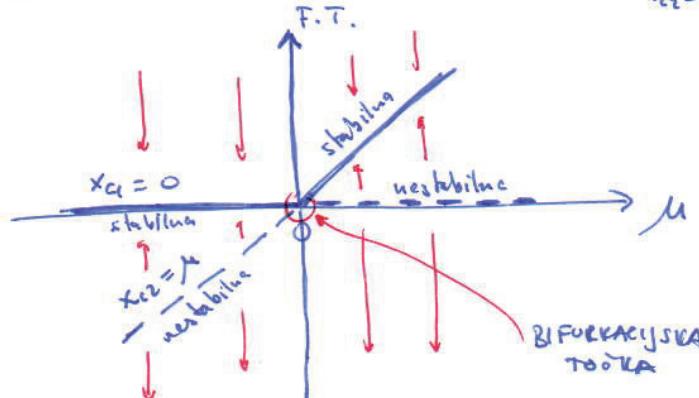
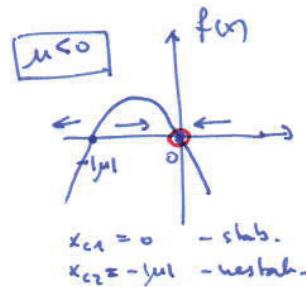
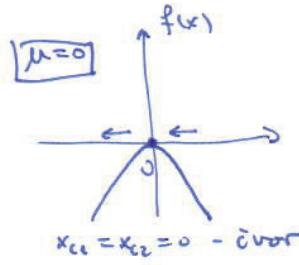
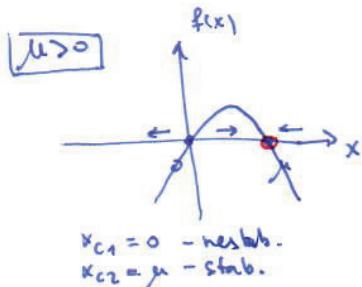
#2

BIFURKACIJA S PROMJENOM STABILNOSTI (zamjenjuje F.T.)

$$\frac{dx}{dt} = \mu x - x^2$$

$$f(x) = \mu x - x^2 \\ = x(\mu - x) \Rightarrow \text{F.T.}$$

$$\begin{cases} x_{c1} = 0 \\ x_{c2} = \mu \end{cases}$$



Fixne točke
 x_{c1} i x_{c2} zamjenile su stabilitet.

#3

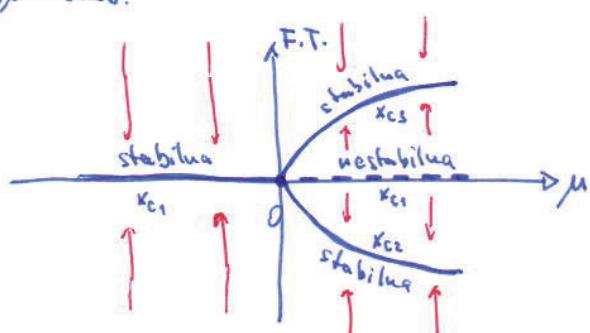
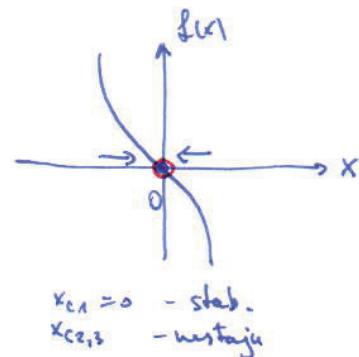
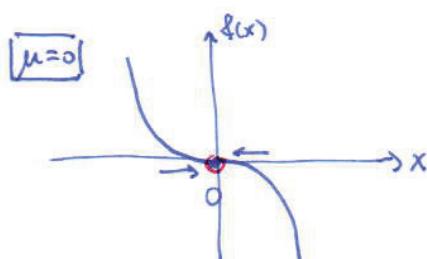
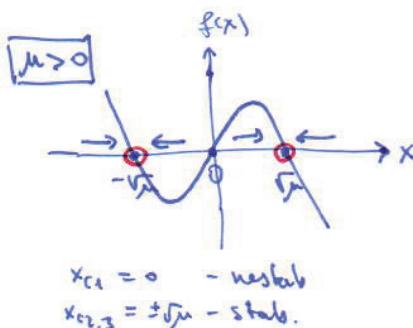
BIFURKACIJA RAŠLJEG
(PITCHFORK BIFURCATION)

$$f(x) = \mu x - x^3 \\ = x(\mu - x^2)$$

$$\frac{dx}{dt} = \mu x - x^3$$

F.T. \Rightarrow

$$\begin{cases} x_{c1} = 0 \\ x_{c2} = -\sqrt{\mu} \\ x_{c3} = +\sqrt{\mu} \end{cases}$$



#4

HOPFOVA BIFURKACIJA

Later! (za višedimenzionalne sustave)

Jz jedne stabilne F.T. nastaju tri, jedne nestabilne i dve stabilne.

Izbor

Izbor ribe :

24

Pitanje: Hocemo li izborom istrijeliti vrstu?

Modelirajmo populaciju ribe (npr. bluefin tane) logističkim modelom a koji dodamo član koji opisuje godišnji izbor:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{N}\right) - R$$

↓ ↓
 konstanta godišnja karta izbora (konstanta)
 rast. podsebe resursa i obolice
 priroda populacije

- model ima 3 parametra: k, N, R

Za $R=0$ imamo stan logistički model gdje je populacija asymptotski približava vrijednosti $P=N$. Poglideljimo što se događa kad uveljimo u model konstanan izbor $R \neq 0$:

$$f_R(P) = kP \left(1 - \frac{P}{N}\right) - R$$

$$= -\frac{k}{N}P^2 + kP - R$$

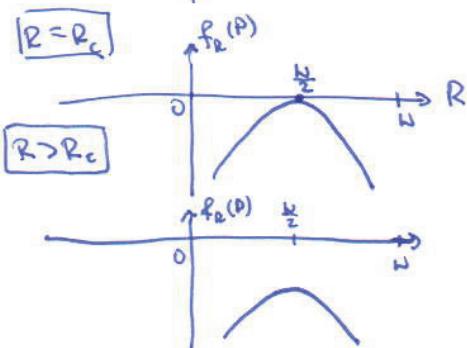
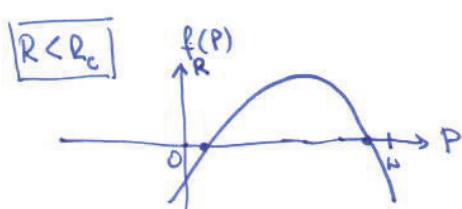
$$f_R(P) = 0 : P^2 - NP + \frac{RN}{k} = 0$$

$$P_{C1,2} = \frac{1}{2} \left(N \pm \sqrt{N^2 - \frac{4RN}{k}} \right) = \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \frac{RN}{k}}$$

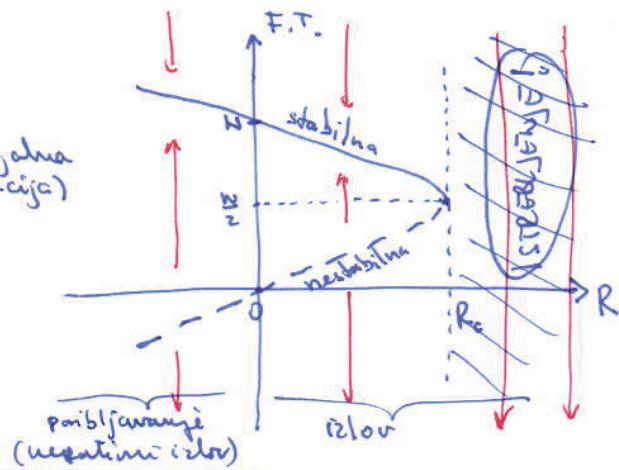
$$= \frac{N}{2} \cdot \left[1 \pm \sqrt{1 - \frac{4R}{kN}} \right]$$

$$R_c = \frac{kN}{4}$$

$R < R_c$; 2 F.T.
 $R = R_c$; 1 F.T.
 $R > R_c$; \emptyset F.T.



(tangencijalna biferacija)
⇒



(*) neponuda verava u rivi:

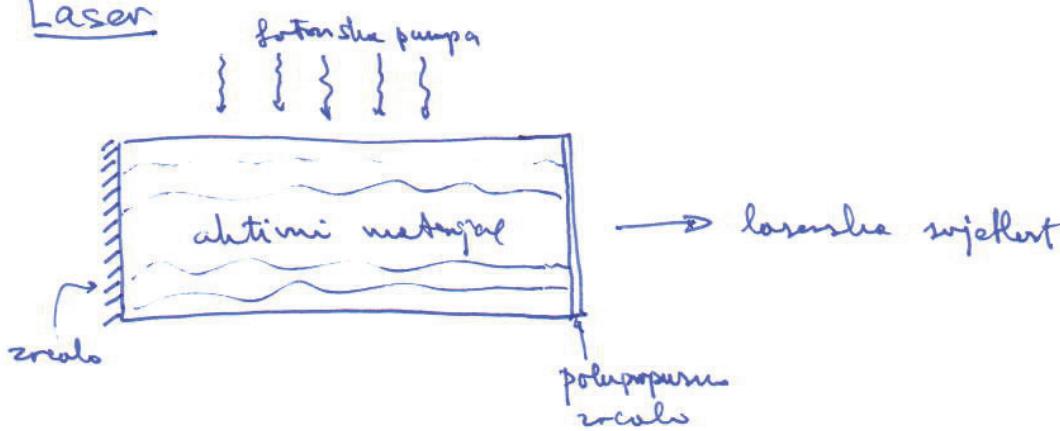
Što now kaže rezultat?

Kako povećavamo kvota izlova R , tako stacionarna ravnotežna populacija riba, od prethće vrijednosti $P_c = N$ bez izlova, poprima novu ravnotežnu vrijednost manju od N koja fazi ka $\frac{N}{2}$ kada povećavamo kvotu izlova R . No, kad dostignemo kvotu izlova $R = R_c = \frac{kN}{4}$, ravnotežna populacija riba je $\frac{N}{2}$. Nakon te vrijednosti dovoljno je samo malo povećati kvotu izlova da ravnotežna populacija riba $> \frac{N}{2}$ padne u \emptyset tj. da potpuno istrijebi vrsta. (Ova rečenica, ali neće F.T. trou u vremenu ka muti, može suprotan pedeset parbole u odnosu na s. primjer i tangencijalnom bifurkacijom.)

Ako se problemu kritičnoj vrijednosti parametra godi se događa bifurkacija, volo opamu je po sustav "čeketi" ales neće. Argument ribara, "dajte samo još malo povećajte kvotu, ništa neće biti" potpuno je pogresan ako se problemu vrijednosti R_c .

zadatak

Laser



Seljčeni optički pumpi (fotoni), atomi u laserskom aktivnom materijalu znače poput dipolnih antena. Za vlastitu slabo pumpanje osciliraju rezonans i znače mogućnost velike potpune uzmjeničnih faza. Uz, učinak što ih u pumpa pumpanja projekti kritičnu vrijednost (parametar i rezonator između 2 zrcala), atomi počnu oscilirati u fazi i od obidove svjetljike nastaje LASER.

model: $n(t)$ - broj fotona u laserskom polju

$$(1) \quad \frac{dn}{dt} = \text{gain} - \text{loss}$$

$$= GnN - kn$$

$\downarrow \text{gain coefficient}$ $\downarrow N(t) = \text{broj polu-} \\ \text{stvenih atoma}$

$\downarrow G > 0$ (parametar) $\downarrow k > 0$: rate con- \\ stant; $T = \frac{1}{k}$ je vrijeme života \\ fotona u lasersu

gain: dobar od stimulirane emisije gdje fotoni u rezonatoru stimuliraju atome u akti- \\ vuom materijalu na zračenje (u fazi)

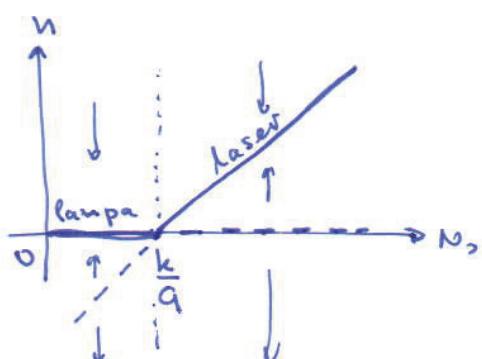
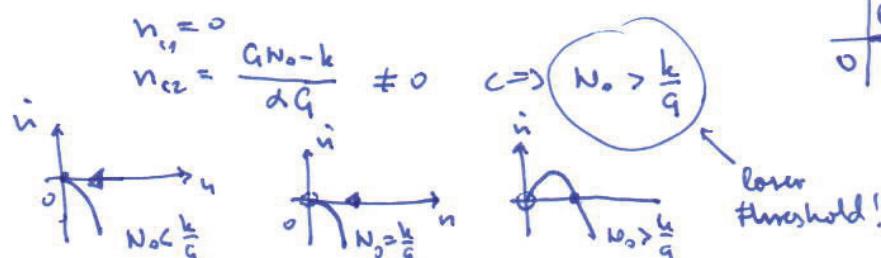
loss: dobar od lošega fotona bez poluproziranog zrcala (i sa stvorenim)

$$(2) \quad N(t) = N_0 - \alpha n - \text{uzimam što ih u foton, atom pada u} \\ \text{excitirajući stanje (inverzna populacija) i ne doprinosi više} \Rightarrow N(t) \text{ peta s brojem} \\ \text{iznicanih fotona} \Rightarrow \text{rate const. } \alpha > 0$$

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$

$$= (G_N_0 - k) n - (\alpha g) \cdot n^2$$

$$\text{F.T.: } n(G_N_0 - k - \alpha g n) = 0$$



§2

LINEARNE DIF. JEDNAĐEŽBE

2.1

LINEARNE DIF. JEDNAĐEŽBE 1. REDA

D.J. 1. reda je linearne ako se može napisati u obliku

$$\frac{dx}{dt} = f(t) \cdot x + g(t), \quad f(t), g(t) \text{ su pravovrijedne funkcije}$$

zavisne varijable uobičajeno u 1. potenciji (linearno)

Istovjetno s drugim matematičkim tehnikama jednađežbu možemo rešiti
koji se daje direktna interpretacija:

$$\frac{dx}{dt} - f(t) \cdot x = g(t)$$

| $\mu(t)$
 ↓ pomoćnu jednađežbu (za rad)
 funkcijom $\mu(t) \equiv \text{„INTEGRACIJSKI FAJTOR“}$

$$\underbrace{\mu(t) \frac{dx}{dt}}_{\text{Sadrži konstantno pravilo o derivaciju definiranju (Leibniz)}} - \mu(t) f(t) \cdot x = \mu(t) g(t)$$

Sadrži konstantno pravilo o derivaciju definiranju (Leibniz)

$$\frac{d}{dt} (\mu(t) x(t)) = \mu'(t) \cdot x(t) + \mu(t) \cdot x'(t)$$

$$\underbrace{\frac{d}{dt} (\mu(t) x(t))}_{\text{ugjet za određivanje}} - \mu'(t) \cdot x(t) - \mu(t) f(t) \cdot x(t) = \mu(t) g(t)$$

$$\underbrace{\mu(t)}_{\text{ugjet za određivanje}} \equiv \phi$$

možemo 2 jednađežbe: (i) $\frac{d}{dt} (\mu \cdot x) = \mu(t) g(t), \quad | \int$

$$(ii) \frac{du}{dt} \cdot x = -\mu \cdot f \cdot x \quad | : x$$

$$\underbrace{\frac{du}{dt}}_{\text{ugjet za određivanje}} = -\mu \cdot f$$

$$(i) \mu \cdot x = \int \mu(t) \cdot g(t) dt + c_1$$

$$x(t) = \frac{1}{\mu(t)} \left[\int \mu(t) g(t) dt + c_1 \right]$$

$$(2) \quad \frac{dx}{x} = -f(t) dt \quad | \int$$

$$\ln x = - \int f(t) dt + c_2$$

$$\mu(t) = e^{- \int f(t) dt + c_2}$$

$$\mu(t) = \tilde{c}_2 e^{- \int f(t) dt} \equiv \tilde{c}_2 \cdot \bar{\mu}$$

(1) & (2):

$$x(t) = \frac{1}{\tilde{c}_2 \bar{\mu}(t)} \left[\int \tilde{c}_2 \bar{\mu}(t) g(t) dt + c_1 \right]$$

$$= \frac{1}{\bar{\mu}(t)} \left[\int \bar{\mu}(t) g(t) dt + \left(\frac{c_1}{\tilde{c}_2} \right) \right] \equiv C$$

Dalle, išme jedinstvenu integracijsku konstantu C , možemo ne oznaci
o odeljivu konstante c_2 . Izaberimo sol preostale da je $\underline{c_2 = 0}$.

Summary metode:

riduzitba: $\frac{dx}{dt} = f(t)x + g(t)$

1.) izračunaj integracijski faktor: $\mu(t) = e^{- \int f(t) dt}$

2.) uvrstimo $\mu(t)$ u izraz za ^{spole} rješenje: $x(t) = \frac{1}{\mu(t)} \left[\int \mu(t) g(t) dt + C \right]$

primjer:

$$\frac{dx}{dt} = t-1 - \frac{2}{t}x \quad \rightarrow \quad f(t) = -\frac{2}{t}, \quad g(t) = t-1$$

lin. jedn.

$$\mu(t) = e^{- \int f(t) dt} = e^{\int \frac{2}{t} dt} = e^{2 \ln t} = (e^{\ln t})^2 = t^2$$

$$x(t) = \frac{1}{t^2} \left[\int t^2(t-1) dt + C \right]$$

$$= \frac{1}{t^2} \left[\frac{1}{4}t^4 - \frac{1}{3}t^3 + C \right]$$

$$= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{C}{t^2} \quad \checkmark$$

Zadatak 1 Zagadčno jezero

28

U oštoto jezeru volumenom 10000 m^3 ulijevaju se 2 potoka A i B, a istječe potok C. Potočni tok potoka je A: $500 \text{ m}^3/\text{dan}$, B: $750 \text{ m}^3/\text{dan}$ (ulijevanje): C: $1250 \text{ m}^3/\text{dan}$ (dalele volumen je očuvan). U $t=0$ lipidi počnu razdvajati jezero u 2 madiće: voda u potoku A onečiste solju za ceste u koncentraciji $5 \text{ kg na } 1000 \text{ m}^3$, i dodatno bacaju krovni netopirni otpad direktno u jezero dotokom $50 \text{ m}^3/\text{dan}$. Tako krov potoka C poneća se na $1300 \text{ m}^3/\text{dan}$ tako da se volumen vode u jezeru s vremenom promjenjije. Odredi koliko će mijenja koncentracije soli u jezeru (pretp. da je voda u jezeru dobro promijetana pa je koncentracija soli u svim točkama jezera ista).

$m(t)$ — ukupna masa soli u jezeru

$$\frac{dm}{dt} = m_{in} - m_{out}$$

$$k \cdot V_A \quad K(t) \cdot V_C$$

$$k = \frac{5}{1000} \frac{\text{kg}}{\text{m}^3} = 0.005 \frac{\text{kg}}{\text{m}^3}$$

$$V_A = 500 \text{ m}^3/\text{dan}$$

$$V_C = 1300 \text{ m}^3/\text{dan}$$

$$V_{k_0} = 50 \text{ m}^3/\text{dan}$$

$$V_0 = 10000 \text{ m}^3$$

$$\frac{dm}{dt} = k V_A - \frac{m}{V_0 - V_{k_0} \cdot t} \cdot V_C$$

$$\frac{dm}{dt} = 0.005 \cdot 500 - \frac{m}{10000 - 50t} \cdot 1300$$

$$\frac{dm}{dt} = 2.5 - \frac{26}{200-t} m$$

→ Voda je u jezeru sve mijeđe. model ovogledno ujedno daje išče vodu u jezeru, a to je do $t=200$ dane.

lin. d.j.:

$$f(t) = \frac{-26}{200-t}, g(t) = 2.5$$

$$m(t) = e^{+26 \int \frac{dt}{200-t}} = e^{-26 \ln(200-t)} = (200-t)^{-26}$$

$$m(t) = \frac{1}{(200-t)^{26}} \left\{ \int (200-t)^{-26} \cdot 2.5 dt + C \right\}$$

$$= (200-t)^{26} \left\{ 2.5 \int \frac{dt}{(200-t)^{26}} + C \right\} = (200-t)^{26} \left\{ 2.5 \frac{1}{-25} (200-t)^{-25} + C \right\}$$

$$= \frac{1}{10} (200-t)^{-25} + C \cdot (200-t)^{26}$$

pocetni uvjet: $m(0) = 0$ (na pocetku $t=0$ presi nije bio sagradjen)

$$0 = \frac{1}{10} \cdot 200 + C \cdot 200^{26} \rightarrow C = \frac{-20}{200^{26}}$$

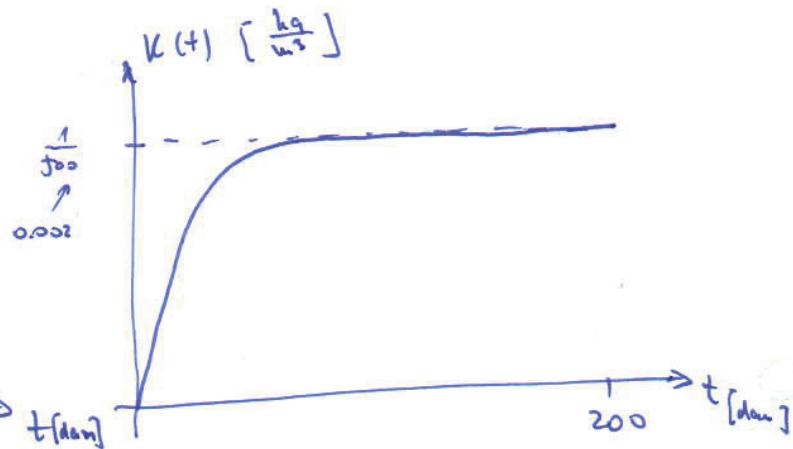
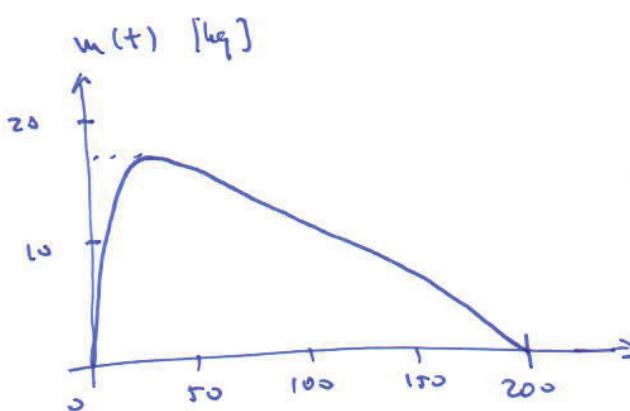
partikularno rješenje:

$$m(t) = \frac{1}{10} (200 - t) - 20 \cdot \left(\frac{200-t}{200}\right)^{26}$$

$t < 200$ dana!

konzentracija soli u serumu:

$$\begin{aligned} K(t) &= \frac{m(t)}{50 \cdot (200-t)} \\ &= \frac{1}{10} \frac{1}{50} - 20 \cdot \frac{1}{50} \frac{(200-t)^{25}}{200^{26}} \\ K(t) &= \frac{1}{500} \left(1 - \left(\frac{200-t}{200}\right)^{25} \right) \end{aligned}$$



Zadatak 2

Određi horizontnu elektronu pod djelovanjem konstantnog električnog polja ako na elektron djeluje i sile terja proporcionalna uglovom brzine.

$$m\ddot{v} = -e\vec{E} - \gamma m\vec{v} \quad | : m \quad \text{velje je } \vec{v} \parallel \vec{E}$$

$$\ddot{v} + \gamma v = -\frac{e}{m} E \quad \rightarrow \text{lin. jedn. 1. reda s konstantnim koefficijentima}$$

homogeni dio: $\ddot{v}_h + \gamma v_h = 0$

vrijedi je oblika $v_h = e^{\lambda t}$

$$\left. \begin{array}{l} \lambda e^{\lambda t} + \gamma e^{\lambda t} = 0 \\ \lambda + \gamma = 0 \\ \lambda = -\gamma \end{array} \right\} \Rightarrow v_h(t) = C e^{-\gamma t}$$

kompletna nehomogeni jednadžba: $v = v_h + v_s$

$v = C e^{-\gamma t} + v_s$

uvjetno da u nehom. jedn. $\ddot{v} + \gamma v = -\frac{e}{m} E$

$$-\gamma C e^{-\gamma t} + \gamma C e^{-\gamma t} + \gamma v_s = -\frac{e}{m} t$$

$$v_s = -\frac{e}{\gamma m} t$$

opće rješenje:

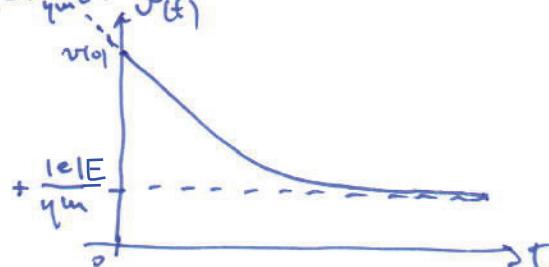
$$v(t) = C e^{-\gamma t} - \frac{e}{\gamma m} t$$

ostvareni je u početku ujeta

(*) Primijetimo koliko su re "homogeni dijelovi" polazili pri izradu v_s . To je zato jer se stacionarna rješenja, a to je ono za to, određuju nehomogenitetom jednadžbe istezaju. v_s koresponduje da može napraviti zanemarivi djelovanjima član u jednadžbi

~~$\ddot{v} + \gamma v = -\frac{e}{m} E \rightarrow v_s = -\frac{e}{\gamma m} t$~~

Homogeni dio ($e^{-\gamma t}$ opisuje prijelaznu pojavu s poč. ujetom koja eksponentno jake u vremenu, a onda se jačaju obiectivo istezaju nehomogenim dijelom $v_s = +\frac{ie}{\gamma m} t$:



LINEARNE DIFERENCIJALNE JEDNADŽBE VIŠEG REDA

$$\underbrace{y^{(n)} + d_{n-1} \underbrace{y^{(n-1)}}_{\uparrow} + \dots + d_0 y}_{\text{diferencijalna jednadžba}} = f(x) ; y(x)$$

- Ako su d_0, d_1, \dots, d_{n-1} konstante, radi se o običajnoj diferencijalnoj jednadžbi, rješavaju se:

homogeni dij: metodom neodređenih koeficijenata

nehomogeni jed: metodom varijacije konstanti (kad znamo rješenje homogenog dijela)

- Ako je bilo koji λ_i funkcija od x , konsti je Frobeniusova metoda.

OBIĆNE V^{LIN.} DIF. JEDNADŽBE (konstantni koeficijenti)

Homogeni dij: $y^{(n)} + d_{n-1} y^{(n-1)} + \dots + d_0 y = 0$

$$\Rightarrow F(\lambda) = \lambda^n + d_{n-1} \lambda^{n-1} + \dots + d_0 \quad - \text{ karakteristični polinom}$$

$$= (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \dots (\lambda - \lambda_k)^{m_k}, \quad m_1 + m_2 + \dots + m_k = n$$

(kraćene nultočke)

→ prostor rješenja: $e^{\lambda_1 x}, x e^{\lambda_1 x}, \dots, x^{m_1-1} e^{\lambda_1 x}$

$$e^{\lambda_2 x}, x e^{\lambda_2 x}, \dots, x^{m_2-1} e^{\lambda_2 x}$$

⋮

$$e^{\lambda_k x}, x e^{\lambda_k x}, \dots, x^{m_k-1} e^{\lambda_k x}$$

rješenje homogene jedn. je linearna kombinacija gornjih vektora:

$$y_H(x) = c_{11} e^{\lambda_1 x} + c_{12} x e^{\lambda_1 x} + \dots + c_{1m_1} x^{m_1-1} e^{\lambda_1 x} + \\ + c_{21} e^{\lambda_2 x} + c_{22} x e^{\lambda_2 x} + \dots \\ \dots \\ + c_{km_k} x^{m_k-1} e^{\lambda_k x}$$

Nehomogena:

polinom n-tog stupnja

Ako je nehomogenost $f(x) = P_n(x) e^{\beta x}$, tad partikularno (stacionarno) rješenje tražimo u obliku $y_p = x^r Q_n(x) e^{\beta x}$ gdje je:

$Q_n(x)$ – polinom n-tog stupnja → neandžljiv koeficijentima

$r = 0$ ako je $\beta \notin \{\lambda_1, \lambda_2, \dots\}$, inčeđe r je konstanta nultočke ($\beta \in \{\lambda_1, \lambda_2, \dots\}$)

za općeniti $f(x)$, partik. rješenje $y_p(x)$ tražimo varijacijom konstanti

OPĆE RJEŠENJE:

$$y(x) = y_H(x) + y_p(x)$$

Ipriner: Nacti 3 lin. uravneua rješenja jednačine

$$y''' + y' - 2y = 0$$

$$\lambda^3 + \lambda - 2 = 0 \quad \rightarrow \quad \lambda^3 + 2\lambda - \lambda - 2 = \lambda(\lambda^2 - 1) + 2(\lambda - 1) = (\lambda - 1)(\lambda^2 + \lambda + 2)$$

pogodnoj mreži korišćeni Hornerovu shemu (mora biti dodeljivo
stvrdnog Elama -2); ±1, ±2

$$\begin{array}{cccc|c}
 (\lambda^3) & (\lambda^2) & (\lambda) & (\lambda^0) & \\
 1 & 0 & 1 & -2 & \\
 \hline
 1 & 1 & 1+0 & 1+1 & 2-2=0 \\
 & & \cancel{\lambda+0} & \cancel{\lambda+1} & \\
 \Rightarrow 1 & 1 & 1 & 1 &
 \end{array}$$

si nultočka $\lambda = 1$ $(\lambda - 1)(\lambda^2 + \lambda + 2) = 0$

$$\lambda_{2,3} = \frac{1}{2} (-1 \pm \sqrt{1-8}) = -\frac{1}{2} \pm i \frac{\sqrt{7}}{2}$$

$$y_1(x) = e^x$$

$$y_2(x) = e^{-\frac{1}{2}x} \sin \frac{\sqrt{7}}{2} x$$

$$y_3(x) = e^{-\frac{1}{2}x} \cos \frac{\sqrt{7}}{2} x$$

$$\underline{\text{HOMOGENA}} : y'' + py' + qy = 0 \rightarrow \lambda^2 + p\lambda + q = 0$$

$\hookrightarrow \lambda_1, \lambda_2$ - kojini

$$1) \lambda_1 \neq \lambda_2 \in \mathbb{R} \Rightarrow y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$2) \lambda_1 = \lambda_2 \in \mathbb{R} \Rightarrow y_h = (c_1 + c_2 x) e^{\lambda_1 x}$$

$$3) \lambda_{1,2} = \alpha \pm \beta i, \beta \neq 0 \Rightarrow y_h = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$$

$$\underline{\text{NEHOMOGENA}} : y'' + py' + qy = f(x)$$

$$(1) f(x) = e^{ax} P_n(x)$$

$$(a) a \notin \{\lambda_1, \lambda_2\} \Rightarrow y_p = e^{ax} Q_n(x)$$

$$(b) a \in \{\lambda_1, \lambda_2\} \Rightarrow y_p = x^r e^{ax} Q_n(x)$$

polynom n-tog stupnja s
neodređenim koeficijentima
 $A + Bx + Cx^2 + \dots + Zx^n$

r = kратnost multoche a (1 ili 2)

$$(2) f(x) = e^{ax} [P_n(x) \cos bx + Q_m(x) \sin bx]$$

$$(a) a \pm bi \notin \{\lambda_1, \lambda_2\} \Rightarrow y_p = e^{ax} [S_N(x) \cos bx + T_N(x) \sin bx]$$

polinomi N-tog stupnja s neodređenim
koeficijentima, gdje je $N = \max\{n, m\}$

$$(b) a \pm bi \in \{\lambda_1, \lambda_2\} \Rightarrow y_p = x^r e^{ax} [S_N(x) \cos bx + T_N(x) \sin bx]$$

$r=1$ za jedn. 2. reda

$$(3) f(x) = \text{lineo str. dugo} \Rightarrow y_p = \text{varijacija konstanti}$$

$$y_p^{(n)} = \underbrace{u_1(x) y_{u_1}(x)}_{\text{rijavajući problem}} + \underbrace{u_2(x) y_{u_2}(x)}_{\text{rijavajući problem}}$$

rijavajući problem:

$$\begin{bmatrix} y_{u_1} & \cdots & y_{u_n} \\ y'_{u_1} & \cdots & y'_{u_n} \\ \vdots & & \vdots \\ y^{(n-1)}_{u_1} & \cdots & y^{(n-1)}_{u_n} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

primjer 1

Metoda nekonstantnih koefficijenata

$$y'' - 2y' + y = e^x + \cos x$$

$f_1(x)$ $f_2(x)$

\Rightarrow Naci čemo partikularne rješenja
 $y_{p1}(x)$ i $y_{p2}(x)$ za svaku nehomogenost posebno i onda ih kombinirati
 princip superpozicije $y_p = y_{p1} + y_{p2}$

$$\text{homogen: } x^2 - 2x + 1 = 0$$

$$\lambda_1 = \lambda_2 = 1 \rightarrow \underline{\text{dvostruka nultočka}}$$

$$\Rightarrow y_{H1} = e^x$$

$$y_{H2} = x e^x$$

nehomogen:

$$1.) f_1(x) = e^x \rightarrow \text{to je sluzaj } (1^0 - b) : f(x) = e^{ax} P_n(x), a=1, n=0$$

$$a=1 \in \{\lambda_1, \lambda_2\} = \{1, 1\} \rightarrow \underline{\text{dvostruka nultočka}} ; r=2$$

$$y_{p1} = x^r e^{ax} Q_n(x) = x^2 e^x \cdot A \quad \text{polinom 0-tog stupnja s realist. koef.}$$

$$y'_{p1} = A(2x e^x + x^2 e^x) = A e^x (2x + x^2)$$

$$y''_{p1} = A e^x (2x + x^2) + A e^x (2 + 2x) = A e^x (2 + 4x + x^2)$$

koristimo u jednadžbi da odredimo koefficijente:

$$A e^x (2 + 4x + x^2) - 2A e^x (2x + x^2) + A e^x x^2 = e^x \quad | : e^x$$

$$A \cdot 2 = 1$$

$$\boxed{A = \frac{1}{2}} \quad \Rightarrow \quad \underline{y_{p1}(x) = \frac{1}{2} x^2 e^x}$$

$$2.) f_2(x) = \cos x \rightarrow \text{sluzaj } (2^0 - a) : f(x) = e^{ax} [P_n(x) \cos bx + \overset{=0}{Q_n(x)} \sin bx]$$

$$\underbrace{a=0}, \underbrace{b=1} \rightarrow \alpha \pm bi = i \notin \{\lambda_1, \lambda_2\} = \{1, 1\}, \quad \boxed{N=0}$$

$$y_{p2} = e^{ax} [S_N(x) \cos bx + T_N(x) \sin bx] = A \cos x + B \sin x$$

$$y'_{p2} = -A \sin x + B \cos x$$

$$y''_{p2} = -A \cos x - B \sin x$$

u jedn:

$$-A \cos x - B \sin x + 2A \cos x - 2B \sin x + A \cos x + B \sin x = \cos x$$

$$2A \sin x = 0 \rightarrow \boxed{A=0}$$

$$2B \cos x = \cos x \rightarrow \boxed{B=-\frac{1}{2}} \quad \Rightarrow \quad \underline{y_{p2}(x) = -\frac{1}{2} \sin x}$$

opće rješenje:

$$\begin{aligned} y(x) &= C_1 e^x + C_2 x e^x + \frac{1}{2} x^2 e^x - \frac{1}{2} \sin x \\ &= e^x (C_1 + C_2 x + \frac{1}{2} x^2) - \frac{1}{2} \sin x \end{aligned}$$

primjer 2

$$y'' + y = \operatorname{tg} x$$

Metoda varijacije konstanti

$$\text{homogene: } y'' + y = 0 \rightarrow \lambda^2 + 1 = 0 \rightarrow \lambda_1 = i \quad \lambda_2 = -i \quad \rightarrow y_{H1} = e^{ix} \\ y_{H2} = e^{-ix}$$

$$\text{nehomogene: } f(x) = \operatorname{tg} x$$

↓

metoda varijacije konstanti

$$\underline{y_p(x) = u_1(x)y_{H1}(x) + u_2(x)y_{H2}(x)}$$

$$y_{H1} = \sin x \\ y_{H2} = \cos x$$

$$\begin{bmatrix} y_{H1} & y_{H2} \\ y'_{H1} & y'_{H2} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$\begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \operatorname{tg} x \end{bmatrix}$$

$$\sin x \cdot u_1' + \cos x \cdot u_2' = 0 \quad | : \cos x$$

$$\cos x \cdot u_1' - \sin x \cdot u_2' = \operatorname{tg} x \quad | : \sin x$$

$$\frac{\sin x}{\cos x}$$

$$\operatorname{tg} x \cdot u_1' + u_2' = 0$$

$$\frac{\cos x}{\sin x}$$

$$\operatorname{tg} x \cdot u_1' - u_2' = \frac{1}{\cos x}$$

$$\frac{\sin^2 x + \cos^2 x}{\cos x \sin x} u_1' = \frac{1}{\cos x} \quad | \cdot \cos x$$

$$\frac{1}{\sin x} u_1' = 1 \rightarrow u_1' = \sin x \quad | \int$$

$$u_1 = -\cos x$$

tejemo je već integracijske konstante. Ona je već uvedena u homogenu rješenju (tako je tu možnosti, da se homogeno rješuje.)

$$u_2' = -\operatorname{tg} x \cdot u_1' = -\operatorname{tg} x \cdot \sin x = -\frac{\sin^2 x}{\cos x} = -\frac{1-\cos^2 x}{\cos x} = \cos x - \frac{1}{\cos x}$$

$$u_2 = \int \cos x dx - \int \frac{dx}{\cos x} = \sin x - \ln \left| \operatorname{tg} \left(\frac{x}{2} + \frac{\pi}{4} \right) \right| =$$

$$= \sin x - \ln \left| \frac{\sin \left(\frac{x}{2} + \frac{\pi}{4} \right)}{\cos \left(\frac{x}{2} + \frac{\pi}{4} \right)} \right| = \sin x - \ln \left| \frac{\sin \frac{x}{2} \cos \frac{\pi}{4} + \cos \frac{x}{2} \sin \frac{\pi}{4}}{\cos \frac{x}{2} \cos \frac{\pi}{4} - \sin \frac{x}{2} \sin \frac{\pi}{4}} \right| \quad | : \cos \frac{x}{2}$$

$$u_2 = \sin x - \ln \left| \frac{1 + \operatorname{tg} \frac{x}{2}}{1 - \operatorname{tg} \frac{x}{2}} \right|$$

$$y_p(x) = -\cos x \sin x + \cos x \sin x - \cos x \cdot \ln \left| \frac{1 + \operatorname{tg} \frac{x}{2}}{1 - \operatorname{tg} \frac{x}{2}} \right| = -\cos x \cdot \ln \left| \frac{1 + \operatorname{tg} \frac{x}{2}}{1 - \operatorname{tg} \frac{x}{2}} \right|$$

opće rješenje:

$$y(x) = C_1 \sin x + C_2 \cos x - \cos x \cdot \ln \left| \frac{1 + \operatorname{tg} \frac{x}{2}}{1 - \operatorname{tg} \frac{x}{2}} \right|$$

Zadatak

$$y'' - y = e^x$$

homogen: $y'' - y = 0 \rightarrow \lambda^2 - 1 = 0 ; \lambda_1 = 1 ; \lambda_2 = -1 \rightarrow y_{H1} = e^x ; y_{H2} = e^{-x}$

neshomogen:

a) metoda neodređenih koeficijenata:

$$f(x) = e^x ; \text{ stoga } f(x) = e^{ax} P_n(x) ; a=1, n=0$$

$$a = 1 \in \{\lambda_1, \lambda_2\} = \{1, -1\} \rightarrow \text{stemušno} \Rightarrow r=1$$

$$y_p = x^r e^{ax} Q_n(x) = x e^x \cdot A$$

$$y'_p = A e^x + A x e^x = A(1+x)e^x$$

$$y''_p = A e^x + A(1+x)e^x = A(2+x)e^x$$

$$A(2+x)e^x - A x e^x = e^x \quad | :e^x$$

$$A(2+x-x) = 1$$

$$\Rightarrow A = \frac{1}{2}, \quad \Rightarrow \underline{y_p = \frac{1}{2} x e^x}$$

opća rešenje:

$$\begin{aligned} y(x) &= C_1 e^x + C_2 e^{-x} + \frac{1}{2} x e^x \\ &= \underbrace{(C_1 + \frac{x}{2}) e^x}_{\text{u}_1(x)} + C_2 e^{-x} \end{aligned}$$

b) metoda varijacije konstanti:

$$y_p = u_1(x) e^x + u_2(x) e^{-x}$$

$$\begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \end{bmatrix}$$

$$\begin{cases} u'_1 e^x + u'_2 e^{-x} = 0 \\ u'_1 e^x - u'_2 e^{-x} = e^x \end{cases} \quad \left| \begin{array}{l} \rightarrow u'_2 e^{-x} = -u'_1 e^x \quad | :e^{-x} \\ u'_2 = -u'_1 e^{2x} = -\frac{1}{2} e^{2x} \end{array} \right. \quad | \int$$

$$2e^x u'_1 = e^x \quad | :e^x$$

$$u'_1 = \frac{1}{2} \quad | \int$$

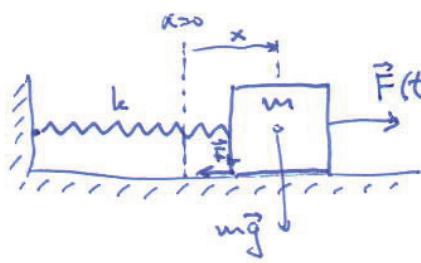
$$u_1(x) = \frac{1}{2} x$$

$$u'_2 = -\frac{1}{2} \int e^{2x} dx = -\frac{1}{4} e^{2x}$$

$$\begin{aligned} y_p(x) &= \frac{1}{2} x e^x - \frac{1}{4} e^{2x} \cdot e^{-x} \\ &= \frac{1}{2} x e^x - \frac{1}{4} e^x = -\frac{1}{4} (1-2x) e^x \end{aligned}$$

opća rešenje:

$$\begin{aligned} y(x) &= C_1 e^x + C_2 e^{-x} - \frac{1}{4} (1-2x) e^x \\ &= \underbrace{(C_1 - \frac{1}{4} + \frac{x}{2}) e^x}_{\tilde{C}_1} + C_2 e^{-x} \\ &= \underbrace{(\tilde{C}_1 + \frac{x}{2}) e^x + C_2 e^{-x}}_{\text{u}_2(x)} \end{aligned}$$



$\vec{F}(t)$ - varjirna sila

$$\begin{aligned} m\ddot{x} &= -kx - F_{fr} + F(t) \\ \ddot{x} + \left(\frac{\eta}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x &= f(t) \\ \ddot{x} + \gamma\dot{x} + \omega^2 x &= f(t) \end{aligned}$$

$\approx \eta \cdot \dot{x}$
 $\approx m \cdot f(t)$
 $\equiv \gamma$
 $\equiv \omega^2$

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = f(t)$$

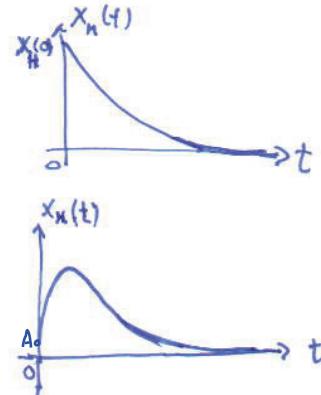
Homogeni (gurjani harmonički oscilator):

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$$

$$\rightarrow \lambda^2 + \gamma\lambda + \omega^2 = 0 \quad ; \quad \lambda_{1,2} = \frac{1}{2} \left[-\gamma \pm \sqrt{\gamma^2 - 4\omega^2} \right]$$

1) $\gamma^2 - 4\omega^2 > 0$ - nathnions gurjije

$$\Rightarrow \lambda_1 \neq \lambda_2 \in \mathbb{R} ; \quad x_h(t) = A e^{\lambda_1 t} + B e^{\lambda_2 t}$$

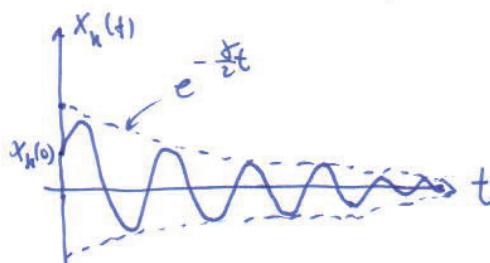


2) $\gamma^2 - 4\omega^2 = 0$ - kritične gurjije

$$\Rightarrow \lambda_1 = \lambda_2 \in \mathbb{R} ; \quad x_h(t) = (A + Bt) e^{-\frac{\gamma}{2}t}$$

3) $\gamma^2 - 4\omega^2 < 0$ - paratnions gurjije

$$\Rightarrow \lambda_{1,2} = \frac{\gamma}{2} \pm i\omega_I \in \mathbb{C} ; \quad x_h(t) = e^{-\frac{\gamma}{2}t} (A \cos \omega_I t + B \sin \omega_I t)$$



Nehomogeni (tijevni H.O.)

- partikularno rješenje $x_p(t)$ zadani obliku $f(t)$

$m \cdot \text{vredostem koef.}$
 $m \cdot \text{varijacije konstanti}$

Ukupno (opće) rješenje:

$$x(t) = x_h(t) + x_p(t) \rightarrow x_p(t)$$

\uparrow
 $(t \rightarrow \infty)$

u prijelaznu
pojavu

exponečijalno
se negira za $t \rightarrow \infty$
dakle je $\gamma \neq 0$

Rješenje jednadžbi za slavoj "sinusidelnuog" tijekom potencija
gušenja

$$f(t) = f_0 \cos \omega t$$

a) pretpostavimo da frekvencija ω nije jednaka $\omega_{1,2}$

- onda raspolaže u slavoj $f(t) = e^{at} [P_n(t) \cos \omega t + Q_m(t) \sin \omega t]$; $a = \phi$

$$\omega \pm i\phi = \pm i\omega \neq -\frac{\gamma}{2} \pm i\sqrt{(\frac{\gamma}{2})^2 - \omega^2}$$

$$\phi = \omega$$

$$P_n \neq 0 \text{ - toga st.}$$

$$Q_m = 0$$

$$\Rightarrow X_p(t) = A \cos \omega t + B \sin \omega t$$

$$\dot{x}_p = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$\ddot{x}_p = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$-\omega^2 A \cos \omega t - \omega^2 B \sin \omega t - \gamma \omega A \sin \omega t + \gamma \omega B \cos \omega t + \omega^2 A \cos \omega t + \omega^2 B \sin \omega t = f_0 \cos \omega t$$

$$-\omega^2 A + \gamma \omega B + \omega^2 A = f_0$$

$$-\omega^2 B - \gamma \omega A + \omega^2 B = 0$$

$$A(\omega^2 - \omega^2) + \gamma \omega B = f_0$$

$$-\gamma \omega A + (\omega^2 - \omega^2) B = 0 \rightarrow B = \frac{\gamma \omega}{\omega^2 - \omega^2} A$$

$$A(\omega^2 - \omega^2) + \frac{(\gamma \omega)^2}{\omega^2 - \omega^2} A = f_0 \quad | \quad (\omega^2 - \omega^2)$$

$$A = \frac{\omega^2 - \omega^2}{(\omega^2 - \omega^2)^2 + (\gamma \omega)^2} f_0 \quad , \quad B = \frac{\gamma \omega}{(\omega^2 - \omega^2)^2 + (\gamma \omega)^2} f_0$$

izvrimo rješenje ponovno amplitudu i fazu:

$$A \cos \omega t + B \sin \omega t \equiv f_0 \cos (\omega t + \varphi)$$

$$= \cos \omega t \cdot \cos \varphi - \sin \omega t \cdot \sin \varphi$$

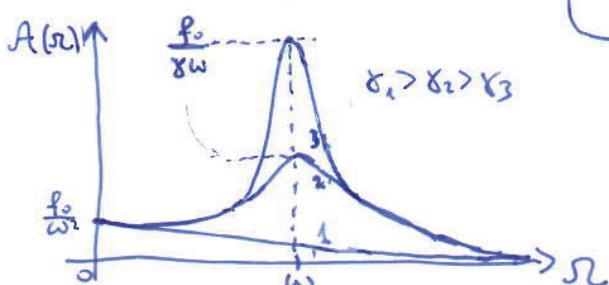
$$\begin{aligned} A &= f_0 \cos \varphi \\ B &= -f_0 \sin \varphi \end{aligned} \quad |^2 \quad |^2 \quad |+$$

$$A^2 = \sqrt{A^2 + B^2}$$

$$f_0 = \sqrt{\frac{(\omega^2 - \omega^2)^2 + (\gamma \omega)^2}{(\omega^2 - \omega^2)^2 + (\gamma \omega)^2}} f_0$$

$$\frac{\sin \varphi}{\cos \varphi} = -\frac{B}{A} \rightarrow \operatorname{tg} \varphi = -\frac{B}{A}$$

$$\varphi = \arctg \left(-\frac{B}{A} \right)$$



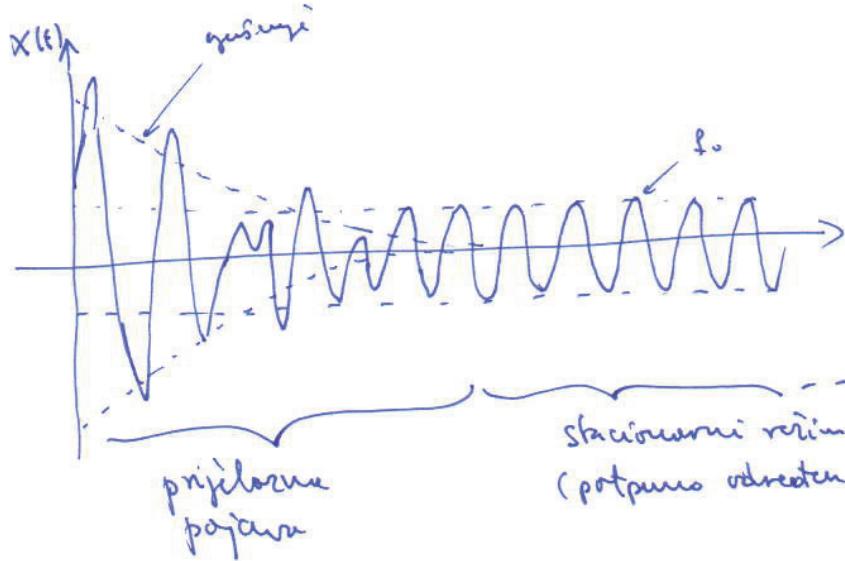
$$\boxed{A = \frac{f_0}{\sqrt{(\omega^2 - \omega^2)^2 + (\gamma \omega)^2}}}$$

$$\boxed{\varphi = \arctg \frac{-\gamma \omega}{\omega^2 - \omega^2}}$$

$$\therefore X_p(t) = A \cos (\omega t + \varphi)$$

Amplitude $X_p(t)$ je ovisna ne počinje $f(t)$ nego
je za $\omega = \omega$, to veća što se jačaju γ manje.

$$x(t) = x_h(t) + x_p(t)$$



b) pretpostavimo da nema gurjenja ($\gamma=0$) i da je $\Omega=\omega$:

$$\lambda_{1,2} = \pm i\omega \quad ; \quad a \pm ib = \pm i\omega \in \{i\omega, -i\omega\} \Rightarrow \underline{\gamma=1}$$

$$\Rightarrow x_p(t) = t \cdot (A \cos \omega t + B \sin \omega t)$$

$$\dot{x}_p = A \cos \omega t + B \sin \omega t + \omega t (-A \sin \omega t + B \cos \omega t) = (A + \omega t B) \cos \omega t + (B - \omega t A) \sin \omega t$$

$$\ddot{x}_p = \omega^2 B \cos \omega t - \omega A \sin \omega t - (\omega^2 t A) \cos \omega t + (\omega^2 t B) \sin \omega t \\ = \omega^2 (2B - \omega^2 t A) \cos \omega t - \omega^2 (2A + \omega^2 t B) \sin \omega t$$

$$\ddot{x} + \omega^2 x = f_0 \cos \omega t$$

$$\omega^2 (2B - \omega^2 t A) \cos \omega t - \omega^2 (2A + \omega^2 t B) \sin \omega t + \omega^2 t A \cos \omega t + \omega^2 t B \sin \omega t = f_0 \cos \omega t$$

$$2\omega^2 B - \omega^4 t A + \omega^4 t A = f_0$$

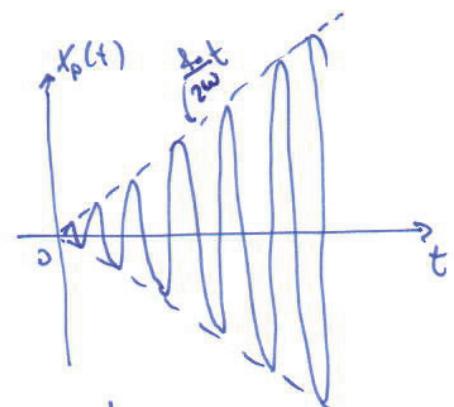
$$-2\omega^2 A - \omega^4 t B + \omega^4 t B = 0$$

$$(\omega^2 - \omega^2) t A + 2\omega^2 B = f_0$$

$$\begin{aligned} -2\omega^2 A + (\omega^2 - \omega^2) t B &= 0 \rightarrow B = \frac{2\omega}{\omega^2 - \omega^2} \frac{1}{t} A \\ (\omega^2 - \omega^2) t A + \frac{(2\omega)^2}{\omega^2 - \omega^2} \frac{1}{t} A &= f_0 \quad | \cdot (\omega^2 - \omega^2) t \\ A &= \frac{\omega^2 - \omega^2}{(\omega^2 - \omega^2) t^2 + (2\omega)^2} t f_0 \end{aligned}$$

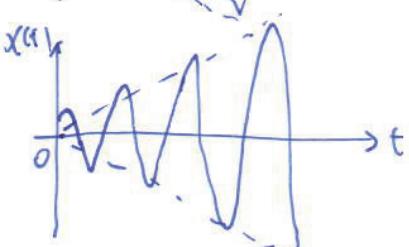
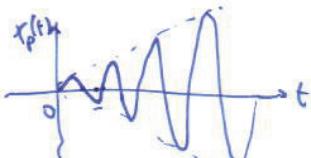
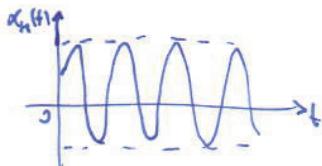
for $\underline{\Omega=\omega}$

$$\begin{aligned} A &= 0 \\ B &= \frac{f_0}{2\omega} \end{aligned} \Rightarrow \boxed{x_p(t) = \frac{f_0}{2\omega} t \sin \omega t}$$



REZONANCIJA! Pjerenje linearnog eksplodiriva kolba t meste.

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{f_0}{2\omega} t \sin \omega t$$



33

DINAMIČKI SUSTAVI VIŠEG REDA

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, t) \rightarrow \begin{array}{l} \text{sustav nezavisivi} \\ \text{dij. jednadžbi} \\ \text{za } x_1, x_2, \dots, x_n \end{array}$$

$$\frac{dx_1}{dt} = F_1(x_1, \dots, x_n, t)$$

⋮

$$\frac{dx_n}{dt} = F_n(x_1, \dots, x_n, t)$$

nepomenu:

Dosta d. jednadžbi višeg reda n (i sve linearne!) daču se supstitucijama
vredni u sustav od n varijabli d. jednadžbi 1. reda!

pr. 1: linearna d. j. n-tog reda

$$\begin{aligned} & x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = 0 \\ \Downarrow \quad & (x \equiv x_1) \\ & x'_1 \equiv x_2 \quad \leftarrow \text{to je } x' \\ & x'_2 \equiv x_3 \quad \leftarrow \text{to je } x'' \\ & \vdots \\ & x'_{n-1} \equiv x_n \quad \leftarrow \text{to je } x^{(n-1)} \\ & x'_n = -a_1 x_n - a_2 x_{n-1} - \dots - a_{n-1} x_2 - a_n x_1 \quad \leftarrow \text{to je } x^{(n)} \end{aligned}$$

pr. 2: $z''' + a z'' + b z' + c z = 0$ uvedimo 3 varijable x_1, x_2, x_3 ;

$$\begin{aligned} z & \equiv x_1 \\ z' & \equiv x_2 \Rightarrow x'_1 = x_2 \\ z'' & \equiv x_3 \Rightarrow x'_2 = x_3 \\ z''' & = -az'' - bz' - cz \Rightarrow x'_3 = -ax_3 - bx_2 - cx_1 \end{aligned}$$

$$\boxed{\begin{array}{l} x'_1 = x_2 \\ x'_2 = x_3 \\ x'_3 = -ax_3 - bx_2 - cx_1 \end{array}} \rightarrow \begin{array}{l} \text{sustav nezavisivi} \\ \text{d. j. za } x_1, x_2, x_3 \end{array}$$

pr. 3: $\ddot{x}^2 + \ddot{x} + x \cdot \dot{x} = 3t + 4$

$x \equiv x_1$

$\dot{x} \equiv x_2 \Rightarrow \dot{x}_1 = x_2$

$\ddot{x} = -\dot{x}^2 - x \cdot \dot{x} + 3t + 4 \Rightarrow \ddot{x}_2 = -x_2^2 - x_1 x_2 + 3t + 4$

$$\boxed{\begin{array}{l} \dot{x}_1 = x_2 \\ \ddot{x}_2 = -x_2^2 - x_1 x_2 + 3t + 4 \end{array}}$$

pr. 4 : obrat

$$(1) \frac{dx}{dt} = x + y + t \quad \rightarrow \quad y = \frac{dx}{dt} - x - t \quad (*)$$

$$(2) \frac{dy}{dt} = -4x - 3y + 2t$$

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 4x - 3\left(\frac{dx}{dt}\right) + 2t + 1$$

substituujeme iž (2)

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - 4x - 3\frac{dx}{dt} + 3x + 3t + 2t + 1$$

substituujeme iž (*)

$$\underline{\ddot{x} - 2\dot{x} + x = 5t + 1}$$

to je reed rješava has dříme
lin. d. j. 2. reda

3.1

FIXNE TOČKE SUSTAVI DIF. JEDNAČIĆI

$$\vec{F}(\vec{x}_c, t) = 0 \Rightarrow$$

$$F_1(x_1, x_2, \dots, x_n, t) = 0$$

$$F_2(x_1, x_2, \dots, x_n, t) = 0$$

:

$$F_n(x_1, x_2, \dots, x_n, t) = 0$$

čvorovi algorskih
jednačiće

npr:

$$\frac{dx}{dt} = x - axy$$

$$x(1-ay) = 0$$

$$\frac{dy}{dt} = -y + bx$$

⇒

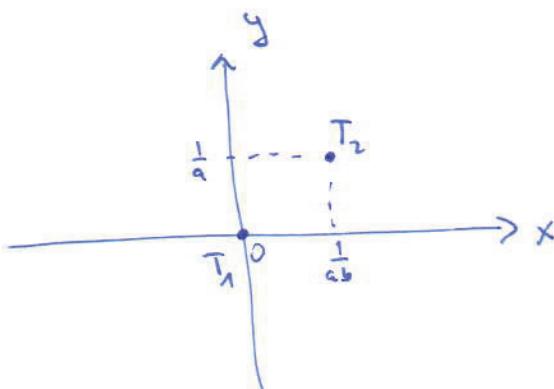
$$-y + bx = 0$$

$$y = bx$$

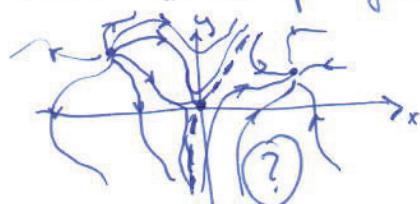
$$x(1-abx) = 0$$

$$T_1: x_{c1} = 0; \quad y_{c1} = 0 : T_1(0,0)$$

$$T_2: x_{c2} = \frac{1}{ab}; \quad y_{c2} = \frac{1}{a} : T_2\left(\frac{1}{ab}, \frac{1}{a}\right)$$



Ali ne znamo još karakter
stoga farazog toka des ujdu. Za sustave 1. reda lako
je lako odrediti karakter F.T. na temelju ponašanja $f(x)$.
Za sustave višeg reda treba pribjeti drugim metodama.



Kvalitativne metode:

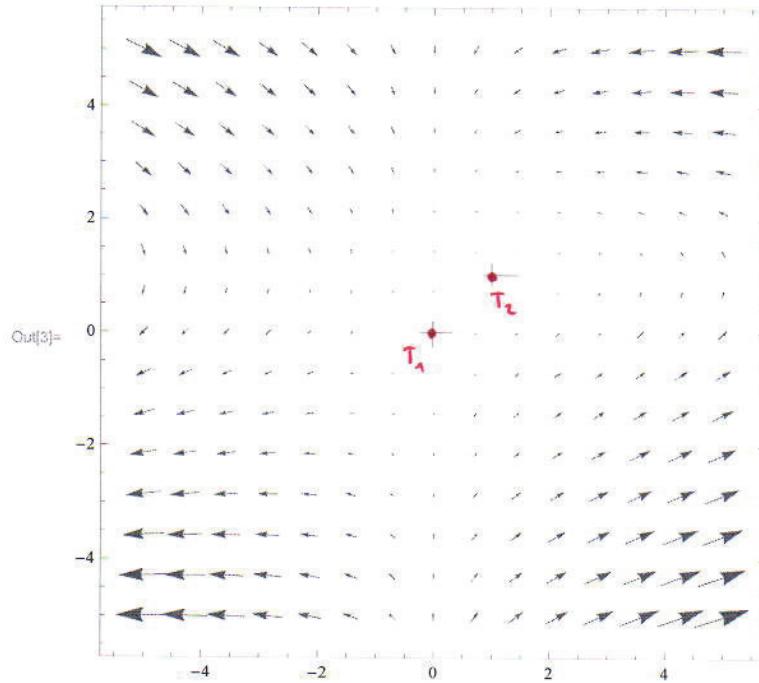
- Tangentno polje
- Dijagram foznog toka

In[1]:= $fx[x_, y_] := x * (1 - y)$
 $fy[x_, y_] := x - y$

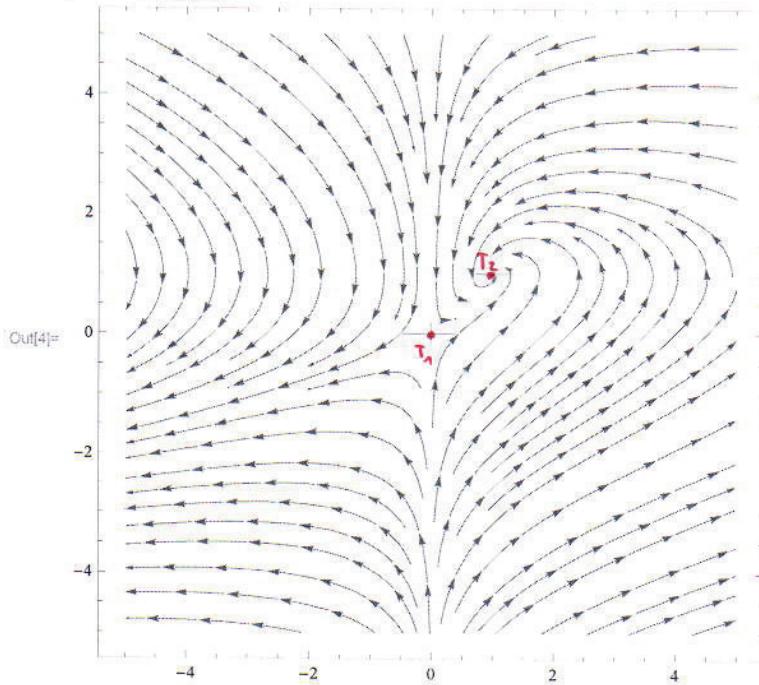
F.T. $T_1(0,0)$

$T_2(1,1)$

In[3]:= VectorPlot[{fx[x, y], fy[x, y]}, {x, -5, 5}, {y, -5, 5}]



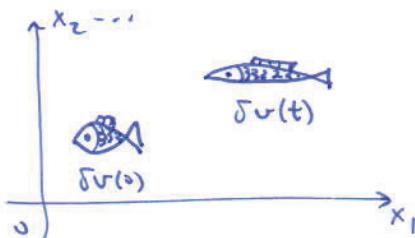
In[4]:= StreamPlot[{fx[x, y], fy[x, y]}, {x, -5, 5}, {y, -5, 5}]



Iz danih dijagrama možemo očijeniti da je T_2 stabilna F.T. tipa parov foznog toka. Tačka T_1 s jedne strane priblazi fozni tok, a s druge ga odvija, tako da radi se o foznoj točki tipa sedla.

No, we su u ovu kvalitativne očijene. Treba vratiti analitičke metode precizne karakterizacije foznih tačaka.

Pitanje: Kolis se mijenja iznos i oblik faruov volumene $\delta v(t)$ u vremenu? (Zahtjev je da Taj dio f.p. ne obuhvaća fixne točke.)



Promjena volumene moramo odrediti preko transformacije koordinata $\rightarrow t=0$ na $t \neq 0$.

$$\delta v(t) = \int \cdot \delta v(0) ; \quad J \equiv \left| \frac{\partial \vec{x}(t)}{\partial \vec{x}(0)} \right|$$

Jacobian
(Jacobiana determinanta)

$$J = \begin{vmatrix} \frac{\partial x_1(t)}{\partial x_1(0)} & \frac{\partial x_2(t)}{\partial x_1(0)} & \dots & \frac{\partial x_n(t)}{\partial x_1(0)} \\ \frac{\partial x_1(t)}{\partial x_2(0)} & \frac{\partial x_2(t)}{\partial x_2(0)} & \dots & \frac{\partial x_n(t)}{\partial x_2(0)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1(t)}{\partial x_n(0)} & \frac{\partial x_2(t)}{\partial x_n(0)} & \dots & \frac{\partial x_n(t)}{\partial x_n(0)} \end{vmatrix}$$

Pustimo $t \rightarrow 0$ i radijus Taylora ravnji da bismo našli infinitesimalnu transformaciju koordinata $\rightarrow t \rightarrow t+dt$.

$$\vec{x}(t) = \vec{x}(0) + \underbrace{\left(\frac{d\vec{x}(t)}{dt} \Big|_{t=0} \cdot t \right)}_{\text{det}} + \sigma(t^2)$$

$\left| \frac{\partial \vec{x}(t)}{\partial \vec{x}(0)} \right| \det$

$\rightarrow \left| \frac{\partial \vec{x}(t)}{\partial \vec{x}(0)} \right| = \sqrt{\left(1 + \frac{\partial \vec{F}(\vec{x})}{\partial \vec{x}} \Big|_{\vec{x}=\vec{x}(0)} \cdot t \right)^2 + \sigma(t^2)}$

$\det = \begin{vmatrix} 1 + \frac{\partial F_1}{\partial x_1} \cdot t & \frac{\partial F_2}{\partial x_1} \cdot t & \dots & & & + \sigma(t^2) \\ \frac{\partial F_1}{\partial x_2} \cdot t & 1 + \frac{\partial F_2}{\partial x_2} \cdot t & \dots & & & \\ \vdots & & & & & \end{vmatrix}$

\rightarrow Linearni u t
bit će samo
doprinosi s
diagonalne, sve
ostalo potpada
pod $\sigma(t^2)$!

$$\left| \frac{\partial \vec{x}(t)}{\partial \vec{x}(0)} \right| = 1 + \text{Tr} \left[\underbrace{\frac{\partial \vec{F}}{\partial \vec{x}}}_{\vec{x}=\vec{x}(0)} \right] \cdot t + \cancel{o(t^2)}$$

zavojimo je $t \rightarrow 0$

$$= \text{div } \vec{F} \Big|_{\vec{x}=\vec{x}(0)} - \text{divergencija vektorskog polja } \vec{F}(\vec{x}) ; \text{ dijagonalni elementi matrice } \frac{\partial \vec{F}}{\partial \vec{x}}$$

$$(\text{div } \vec{f} = \vec{V} \cdot \vec{f} \equiv \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} + \dots)$$

$$\gamma \approx 1 + \text{div } \vec{F}(\vec{x}) \Big|_{\vec{x}=\vec{x}(0)} \cdot t$$

$$\frac{\partial \delta V^{(t)}}{\partial t} = \frac{\partial}{\partial t} \left((1 + \text{div } \vec{F}(\vec{x}) \Big|_{\vec{x}=\vec{x}(0)} \cdot t) \cdot \delta V(0) \right) = \text{div } \vec{F}(\vec{x}) \Big|_{\vec{x}=\vec{x}(0)} \cdot \delta V(0)$$

Relativne promjene formog volumena:

$$\frac{1}{\delta V(0)} \frac{\partial \delta V}{\partial t} = \text{div } \vec{F} \Big|_{\vec{x}=\vec{x}(t)} = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} \Big|_{\vec{x}(0)} = \sum_{i=1}^n \frac{\partial \dot{x}_i}{\partial x_i} \equiv L$$

LIEVA DERIVACIJA

Sustav formog prostora:

- | | |
|--|-------------------------------------|
| hamiltonski
(konzervativni)
sustav | $L < 0$; δV se kontralira |
| dispativni
sustav | $L = 0$; δV ostaje nečvren |
| | $L > 0$; δV se expandira |

(*) $L(\vec{x}(t))$ - avisi

točki u formom prostora $\vec{x}(t)$, dešće svihe točke F.P. može imati drugačiji iznos lijeve derivacije.
Postoje sustavi za koji je $L = \text{const.} + \vec{x}(0)$.

Liouvilloov teorem:

Lijsa derivacija za "hamiltonski" (konzervativni) sustav iznosi $L=0$ u svim točkama F.P.

Hamiltonski sustav:

$$H(\vec{q}, \vec{p})$$

Hamiltonove jednačine:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$L = \sum_{i=1}^n \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right)$$

$$= \sum_{i=1}^n \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = \phi \checkmark$$

upr.

$$\frac{dx}{dt} = x - axy$$

$$F_x(x,y) = x - axy$$

$$a,b > 0$$

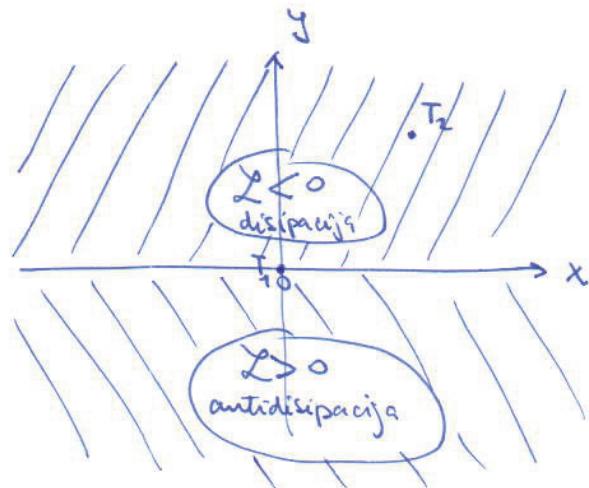
$$\frac{dy}{dt} = -y + bx$$

$$F_y(x,y) = -y + bx$$

$$L = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$$

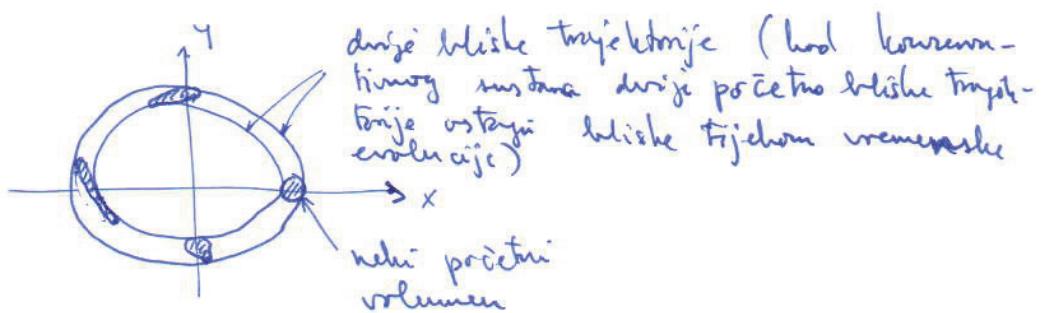
$$= x - ay - x + 0$$

$$= -ay$$



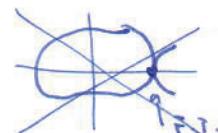
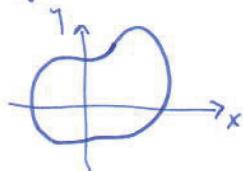
Da, sačuvanje volumene ne implicira i racionalni oblik !

Ovih formi volumene su se tijekom evolucije u vremenu mijenja i za konzervativan sustav.

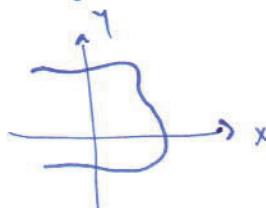


(*) Autonomni sustav :

- zatvorene trajektorije predstavljaju periodične gibanje
(prep. je da nema zatvorene točke na trajektoriji)



- otvorene trajektorije predstavljaju nepotpisne gibanje



Pitanje prekucja biloog volumene elementars je pitanje stabilnosti trajektorije.

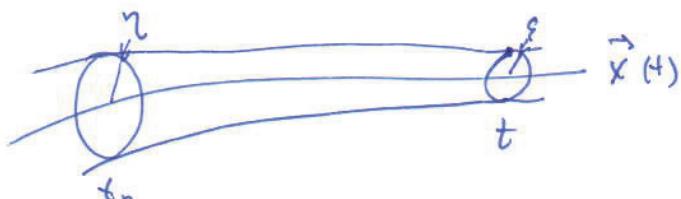
#1 Lyapunov stabilnost

Trajektorija je stabilna po Lyapunovu ako \forall trajektorija koja počeli u točkom koja je bliza u t_0 , tijekom vremena $t > t_0$ joj postaju sve blize.

$$\vec{x}(t); \forall \varepsilon > 0, t_0; \exists \eta(t_0, \varepsilon) \text{ tako da}$$

$$|\vec{y}(t) - \vec{x}(t)| < \eta \Rightarrow |\vec{y}(t) - \vec{x}(t)| < \varepsilon, \text{ za } t > t_0$$

$$\varepsilon < \eta$$



#2 Asimptotska stabilnost (velika jaka stabilitet)

$$\lim_{t \rightarrow \infty} |\vec{y}(t) - \vec{x}(t)| = 0$$

(*) karakteristika samo dissipativnih sustava!

(*) Takođe trajektorija upije određeni dio biloog prostora. Fixne točke u logi se on upije zove se attractor, a dio biloog prostora koji se tijekom vremena $t \rightarrow \infty$ upije zove se bazu atrakcije.

#3 Orbitalna stabilitet (velika od Lyapunovove)

Kriterij je sličan Lyapunovovom, trajektorije iz γ -okolice se prate u biloog prostoru, ali ne moraju u vremenu.



(*) Strukturalno stabilan din. sistem: F.P. sačuva strukturu (zatvorene trajektorije ostaju zatvorene, otvorene ostaju otvorene, bliske trajektorije ostaju bliske, aks uključujuvi neke perturbacije, trajektorije se samo malo pomaknu).

(*) Upr. matematički ujednačenje je dinamički sustav karakteriziran orbitalnom stabilitetu

$$\dot{\vec{x}}(t) = \vec{F}(\vec{x}, t) \rightarrow \dot{x}_i = F_i(\vec{x}, t) \quad ; \quad i = 1, 2, \dots, n$$

- Metoda:
- 1.) načinom Fixne točke
 - 2.) preko linearizacije sustava u okolini F.T.

$$F_i(\vec{x}, t) = 0 \rightarrow x_{ci}(t) \leftarrow \text{Fixna točka } \vec{x}_c^H = \begin{pmatrix} x_{c1} \\ x_{c2} \\ \vdots \\ x_{cn} \end{pmatrix}$$

↑
način sre koordinate F.T. (npr. $i=1, \dots, n$)
i radius linearizacije u okolini \vec{x}_c

Promatrano trajektoriju blisku fixnoj točki:

$$x_i(t) = x_{ci}(t) + \xi_i(t), \quad t \gg \xi_i \ll$$

└ mali posek od fix. točke; nova koordinate za kog je F.T. ishodiste

Pitamo se koliko će jednostavno "podnesiti" ξ_i za $t > t_0$, da bi
će bliske trajektorija ostati bliske. Uvjetom je u rešenju:

$$\dot{x}_{ci} + \dot{\xi}_i = F_i(\vec{x}_c + \vec{\xi}, t)$$

$\vec{x} = \vec{x}_c$ (po def. F.T.)

$$\dot{x}_{ci} + \dot{\xi}_i \approx F_i(\vec{x}_c) + \sum_k \frac{\partial F_i}{\partial \xi_k} \cdot \xi_k$$

\vec{x}_c

└ jer je $\dot{x}_i = F_i(\vec{x}, t)$

$$\dot{\xi}_i = \sum_{k=1}^n \frac{\partial F_i}{\partial \xi_k} \cdot \xi_k \Rightarrow \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial \xi_1} & \frac{\partial F_1}{\partial \xi_2} & \cdots & \frac{\partial F_1}{\partial \xi_n} \\ \frac{\partial F_2}{\partial \xi_1} & \frac{\partial F_2}{\partial \xi_2} & \cdots & \frac{\partial F_2}{\partial \xi_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial \xi_1} & \frac{\partial F_n}{\partial \xi_2} & \cdots & \frac{\partial F_n}{\partial \xi_n} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$$

→ Taylorov razvoj oks
F.T. $\vec{x} = \vec{x}_c$

Općenito, ove funkcije ovise o vremenu, dokle svoju stabilnost i topologiju sustava mijenjaju se u vremenu i generiraju predstavljanje zadatak problem rješenja u nekim slučajevima (npr. linearizacija periodičnih rješenja radi na koštaju periodičnih ovise o vremenu - Hillov problem itd.)

Uzimimo da se radi o autonomnom sustavu tj. da su koeficijenti u razvoju konstantni

$$\boxed{A_{ik} = \left. \frac{\partial F_i(\vec{x})}{\partial x_k} \right|_{\vec{x}_c} = \text{const}} \quad , \quad \text{za } \frac{\partial \vec{x}}{\partial t} = \vec{F}(\vec{x})$$

U tom slučaju rješenja linearnog sustava s konstantnim koeficijentima su:

$$\boxed{\dot{x}_i(t) = x_i(0) e^{\lambda t}}$$

uvestimo ih u jednadžbu:

$$(\dot{x}_i = \lambda x_i(0) e^{\lambda t} = \lambda x_i)$$

$$x_i : \lambda x_i = \sum_k A_{ik} x_k \rightarrow$$

$$\lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

A - Jacobian!

(*) $\det A \neq 0$

(Ako je $\det A = 0$ tj. A je singularna, ne možemo konstruktivno Jacoliana u ovim koordinatama za analizu stalnosti.)

homogeni sustav nije det.
mora isčarati:

$$\begin{bmatrix} A_{11}-\lambda & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22}-\lambda & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn}-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0$$

$\underbrace{A_\lambda}_{A_\lambda}$

$$\boxed{\det A_\lambda = 0} !$$

Problem se sudi na dijagonalizaciju matrice A (travnuje vlastitih vrijednosti i vlastitih vektora).

Spektar svojstvenih vrijednosti

$$\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$$

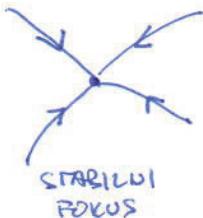
↑ općenito, sv. vrijednosti
 λ_i su kompleksni brojevi

Ovo sve se događa ako sedne sime točke \vec{x}_c .

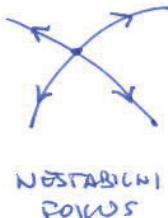
Index i u λ_i : označava $i = 1, 2, \dots, n$ raspoređenih ujedno sa pripadajućim sv. vrijednostima λ_i , gdje je n dimenzije F.P., iz kojih se formi tako giba da \vec{x}_c

Ovisno o dobivenom spektru $\lambda_i, i=1 \dots n$ varibujem u videlicu
tipove stabilnosti: za sve $i=1 \dots n$ mora biti $\operatorname{Re}(\lambda_i) < 0$

- 1.) ako su $\operatorname{Re}(\lambda_i) < 0, \forall i \Rightarrow$ F.T. \vec{x}_0 je ASIMPTOTSKI STABILAN
(stabilni FOKUS)
- 2.) ako su $\operatorname{Re}(\lambda_i) > 0, \forall i \Rightarrow$ F.T. \vec{x}_0 je NESTABILNA (nest. Fokus)
- 3.) ako su $\operatorname{Re}(\lambda_i) = 0, \forall i \Rightarrow$ F.T. \vec{x}_0 je tzv. CENTAR (ELIPTICNA F.T.)
- 4.) ako su $\operatorname{Re}(\lambda_i) < 0$ za neke i , a $\operatorname{Re}(\lambda_j) > 0$ za ostale indekse \Rightarrow
F.T. je SEDOLO (HIPERBOLNA F.T.)



STABILNI FOKUS



NESTABILNI FOKUS



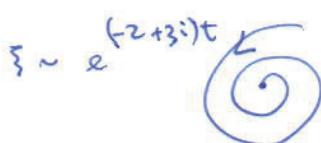
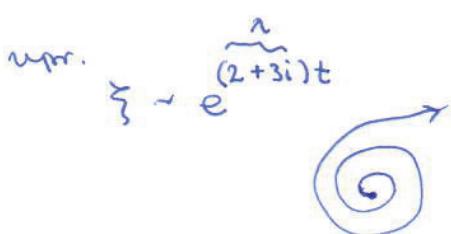
ELIPTICNA F.T.



SEDOLO (HIPERBOLNA F.T.)

dijagrami:

$$\xi_i \sim e^{\lambda_i t} = e^{\operatorname{Re}(\lambda_i)t + i \operatorname{Im}(\lambda_i)t} = \underbrace{e^{\operatorname{Re}(\lambda_i)t}}_{\text{određuje eksponentijski karakter rješenja u vremenu (trenje za } < 0 \text{ ili ekspozicija za } > 0\text{)}} \cdot \underbrace{e^{i \operatorname{Im}(\lambda_i)t}}_{\text{vrednuje osi katorni konflikti rješenja}}$$



Napistimo $\vec{\xi}_i(t), i=1 \dots n$ kao vektor $\vec{\xi}(t)$ kada je s poč. vektor $\vec{\xi}(0)$ uzimaju prema linearog operatora $L(t)$:

$$\vec{\xi}(t) = L(t) \vec{\xi}(0) ; L(t) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & 0 \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

Proustimo:

$$\operatorname{Tr}[L^*(t) L(t)] = \sum_{i=1}^n e^{(\lambda_i + \lambda_i^*)t} = \sum_{i=1}^n e^{2\operatorname{Re}(\lambda_i)t}$$

- U ovaj sumi neki $\operatorname{Re}(\lambda_j)$ je najveći $\Rightarrow \operatorname{Re}(\lambda_j) = \max_{i=1,\dots,n} \{\operatorname{Re}(\lambda_i)\}$
- Putinovo da $t \rightarrow \infty$: Kada je $t \rightarrow \infty$, tada je doprinos $e^{2\operatorname{Re}(\lambda_j)t}$ puno veći od doprinosa svih ostalih $e^{2\operatorname{Re}(\lambda_i)t}, \forall i \neq j$ i ne mijenja nijeli rezultanti!

def:

$$\bar{\lambda} = \lim_{t \rightarrow \infty} \frac{1}{2t} \ln (\operatorname{Tr}[L^+ L]) = \max_{i=1,\dots,n} \{\operatorname{Re}(\lambda_i)\}$$

↳ Lyapunovov exponent ("Ly-exponent")

Pogledajmo sad tipove stabilitetu osim o Ly-exponentu:

- 1) $\bar{\lambda} < 0$ - arimptotska stabilitet
- 2) $\bar{\lambda} > 0$ - nestabilnost
- 3) $\bar{\lambda} = 0$ - periodičke oscilacije

primjer:

$$\begin{aligned} \frac{dx}{dt} &= x - axy \\ \frac{dy}{dt} &= -y + bx \end{aligned} \quad \rightarrow \quad \begin{aligned} F_x &= x - axy \\ F_y &= -y + bx \end{aligned} \quad \rightarrow \quad A = \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1-ay & -ax \\ b & -1 \end{bmatrix}$$

F.T. $T_1(0,0)$:

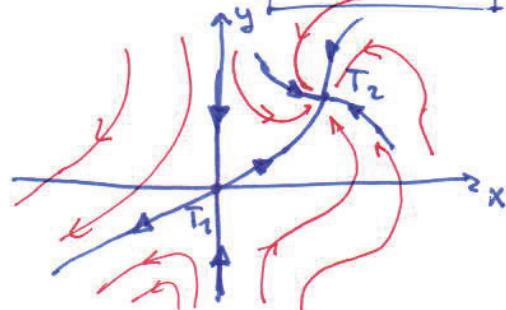
$$A|_{T_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ; \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = -(1+\lambda)(1-\lambda) = \lambda^2 - 1 = 0 ; \boxed{\begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -1 \end{array}}$$

$$\begin{array}{ll} \operatorname{Re}(\lambda_1) > 0 \\ \operatorname{Re}(\lambda_2) < 0 \end{array} \Rightarrow \text{sedlo!} \quad ; \quad \bar{\lambda} = 1 \text{ (nestabilnost)}$$

F.T. $T_2\left(\frac{1}{ab}, \frac{1}{a}\right)$:

$$A|_{T_2} = \begin{bmatrix} 0 & -\frac{1}{b} \\ b & -1 \end{bmatrix} ; \begin{vmatrix} -\lambda & -\frac{1}{b} \\ b & -1-\lambda \end{vmatrix} = \lambda(\lambda+1)+1 = \lambda^2 + \lambda + 1 = 0 ; \boxed{\lambda_1 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i}$$

$$\begin{array}{ll} \operatorname{Re}(\lambda_1) < 0 \\ \operatorname{Re}(\lambda_2) < 0 \end{array} \Rightarrow \text{stabilni lokus!} \quad ; \quad \bar{\lambda} = -\frac{1}{2} \text{ (stabilnost)}$$



4.1

Klasifikacija fizičkih trouha

$$\dot{x}_1 = F_1(x_1, x_2)$$

$$\dot{x}_2 = F_2(x_1, x_2)$$

def:

$$\frac{dx_1}{dx_2} = \frac{F_1(x_1, x_2)}{F_2(x_1, x_2)}$$

→ definisano je svakog osim u situaciji gdje se javlja $\frac{0}{0}$.

(tako i $\frac{dx_1}{0} \Rightarrow$ je definisano)

To su singularne točke, a trouha u kojima su vektori nulltrični
 $F_1(x_1, x_2) = 0 \quad \& \quad F_2(x_1, x_2) = 0.$

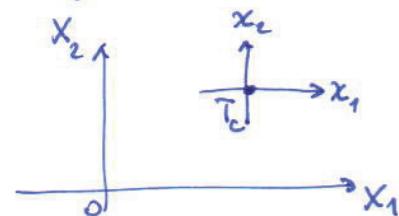
Uzimajući $F.T. (x_{1c}, x_{2c})$ i ispitujemo

je li okolina (npr. da se radi o izoliranoj F.T. a ne veliči kontinuirajućim ringom strukturi).

$T_c(x_{1c}, x_{2c}) : F.T.$

$$x_1 = x_{1c} + x_1$$

$$x_2 = x_{2c} + x_2$$



linearizacija: $\dot{x}_1 = a_{11} x_1 + a_{12} x_2$

$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2$$

$$a_{11} = \left. \frac{\partial F_1}{\partial x_1} \right|_{(x_{1c}, x_{2c})}$$

$$a_{12} = \left. \frac{\partial F_1}{\partial x_2} \right|_{(x_{1c}, x_{2c})}$$

$$\underline{A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

$$a_{21} = \left. \frac{\partial F_2}{\partial x_1} \right|_{(x_{1c}, x_{2c})}$$

$$a_{22} = \left. \frac{\partial F_2}{\partial x_2} \right|_{(x_{1c}, x_{2c})}$$

rešenje lin. sustava:

$$x_1(t) = x_{01} e^{\lambda_1 t}$$

$$x_2(t) = x_{02} e^{\lambda_2 t}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

karakteristični polinom

$$\Rightarrow \lambda^2 - T\lambda + \Delta = 0$$

$$T \equiv T_r[A] = a_{11} + a_{22}$$

$$\Delta \equiv \det[A] = a_{11}a_{22} - a_{12}a_{21}$$

$$\lambda_{1,2} = \frac{1}{2} (T \pm \sqrt{T^2 - 4\Delta})$$

\Downarrow diskriminanta

rješenja:
 $x_1 = x_{10} e^{\lambda_1 t}$
 $x_2 = x_{20} e^{\lambda_2 t}$

Klasificacija rješenja:

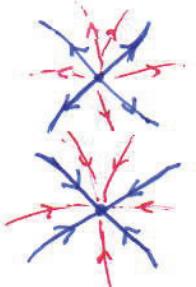
(1) $\lambda_1, \lambda_2 \in \mathbb{R}$ - reale svojstvene vrijednosti

ugjet: $\Leftrightarrow \boxed{\Delta = T^2 - 4\Delta \geq 0}$

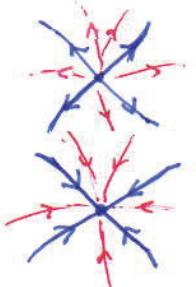
(1a) λ_1, λ_2 su istog predznaka

ugjet: $\lambda_{1,2} = \frac{1}{2} (T \pm \sqrt{T^2 - 4\Delta}) \rightarrow |T| > |\sqrt{T^2 - 4\Delta}| \Leftrightarrow \Delta > 0$
 (& $T^2 \geq 4\Delta$)

(1a') $\lambda_1 > 0 \& \lambda_2 > 0 \Leftrightarrow T > 0$: NESTABILNI ČVOR
 (node)



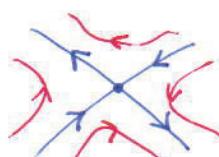
(1a'') $\lambda_1 < 0 \& \lambda_2 < 0 \Leftrightarrow T < 0$: STABILNI ČVOR



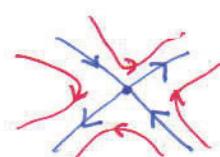
(1b) λ_1, λ_2 su različitog predznaka

$|T| < |\sqrt{T^2 - 4\Delta}| \Leftrightarrow \Delta < 0$: SEDLENA (HIPERBOLNA) F.T.

(1b') $\lambda_1 > 0, \lambda_2 < 0$



(1b'') $\lambda_1 < 0, \lambda_2 > 0$



(2) $\lambda_1, \lambda_2 \in \mathbb{C} \Rightarrow$ uvijek je $\lambda_1 = \lambda_2^*$. tj. λ_1, λ_2 su kompleksni konjugirani \rightarrow drugi mogućnosta nisu realne jednolike - karakteristiku polinoma

uvjet:

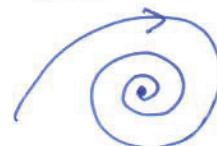
$$\Re < 0$$

automatski slijedi da je

$$\Re(\lambda_1) = \Re(\lambda_2) = T$$

$\Re(\lambda_1) : \Re(\lambda_2)$ su istog predznaka

$$(2a) \quad \Re(\lambda_1) = \Re(\lambda_2) < 0 \quad \Leftrightarrow \quad T < 0 : \text{STABILNI FOKUS}$$



$$(2b) \quad \Re(\lambda_1) = \Re(\lambda_2) > 0 \quad \Leftrightarrow \quad T > 0 : \text{NESTABILNI FOKUS}$$



$$(2c) \quad \Re(\lambda_1) = \Re(\lambda_2) = 0 \quad \Leftrightarrow \quad T = 0 : \text{CENTAR (ELIPTIČNA F.T.)}$$

(λ_1, λ_2 su imaginarni)



$$(3) \quad \lambda_1 = \lambda_2 \quad (\text{automatski su realni})$$

$$\text{uvjet: } \Im = 0 \quad \Leftrightarrow \quad T^2 - 4\Delta = 0 \rightarrow (a_{11} - a_{22})^2 + 4a_{12}a_{21} = 0$$

$$\text{izaberimo mogućnost } a_{11} = a_{22} \equiv a \neq 0$$

↑

Ovo se može postići pomerajući vremenske stope tj. rotacijsku situaciju motora svaki na ovu.

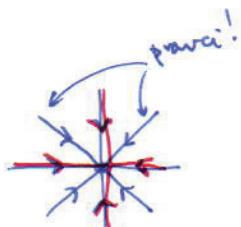
$$(3a) \quad \underline{a_{12} = a_{21} = 0 \quad \text{z} \quad a \neq 0}$$

tada je: $\dot{x}_1 = ax_1$ $\dot{x}_2 = ax_2$ \rightarrow

$$\begin{aligned} x_1 &= x_{10} e^{at} \\ x_2 &= x_{20} e^{at} \end{aligned}$$

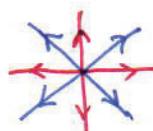
$$\xrightarrow{\quad} x_1 = \frac{x_{10}}{x_{20}} x_2$$

$$\underline{a < 0}$$



STABILNA ZVIJEZDA

$$\underline{a > 0}$$

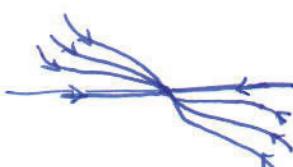


NESTABILNA ZVIJEZDA

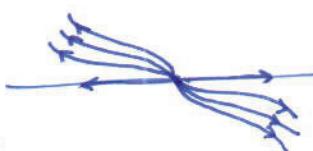
$$(3b) \quad \underline{a_{12} \neq 0, \quad a_{21} = 0 \quad \text{z} \quad a \neq 0}$$

$$\begin{aligned} \dot{x}_1 &= ax_1 + a_{12}x_2 && \xrightarrow{\text{inhomogeni problem}} x_1 = x_{10} e^{at} + a_{12}x_{20} \cdot t e^{at} \\ \dot{x}_2 &= ax_2 && \xrightarrow{\text{homogeni problem}} x_2 = x_{20} e^{at} \end{aligned}$$

$$\underline{a < 0}$$



$$\underline{a > 0}$$



čvorovi s jedinstvenom tangencijom

usporenje:

- Stabilan čvor je asymptotski stabilan.
- Nestabilan čvor i sedlina točka su nestabilni.
- Stabilni fokus je asymptotski stabilan.
- Nestabilni fokus je nestabilan.
- Centar nije asymptotski stabilan, ali je stabilan po Lyapunovu.
- Traj T je u biti Lieva derivacija u okolini F.T.: $T = L_{F.T.}$
- ↳ Konservativni sustavi mogu imati samo LIJELIĆE i SEDLICE fixne točke (bez Liouvillove funkcije).

- Determinante Δ je u biti Hessian sustava ako ga prouznamo potencijalni dinamidni sustav

$$\dot{x}_1 = F_1(x_1, x_2) = -\frac{\partial U(x_1, x_2)}{\partial x_1}$$

$$\dot{x}_2 = F_2(x_1, x_2) = -\frac{\partial U(x_1, x_2)}{\partial x_2}$$

$$\vec{\dot{x}} = \vec{F}(\vec{x}) = -\text{grad } U(\vec{x})$$

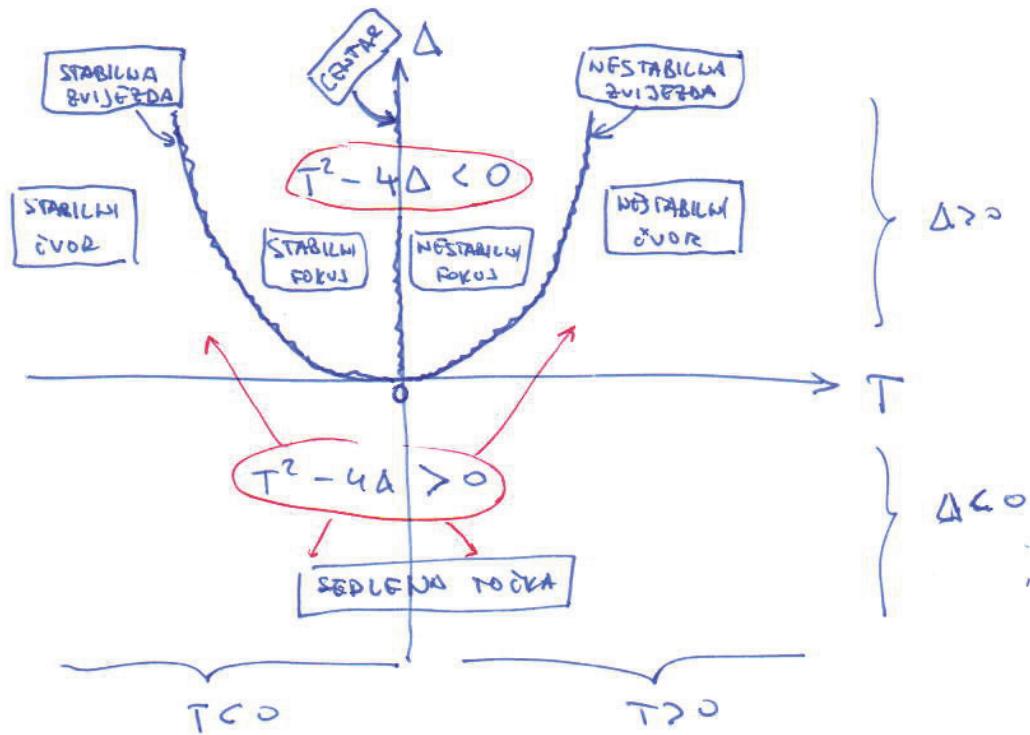
Hessian (Hesijan determinanta) : $H = \left| \frac{\partial^2 U}{\partial x_i \partial x_j} \right|, i, j \in \{1, 2, \dots, n\}$

$$\alpha_{ij} = \frac{\partial F_i}{\partial x_j} = -\frac{\partial^2 U}{\partial x_i \partial x_j}; \quad \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ je Hessian za potku } U(x_1, x_2)$$

$$\Delta = a_{11}a_{22} - a_{12}^2 > 0 \quad \begin{cases} a_{11}, a_{22} < 0 \Rightarrow \min. \\ a_{11}, a_{22} > 0 \Rightarrow \max. \end{cases} \quad (a_{11} \text{ i } a_{22} \text{ su istih predznaka})$$

$$\Delta = -++ < 0; \quad a_{11} \text{ i } a_{22} \text{ su suprotvrsi predznaci} \Rightarrow \underline{\text{sedlo}}$$

Obréđinimo ove situacije na dijagramu:



(*) Konzervativni sustavi su samo ne oordinati ($T=0$) i

to $\Delta > 0 \Rightarrow$ CENTAR ; $\Delta < 0 \Rightarrow$ SEDLO.

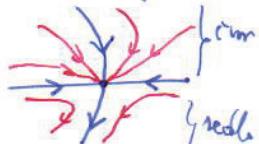
(**) Prijedav predio $T=0$ liniji velykoga promjera premašuje lijeve derivačije.

(***) Prijedav kroz $\Delta=0$ (čvor \Leftrightarrow sedlo) može biti u oba smjera:

naš primjer:

$$\frac{dx}{dt} = x - axy$$

$$\frac{dy}{dt} = -y + bxy$$



F.T. $T_1(0,0)$

$$A = \begin{pmatrix} 1 & 0 \\ b & -1 \end{pmatrix}$$

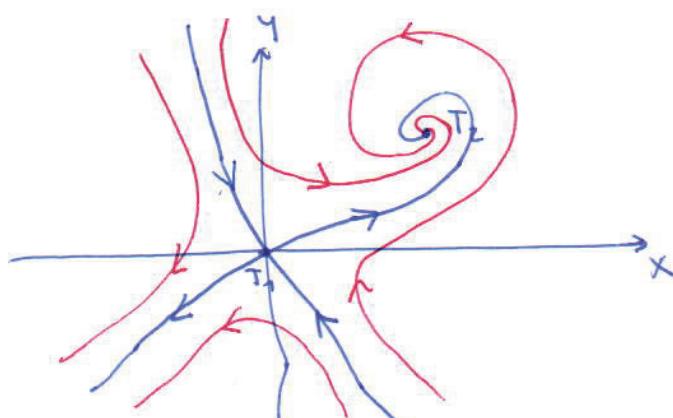
$$\begin{aligned} T &= 0 \\ \Delta &= -1 < 0 \\ D &= 4 > 0 \end{aligned}$$

sedlo!

F.T. $T_2\left(\frac{1}{ab}, \frac{1}{a}\right)$

$$A = \begin{pmatrix} 0 & -\frac{1}{b} \\ b & -1 \end{pmatrix} : \quad \begin{aligned} T &= -1 < 0 \\ \Delta &= 1 > 0 \\ D &= -3 < 0 \end{aligned}$$

stabilni fokus!



Linearizacijom sustava oko fixne točke dobili smo matricu koeficijenata A i rješenja u bliskoj okolini F.T. $x_1 = e^{\lambda_1 t}, x_2 = e^{\lambda_2 t}$ odredene majstrenim vrijednostima matrice A : λ_1 i λ_2 .

Svojstveni vektori matrice A određuju one lineare kombinacije x_1 i x_2 za koje se sustav varnih jednadžbi rješava. (Oni su autonomske rješenja jer sustav je linearan, a u lin. sustavu svaka lin. kombinacija rješenja također je rješenje.)

vijedi:

$$A \cdot \vec{v}_i = \lambda_i \vec{v}_i$$

$$\vec{v}_1 = \begin{pmatrix} A \\ B \end{pmatrix}_{\text{f}}^{\text{r}}$$

$$\vec{v}_2 = \begin{pmatrix} C \\ D \end{pmatrix}_{\text{f}}^{\text{r}}$$

treba odrediti

Za nečete $\vec{v}_1 : \vec{v}_2$ opće rješenje može se napisati da je:

$$\underbrace{\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2}_{\text{u}}$$

primjer: Uzmimo novi primjer

$$A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}; \lambda_{1,2} = \pm 1$$

načini majstrene vektore

$$\lambda_1 = 1: \underbrace{[(1 \ 0) - 1 \cdot (0 \ 1)] \begin{pmatrix} A \\ B \end{pmatrix}}_{(1-1 \ 0) \ (A \ B)} = 0$$

(*) polovitina:
 $A \vec{v}_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \checkmark$

$$(1-\lambda_1 \ 0) \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \Rightarrow \begin{array}{l} \cancel{A+0B=0} \\ \cancel{A-2B=0} \end{array} \rightarrow A=2B$$

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

slabodjni parametar
 $B=1$ $\Delta \vec{v}_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \checkmark$

$$\lambda_2 = -1: \underbrace{(1-\lambda_2 \ 0) \ (C \ D)}_{(1+1 \ 0) \ (C \ D)} = 0$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = 0 \Rightarrow \begin{array}{l} 2C=0, +D \\ C=0, +D \end{array} \rightarrow D=L \Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\vec{x}(t) = c_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{u}}$$

→ opće rješenje (vidi karakter redne zbroje e^t, e^{-t})

No., pagildojimo rješenje u obliku F.T. $T_2(1,1)$:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} ; \quad \lambda_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \rightarrow \text{kompleksne sv. vrijednosti!}$$

sv. vektori:

$$\lambda_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} : \quad \begin{pmatrix} 0-\lambda_1 & -1 \\ 1 & -1-\lambda_1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{1}{2}-i\frac{\sqrt{3}}{2} & -1 \\ 1 & -\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \rightarrow \begin{array}{l} \alpha A - B = 0 \\ A - \alpha^* B = 0 \end{array} \rightarrow A = \alpha^* B = \alpha^* : \quad \begin{array}{l} \overrightarrow{v}_1 = \begin{pmatrix} \frac{1}{2}+i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \\ B=1 \end{array}$$

$$\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} : \quad \begin{pmatrix} 0-\lambda_2 & -1 \\ 1 & -1-\lambda_2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{1}{2}+i\frac{\sqrt{3}}{2} & -1 \\ 1 & -\frac{1}{2}+i\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = 0 \rightarrow \begin{array}{l} \alpha^* C - D = 0 \\ C - \alpha D = 0 \end{array} \rightarrow C = \alpha D = \alpha : \quad \begin{array}{l} \overrightarrow{v}_2 = \begin{pmatrix} \frac{1}{2}-i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} \\ D=1 \end{array}$$

$$\vec{x}(t) = C_1 e^{(-\frac{1}{2}+i\frac{\sqrt{3}}{2})t} \begin{pmatrix} \frac{1}{2}+i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix} + C_2 e^{(-\frac{1}{2}-i\frac{\sqrt{3}}{2})t} \begin{pmatrix} \frac{1}{2}-i\frac{\sqrt{3}}{2} \\ 1 \end{pmatrix}$$

Imajući dva kompleksna rješenja je možno kompleksnu koordinatu u pomoći rješenju?

* Pitajući: Kako kompleksnu rješenju operativi reale fizikalne sustave?

iskoristimo:

$$1.) e^{ax+ib} = e^a \cdot e^{ib} = e^a (\cos b + i \sin b) \quad (\text{Eulerova formula})$$

$$= \underbrace{e^a \cos b}_{\text{Re } (\vec{x})} + i \underbrace{e^a \sin b}_{\text{Im } (\vec{x})}$$

\Rightarrow Rješenje $\vec{x}(t)$ može se napisati kao: $\vec{x}(t) = \vec{x}_R(t) + i \vec{x}_I(t)$

2.) Nda je $\vec{x}(t)$ kompleksna rješenja (linearizirano) sustava

$$\frac{d\vec{x}}{dt} = A \vec{x}$$

realna matrica koeficijenata a_{ij}

$$\vec{x} = \vec{x}_R + i \vec{x}_I$$

$$\Rightarrow \frac{d\vec{x}_R}{dt} + i \frac{d\vec{x}_I}{dt} = A \vec{x}_R + i A \vec{x}_I \Rightarrow \begin{cases} \frac{d\vec{x}_R}{dt} = A \vec{x}_R \\ \frac{d\vec{x}_I}{dt} = A \vec{x}_I \end{cases}$$

Sustav može zapisati bio je 2 odvojene, nevezane jednadžbe s istim "dinamičkim potencijalom" tj. matricom koeficijenata A. Dakle, to je ista jednadžba koja daje 2 rješenja: $\vec{x}_2(t) : \vec{x}_1(t)$. Ta dva rješenja su linearno nezavisna: mogu se iskombinirati, po principu lineare superpozicije koji vrijedi u linearnom sustavu, za zapis općeg rješenja = obliku:

$$\underbrace{\vec{x}(t) = C_1 \vec{x}_2(t) + C_2 \vec{x}_1(t)}_{\text{broj } n \text{ "realne funkcije"}}, \quad t \in \mathbb{R}$$

(*) dokaz lin. nezavisnosti $\vec{x}_2 : \vec{x}_1$:

- Isti postupak provodi se za realne komponente vektora $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Uzmimo jednu komponentu: oznacimo ju x :

$$\begin{aligned} x(t) &= e^{\lambda_1 t} = e^{(\lambda_1 + i\lambda_2)t} = e^{\lambda_1 t} (\cos \lambda_2 t + i \sin \lambda_2 t) \\ &= \underbrace{e^{\lambda_1 t} \cos \lambda_2 t}_{x_2} + i \cdot \underbrace{e^{\lambda_1 t} \sin \lambda_2 t}_{x_1} \end{aligned}$$

wronskian

Rješenja x : y su lin. nezavisna ako $W = \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} \neq 0$!

$$\begin{aligned} x_2 &= e^{\lambda_1 t} \cos \lambda_2 t ; \quad \dot{x}_2 = \lambda_2 e^{\lambda_1 t} \cos \lambda_2 t - \lambda_1 e^{\lambda_1 t} \sin \lambda_2 t = \\ &= e^{\lambda_1 t} (\lambda_2 \cos \lambda_2 t - \lambda_1 \sin \lambda_2 t) \end{aligned}$$

$$x_1 = e^{\lambda_1 t} \sin \lambda_2 t ; \quad \dot{x}_1 = e^{\lambda_1 t} (\lambda_2 \sin \lambda_2 t + \lambda_1 \cos \lambda_2 t)$$

$$W = \begin{vmatrix} e^{\lambda_1 t} \cos \lambda_2 t & e^{\lambda_1 t} \sin \lambda_2 t \\ e^{\lambda_1 t} (\lambda_2 \cos \lambda_2 t - \lambda_1 \sin \lambda_2 t) & e^{\lambda_1 t} (\lambda_2 \sin \lambda_2 t + \lambda_1 \cos \lambda_2 t) \end{vmatrix}$$

$$= e^{2\lambda_1 t} (\lambda_2 \sin \lambda_2 t \cdot \cos \lambda_2 t + \lambda_1 \cos \lambda_2 t \cdot \sin \lambda_2 t - \lambda_2 \sin \lambda_2 t \cdot \cos \lambda_2 t + \lambda_1 \sin \lambda_2 t) = \lambda_2 e^{2\lambda_1 t} \neq 0 \checkmark$$

$$\begin{aligned} \Rightarrow \vec{x}(t) &= c_1 e^{\lambda_1 t} \begin{pmatrix} \cos \lambda_2 t \\ 1 \end{pmatrix} + c_2 e^{\lambda_1 t} \begin{pmatrix} \sin \lambda_2 t \\ 1 \end{pmatrix} \\ &= c_1 e^a (\cos b + i \sin b) \begin{pmatrix} \cos d \\ 1 \end{pmatrix} + c_2 e^a (\cos b - i \sin b) \begin{pmatrix} \sin d \\ 1 \end{pmatrix} \\ &= e^a \left\{ c_1 \begin{pmatrix} \cos b + i \sin b & \cos b - i \sin b \\ \cos b & i \sin b \end{pmatrix} + c_2 \begin{pmatrix} -i & 1 \\ 0 & 1 \end{pmatrix} \right\} \\ &= (c_1 + c_2) \cdot e^a \begin{pmatrix} \cos b - i \sin b \\ \cos b \end{pmatrix} + i(c_1 - c_2) e^a \begin{pmatrix} \sin b & \cos b \\ \sin b & \cos b \end{pmatrix} \end{aligned}$$

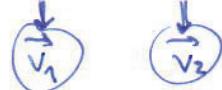
$$\rightarrow \vec{x}(t) = e^a \left(K_1 \begin{pmatrix} \cos b & \sin b \\ \cos b & \cos b \end{pmatrix} + K_2 \begin{pmatrix} \sin b & \cos b \\ \sin b & \cos b \end{pmatrix} \right)$$

(*) na način prijeđe je:
 $a = -\frac{1}{2}t$, $b = \frac{\sqrt{3}}{2}t$
 $c = \frac{1}{2}$, $d = \frac{\sqrt{3}}{2}$

stabilni fokus

Ovo je poseban slučaj kada je $\lambda_1 = 0$ ili $\lambda_2 = 0$. U tom slučaju cijela linija (kontinuirani skup točaka) ne leži na leži F.T. (dok leži nisu linearnizirali) u smjeru svrhistvenog vektora, koji pripada isčrpanoj vrijednosti, postaje singularna.

- uvek je $\frac{d\vec{x}}{dt} = A\vec{x}$ i $\lambda_1 = 0, \lambda_2 \neq 0$ (to se dogodi kada je $\det(A) = 0$)



rješenje:

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

} - za $\lambda_2 < 0$ asymptotski tok stijene se ka vanotokom linija \vec{v}_2 duž smjera \vec{v}_2
- za $\lambda_2 > 0$ tok izvire iz linije \vec{v}_1 duž \vec{v}_2

konstante \Rightarrow vrlo rješenje $\rightarrow c_2 = 0$
je vanotok \Rightarrow kontinuirani niz točaka (singularna linija duž \vec{v}_1)

primjer: $\frac{d\vec{x}}{dt} = A\vec{x}, A = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$

$$\det A = 0!, T = -4, \Delta = 0 ; \lambda_{1,2} : (-3-\lambda)(-1-\lambda) - 3 = 0 \\ 8+3\lambda+\lambda+\lambda^2 - 8 = 0 \\ \lambda(\lambda+4) = 0 \Rightarrow \boxed{\lambda_1 = 0 \\ \lambda_2 = -4}$$

sv. vektori:

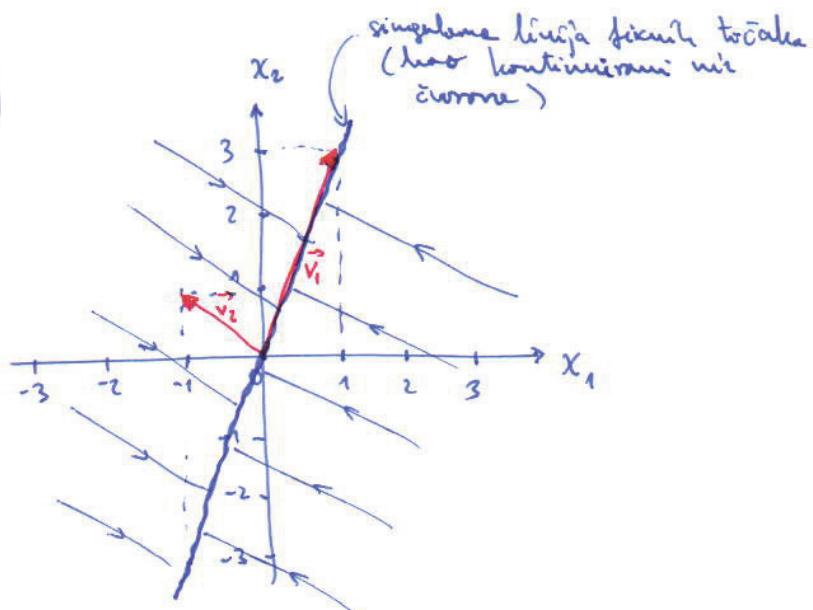
$$1: \lambda_1 = 0 \\ \begin{pmatrix} -3+0 & 1 \\ 3 & -1+0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \rightarrow \begin{array}{l} -3A+B=0 \\ 3A-B=0 \end{array} \rightarrow \begin{array}{l} 3A=B \\ B=3 \end{array} \rightarrow A=1 \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$2: \lambda_2 = -4 \\ \begin{pmatrix} -3+4 & 1 \\ 3 & -1+4 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \\ \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \rightarrow \begin{array}{l} A+B=0 \\ 3A+3B=0 \end{array} \rightarrow \begin{array}{l} A=-B \\ B=1 \end{array} \rightarrow A=-1 \rightarrow \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

rješenje:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

* Ovoj slučaju varan je za objašnjenje poslednjih tipova strukturnih nestabilnosti poput prijelaza iz fixne točke u granicnu kolo (limit cycle) i Hopfove bifurkacije!



primjer 1 Lovač - loviva sustav ("zecari i lisice")

Modeliramo evoluciju 2 vrste na istom staništu koje su u interakciji kao predator i plijen. Dakle imamo 2 varijable.

(*) pretpostavke modela:

- Ako nema lisica, zecari se razmnožavaju bezimom proporcionalnoj njihovom broju.
- Lisice love i ždu zecare bezimom koja je proporcionalna frekvenciji susreta lisice sa zecom.
- Ako nema zecara, populacija lisice opada bezimom proporcionalnom rebi suradji (sto ih je više, to ih više umre od gledi).
- Brojna razmnožavanja lisice proporcionalna je broju pojedenih zecara (koji je po drugoj metri. proporcionalna frekvenciji susreta).

t - vremje

$Z(t)$ - broj zecara

$L(t)$ - broj lisica

α - koef. prirosti zecara

β - frekvencija susreta zec-lisica u kojem je zec pojeden

γ - koef. smrtnosti lisica

δ - koef. prirosti populacije lisica po pojedenom zecu

$$\begin{aligned} \frac{dZ}{dt} &= \underbrace{\alpha Z}_{=F_Z(Z, L)} - \beta Z \cdot L \\ \frac{dL}{dt} &= \underbrace{-\gamma L}_{=F_L(Z, L)} + \delta Z \cdot L \end{aligned}$$

$$\alpha, \beta, \gamma, \delta > 0$$

F.T.

$$\begin{aligned} Z(\alpha - \beta L) &= 0 \\ L(-\gamma + \delta Z) &= 0 \end{aligned} \Rightarrow$$

$$\begin{aligned} 1.) \quad Z=0, L=0 &\Rightarrow T_1(0, 0) \\ 2.) \quad Z=\frac{\gamma}{\delta}, L=\frac{\alpha}{\beta} &\Rightarrow T_2\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) \end{aligned}$$

$$\left. \begin{array}{l} \beta - \text{broj zecara} \\ \gamma - \text{broj lisica} \\ \delta - \text{koef. prirosti lisica} \end{array} \right\} > 0$$

- Volterro model za 2 interagirajuće predator-plijen populacije

(*) napomena: Staviti što više parametara u idealni; npr. $T \equiv dt$ (polifleti jednadžbe s Δ) \Rightarrow reducirati broj parametara s 4 na 3

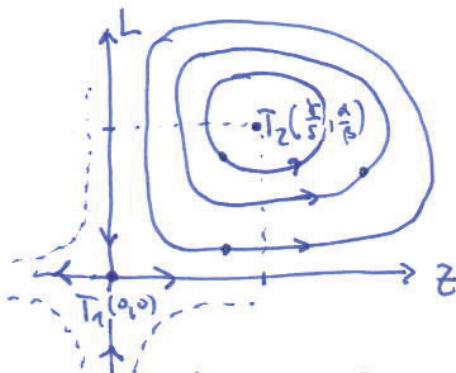
linearizacija:

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial L} & \frac{\partial F_1}{\partial Z} \\ \frac{\partial F_2}{\partial L} & \frac{\partial F_2}{\partial Z} \end{bmatrix} = \begin{bmatrix} \alpha - \beta L & -\beta Z \\ \delta L & -\gamma + \delta Z \end{bmatrix}$$

$T_1(0,0)$: $A = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix} \rightarrow T = \alpha - \gamma$
 $\Delta = -\alpha\gamma < 0$
 $D = (\alpha - \gamma)^2 + 4\alpha\gamma = (\alpha + \gamma)^2 > 0 \quad \Rightarrow \text{sedlo!}$

$T_2\left(\frac{\alpha}{\delta}, \frac{\gamma}{\delta}\right)$: $A = \begin{bmatrix} \alpha - \beta \frac{\alpha}{\delta} & -\beta \frac{\gamma}{\delta} \\ \delta \frac{\alpha}{\delta} & -\gamma + \delta \frac{\gamma}{\delta} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\delta} & 0 \end{bmatrix} \rightarrow T = 0$
 $\Delta = 2\gamma > 0$
 $D = -4\alpha\gamma < 0 \quad \Rightarrow \text{centar!}$

sljice u kojim prostoru:



→ Populacije L i Z orbitiraju oko centra; to je periodičke oscilacije u kojim linička i rečena linijski prati jedne druge.

analitičko rješenje putanjë (L(z)):

$$\frac{dL}{dz} = \frac{\dot{L}}{\dot{z}} = \frac{F_L(z,L)}{F_Z(z,L)}$$

$$\frac{dL}{dt} = \frac{-\gamma L + \delta LZ}{\alpha z - \beta LZ} = \frac{-\frac{\gamma}{z} + \delta}{\frac{\alpha}{L} - \beta} \quad \left| \begin{array}{l} (\text{reparacija}) \\ (\frac{\alpha}{L} - \beta)dz \end{array} \right.$$

$$\left(\frac{\alpha}{L} - \beta \right) dL = \left(-\frac{\gamma}{z} + \delta \right) dz \quad | \int$$

$$d \ln L - \beta L = -\gamma \ln z + \delta z + C$$

$$\ln \frac{d}{L} + \ln z^\delta - \beta L - \delta z = C$$

implicitna zadana rješenje
 $L(z)$

konstanta integracije; sadrži početne uvjete tj. vrijednosti varijabli trajektorije kosi orbitirajući oko F.T. T_2

Modificirano model ne sljedeći način: pretpostavimo da je u rastućoj lovackoj populaciji lovine opisana uvođeno eksponentijalnim logističkim modelom

$$\frac{dz}{dt} = \alpha z \left(1 - \frac{z}{N}\right) - \beta z \cdot L \rightarrow F_z(z, L)$$

$$\frac{dl}{dt} = -\gamma L + \delta z \cdot L \rightarrow F_L(z, L)$$

$$F_z \cdot F_L \cdot (1) \quad \alpha z \left(1 - \frac{z}{N} - \frac{\beta}{\alpha} L\right) = 0$$

$$(1) \quad \gamma L \left(-1 + \frac{\delta}{\gamma} z\right) = 0$$

$$T_1: \quad z=0, \quad L=0 \\ T_1(0,0)$$

 $\bar{T}_3:$

$$T_2: \quad z=N, \quad L=0 \\ T_2(N,0)$$

$$\begin{aligned} & \text{iz (1): } L=0 \text{ ili } z = \frac{N}{\alpha} \\ & \text{u (1): a) } L=0 \text{ i } \alpha z \left(1 - \frac{z}{N}\right) = 0 \Rightarrow z=0 \text{ ili } z=N \\ & \text{b) } z = \frac{N}{\alpha}; \quad \frac{\alpha}{\delta} \left(1 - \frac{N}{\alpha} - \frac{\beta}{\alpha} L\right) = 0 \Rightarrow L = \frac{\alpha}{\beta} \left(1 - \frac{N}{\alpha}\right) \end{aligned}$$

$$A = \begin{bmatrix} \alpha \left(1 - \frac{z}{N}\right) - \beta L & -\beta z \\ \delta L & -\gamma + \delta z \end{bmatrix};$$

$$\begin{cases} N > \frac{N}{\alpha} \\ T_1(0,0) \end{cases}$$

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix} \rightarrow T = \alpha - \gamma \\ \Delta = -\alpha \gamma < 0 \\ D = (\alpha - \gamma)^2 + 4\alpha\gamma = \\ = (\alpha + \gamma)^2 > 0 \\ \Rightarrow \text{sedlo!} \end{cases}$$

$$\begin{cases} T_2(N,0) \\ A = \begin{bmatrix} -\alpha & -\beta N \\ 0 & -\gamma + \delta N \end{bmatrix} \\ \rightarrow T = -\left(\alpha + \left(1 - \frac{\delta N}{\alpha}\right)\right) = ? \\ \Delta = \alpha\gamma \left(1 - \frac{\delta N}{\alpha}\right) < 0 \\ D = \left(\alpha - \gamma \left(1 - \frac{\delta N}{\alpha}\right)\right)^2 > 0 \\ \Rightarrow \text{sedlo!} \end{cases}$$

$$\bar{T}_3 \left(\frac{N}{\alpha}, \frac{\alpha}{\beta} \left(1 - \frac{N}{\alpha}\right) \right);$$

$$A = \begin{bmatrix} \alpha \left(1 - \frac{N}{\alpha}\right) - \beta \frac{N}{\alpha} & -\beta \frac{N}{\alpha} \\ \delta \frac{N}{\alpha} & -\gamma + \delta \frac{N}{\alpha} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha\gamma}{\alpha} & -\frac{\beta\gamma}{\alpha} \\ \frac{\alpha\delta}{\beta} \left(1 - \frac{N}{\alpha}\right) & 0 \end{bmatrix}$$

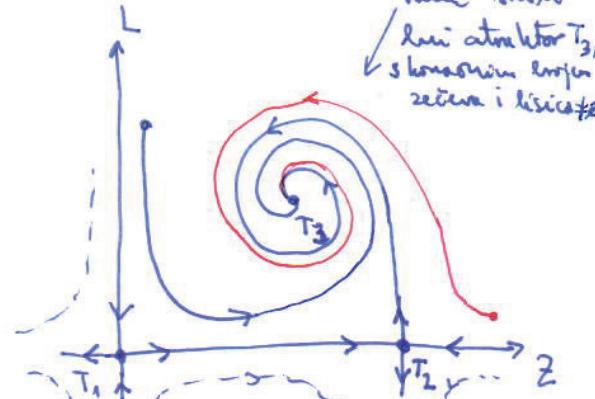
$$\hookrightarrow T = -\frac{\alpha\gamma}{\alpha} < 0$$

$$\Delta = \alpha\gamma \left(1 - \frac{N}{\alpha}\right) > 0$$

$$D = \left(\frac{\alpha\gamma}{\alpha}\right)^2 - 4\alpha\gamma \left(1 - \frac{N}{\alpha}\right)$$

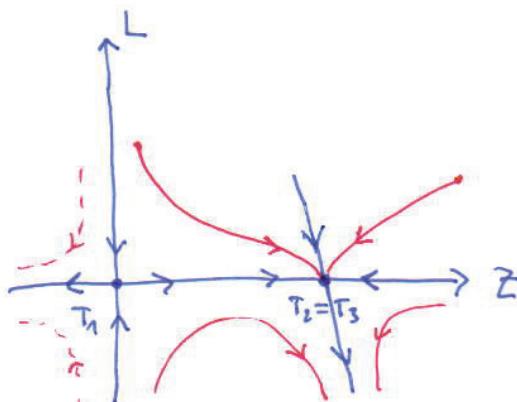
stabilni fokus!
(ili ovor)

Uvjet je $1 - \frac{N}{\alpha} < 0$, ovdje je F.T.
 $T_3 < 0$ tj. u neizikliznom
primjeru



$$(2^*) \quad N = \frac{\delta}{\sigma}$$

$$\begin{aligned} T_1(0,0) &= \text{sедло (instable ista)} \\ T_2(N,0) &= \left(\frac{\delta}{\sigma}, 0\right) \\ T_3\left(\frac{\delta}{\sigma}, 0\right) &= T_2 \end{aligned}$$



$$(3^*) \quad N < \frac{\delta}{\sigma}:$$

$$T_1(0,0) = \text{sедло (instable ista)}$$

$$T_2(N,0): \quad T = -(\alpha + \gamma |1 - \frac{\delta N}{\sigma}|) < 0 \\ \Delta = 2\gamma |1 - \frac{\delta N}{\sigma}| > 0 \rightarrow \text{stab. čvor!}$$

$$\mathcal{D} = (\alpha - \gamma |1 - \frac{\delta N}{\sigma}|)^2 > 0$$

$$T_3\left(\frac{\delta}{\sigma}, -\frac{\alpha}{\beta} |1 - \frac{\delta N}{\sigma}|\right): \quad T = -\frac{\alpha}{\beta N} < 0 \\ \Delta = -2\gamma |1 - \frac{\delta N}{\sigma}| < 0 \quad \downarrow \rightarrow \text{sedlo!} \\ \mathcal{D} = \left(\frac{\alpha}{\beta N}\right)^2 + 4\alpha\gamma |1 - \frac{\delta N}{\sigma}| > 0$$

$$\begin{aligned} T_2: \quad T &= -\alpha < 0 \\ \Delta &= 0 \\ \mathcal{D} &= \alpha^2 > 0 \end{aligned} \rightarrow \text{priječar čvor-sedlo!}$$

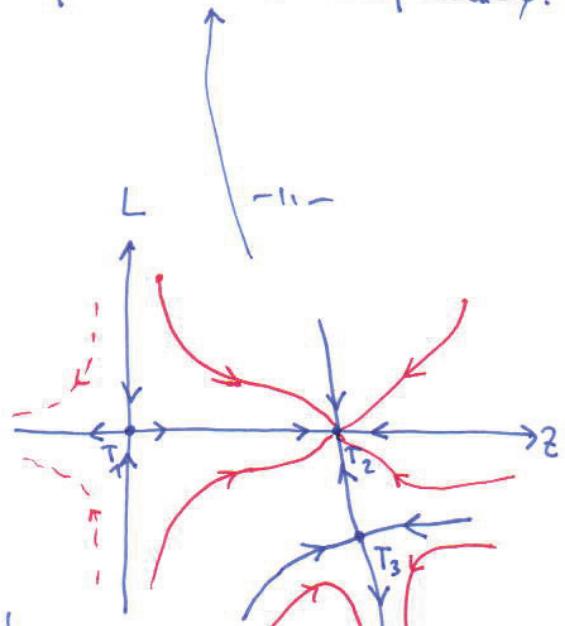
$$(T_3: \quad T = -\frac{\alpha x}{\beta N} = -\alpha < 0 \\ \Delta = 0 \\ \mathcal{D} = \alpha^2 > 0)$$

BIFURKACIJA
ČVOR-SEDLO!

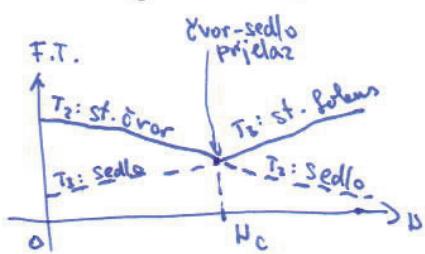
(saddle-node bifurcation)

($\Delta = 0 \Leftrightarrow \lambda_1 = 0, \lambda_2 \neq 0$ sljedi)

→ Populacija se zaglavljuje u stabilnom stanju T_2 gdje je broj rečera = N , a broj lisica = ϕ .



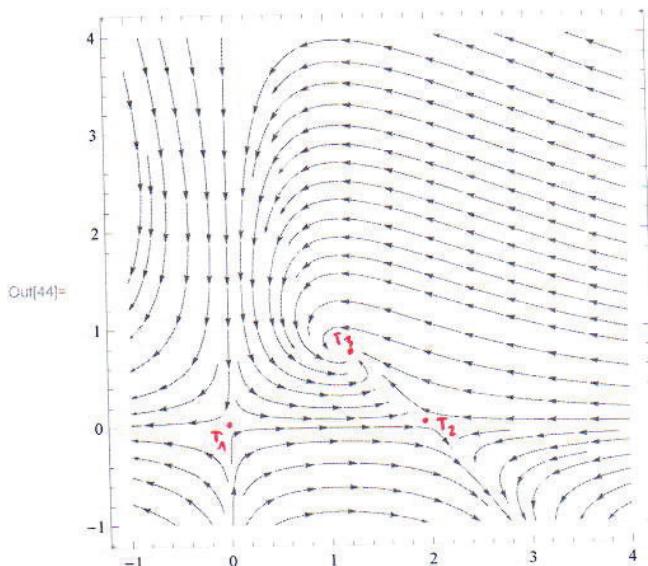
Model predviđa bifurkaciju kada je $N_c = \frac{\delta}{\sigma}$. Kada je $N > N_c$, tada je stabilna populacija s konstantnim brojem rečera i lisica (t.i. T_2). Kad parametar N dostigne vrijednost $N = N_c$ ili manji ($N < N_c$), broj rečera se ne može max. vrijednost N , a broj lisica pada na ϕ (nema dovoljno rečera da se lisice odvori).



```
In[37]:= fx[x_, y_] := a*x*(1 - x/n) - b*x*y
fy[x_, y_] := -c*y + d*x*y
(* T1=sedlo, T2=sedlo, T3=st. fokus *)

In[39]:= a = 2;
b = 1.2;
c = 1;
d = .9;
n = 2;

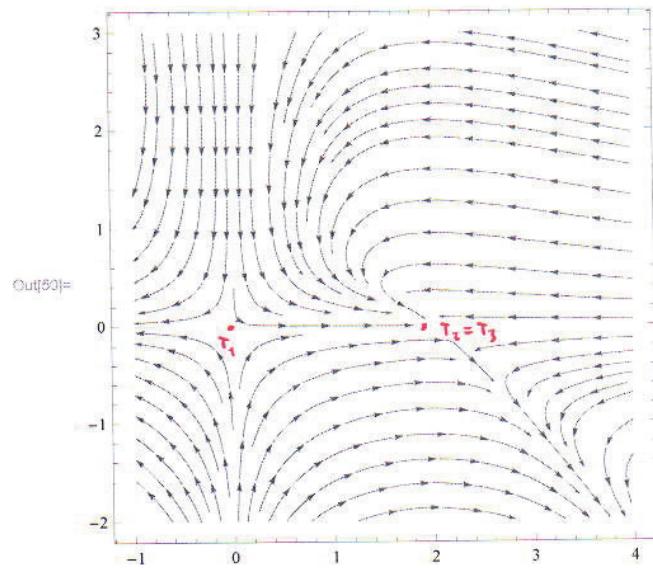
In[44]:= StreamPlot[{fx[x, y], fy[x, y]}, {x, -1, 4}, {y, -1, 4}]
```



```
(* T1=sedlo, T2=T3=čvor-sedlo *)

In[45]:= a = 2;
b = 1.2;
c = 2*.9;
d = .9;
n = 2;
```

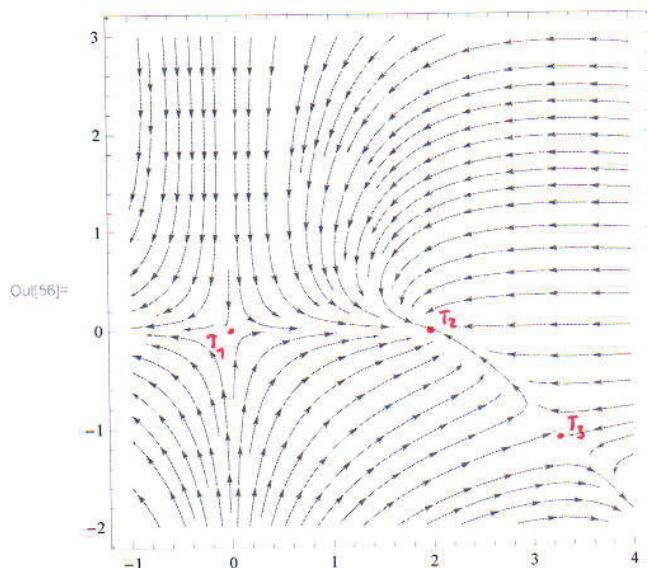
In[50]:= StreamPlot[{fx[x, y], fy[x, y]}, {x, -1, 4}, {y, -2, 3}]



(* T_1 =sedlo, T_2 =čvor, T_3 =sedlo *)

```
In[51]:= a = 2;
b = 1.2;
c = 3;
d = .9;
n = 2;
```

In[56]:= StreamPlot[{fx[x, y], fy[x, y]}, {x, -1, 4}, {y, -2, 3}]



① EULEROVA METODA

$$\frac{d\vec{x}}{dt} = \vec{f}(t, \vec{x}) \rightarrow \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} \approx \vec{f}(t, \vec{x}(t)) \cdot \Delta t$$

$$\vec{x}(t+\Delta t) = \vec{x}(t) + \vec{f}(t, \vec{x}(t)) \cdot \Delta t$$

def: $\vec{f}(t, \vec{x}) \rightarrow N$ komponenti

input: t_0, t, \vec{x}_0, n

$$\Delta t = \frac{t-t_0}{n}$$

$$\begin{cases} \Delta \vec{x}_i = \vec{f}(t_i, \vec{x}_i) \cdot \Delta t, i=1, 2, \dots, n \\ t_i = t_0 + i \cdot \Delta t \\ \vec{x}_{i+1} = \vec{x}_i + \Delta \vec{x}_i \end{cases}$$

(poljija)

□

za sustav od 2 jednađebla ($n=2$):

$$\text{def. } f_x(t, x, y)$$

$$f_y(t, x, y)$$

input: t_0, t, x_0, y_0, n

$$\Delta t = \frac{t-t_0}{n}$$

$$\Delta x_i = f_x(t_i, x_i, y_i) \cdot \Delta t$$

$$\Delta y_i = f_y(t_i, x_i, y_i) \cdot \Delta t$$

$$t_i = t_0 + i \cdot \Delta t$$

$$x_{i+1} = x_i + \Delta x_i$$

$$y_{i+1} = y_i + \Delta y_i$$

$$\begin{aligned} \vec{x} &= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ \vec{f} &= \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \end{aligned}$$

② RUNG-KUTTA IV

def: $\vec{f}(t, \vec{x})$

input: t_0, t, \vec{x}_0, n

$$\Delta t = \frac{t-t_0}{n}$$

$$\vec{k}_i^{(1)} = \vec{f}(t_i, \vec{x}_i) \cdot \Delta t$$

$$\vec{k}_i^{(2)} = \vec{f}\left(t_i + \frac{\Delta t}{2}, \vec{x}_i + \frac{\vec{k}_i^{(1)}}{2}\right) \cdot \Delta t$$

$$\vec{k}_i^{(3)} = \vec{f}\left(t_i + \frac{\Delta t}{2}, \vec{x}_i + \frac{\vec{k}_i^{(2)}}{2}\right) \cdot \Delta t$$

$$\vec{k}_i^{(4)} = \vec{f}(t_i + \Delta t, \vec{x}_i + \vec{k}_i^{(3)}) \cdot \Delta t$$

$$\Delta \vec{x}_i = \frac{1}{6} (\vec{k}_i^{(1)} + 2\vec{k}_i^{(2)} + 2\vec{k}_i^{(3)} + \vec{k}_i^{(4)})$$

$$t_i = t_0 + i \cdot \Delta t$$

$$\vec{x}_{i+1} = \vec{x}_i + \Delta \vec{x}_i$$

□

```
(*Euler*)
fx[x_, y_, t_] := (2 - 1.2 * y) * x
fy[x_, y_, t_] := (-1 + 0.9 * x) * y

Euler[t_, t0_, x0_, y0_, n_] := Module[{i, xx, dx, yy, dy, tt = t0, dt},
  dt = (t - t0) / n;
  xx = x0;
  yy = y0;
  For[i = 1, i ≤ n, i++,
    dx = fx[xx, yy, tt] * dt;
    dy = fy[xx, yy, tt] * dt;
    tt = t0 + i * dt;
    xx = xx + dx;
    yy = yy + dy
  ];
  {xx, yy} // N
]

Euler[5, 0, 1, 0.5, 400]
{0.874093, 0.46438}
```

```
(*Runge-Kutta IV*)
fx[x_, y_, t_] := (2 - 1.2 * y) * x
fy[x_, y_, t_] := (-1 + 0.9 * x) * y

RK[t_, t0_, x0_, y0_, n_] :=
Module[{i, xx, dx, yy, dy, tt = t0, dt, kx1, kx2, kx3, kx4, ky1, ky2, ky3, ky4},
  dt = (t - t0) / n;
  xx = x0;
  yy = y0;
  For[i = 1, i ≤ n, i++,
    kx1 = fx[xx, yy, tt] * dt;
    kx2 = fx[xx + kx1 / 2, yy + ky1 / 2, tt + dt / 2] * dt;
    kx3 = fx[xx + kx2 / 2, yy + ky2 / 2, tt + dt / 2] * dt;
    kx4 = fx[xx + kx3, yy + ky3, tt + dt] * dt;
    ky1 = fy[xx, yy, tt] * dt;
    ky2 = fy[xx + kx1 / 2, yy + ky1 / 2, tt + dt / 2] * dt;
    ky3 = fy[xx + kx2 / 2, yy + ky2 / 2, tt + dt / 2] * dt;
    ky4 = fy[xx + kx3, yy + ky3, tt + dt] * dt;
    dx = (kx1 + 2 * kx2 + 2 * kx3 + kx4) / 6;
    dy = (ky1 + 2 * ky2 + 2 * ky3 + ky4) / 6;
    tt = t0 + i * dt;
    xx = xx + dx;
    yy = yy + dy
  ];
  {xx, yy} // N
]

RK[5, 0, 1, 0.5, 40]
{0.957298, 0.478451}
```

```
(*drawing*)

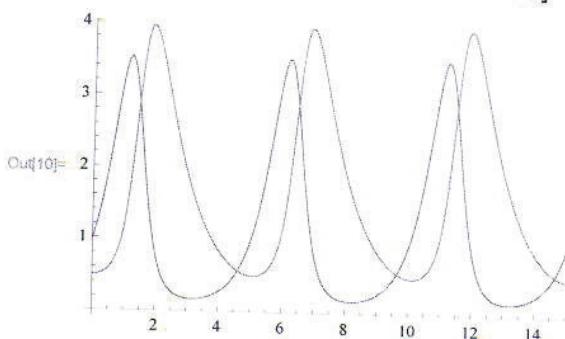
In[1]:= fx[x_, y_, t_] := (2 - 1.2 * y) * x
fy[x_, y_, t_] := (-1 + 0.9 * x) * y

RK[t_, t0_, x0_, y0_, n_] :=
Module[{i, xx, dx, yy, dy, tt = t0, dt, kx1, kx2, kx3, kx4, ky1, ky2, ky3, ky4, T},
dt = (t - t0) / n;
xx = x0;
yy = y0;
T = Table[{0, 0, 0}, {n + 1}];
T[[1]] = {t0, x0, y0};
For[i = 1, i <= n, i++,
kx1 = fx[xx, yy, tt] * dt;
kx2 = fx[xx + kx1 / 2, yy + ky1 / 2, tt + dt / 2] * dt;
kx3 = fx[xx + kx2 / 2, yy + ky2 / 2, tt + dt / 2] * dt;
kx4 = fx[xx + kx3, yy + ky3, tt + dt] * dt;
ky1 = fy[xx, yy, tt] * dt;
ky2 = fy[xx + kx1 / 2, yy + ky1 / 2, tt + dt / 2] * dt;
ky3 = fy[xx + kx2 / 2, yy + ky2 / 2, tt + dt / 2] * dt;
ky4 = fy[xx + kx3, yy + ky3, tt + dt] * dt;
dx = (kx1 + 2 * kx2 + 2 * kx3 + kx4) / 6;
dy = (ky1 + 2 * ky2 + 2 * ky3 + ky4) / 6;
tt = t0 + i * dt;
xx = xx + dx;
yy = yy + dy;
T[[i + 1]] = {tt, xx, yy}
];
T // N
]

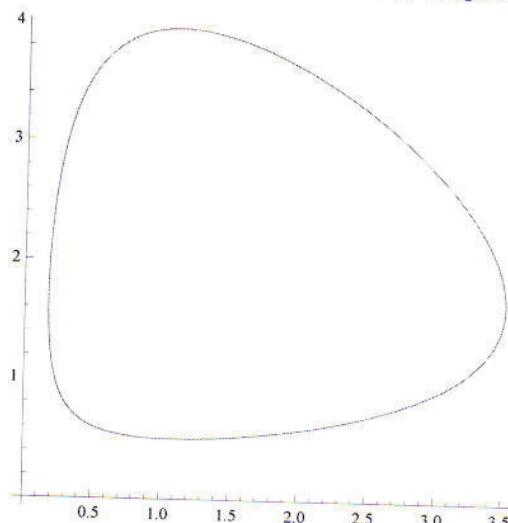
In[4]:= n = 4000;
T = RK[15, 0, 1, 0.5, n];

In[6]:= Tx = Table[{0, 0}, {n + 1}];
Ty = Table[{0, 0}, {n + 1}];
TxTy = Table[{0, 0}, {n + 1}];
For[i = 1, i <= n + 1, i++,
Tx[[i]][[1]] = T[[i]][[1]];
Tx[[i]][[2]] = T[[i]][[2]];
Ty[[i]][[1]] = T[[i]][[1]];
Ty[[i]][[2]] = T[[i]][[3]];
TxTy[[i]][[1]] = T[[i]][[2]];
TxTy[[i]][[2]] = T[[i]][[3]]
]
]
```

```
In[10]:= ListPlot[{Tx, Ty}, Joined → True]
```



```
ListPlot[Txy, Joined → True, AspectRatio → 1]
```



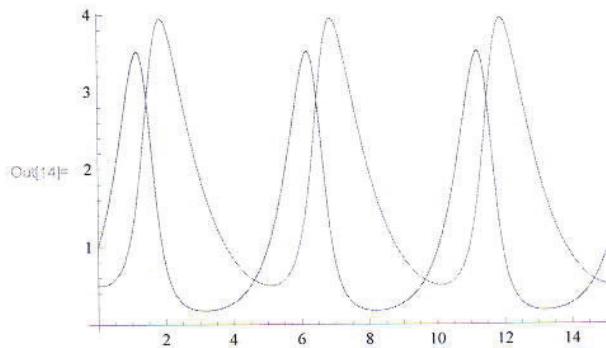
(* NDSolve *)

```
In[11]:= fx[x_, y_, t_] := (2 - 1.2 * y) * x
fy[x_, y_, t_] := (-1 + 0.9 * x) * y
```

```
s = NDSolve[{x'[t] == fx[x[t], y[t], t],
y'[t] == fy[x[t], y[t], t], x[0] == 1, y[0] == 0.5}, {x, y}, {t, 0, 15}]
```

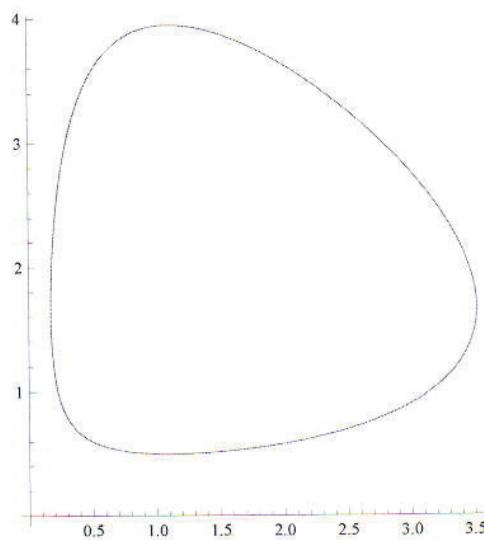
```
Out[13]= {x → InterpolatingFunction[{{0., 15.}}, <>],
y → InterpolatingFunction[{{0., 15.}}, <>]}}
```

```
In[14]:= Plot[Evaluate[{x[t], y[t]} /. s], {t, 0, 15}]
```

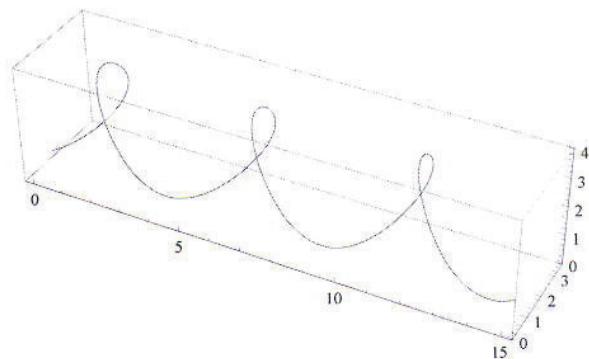


```
Out[14]=
```

```
ParametricPlot[Evaluate[{x[t], y[t]} /. s, {t, 0, 15}]]
```



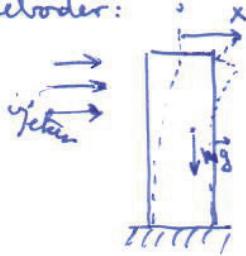
```
ParametricPlot3D[Evaluate[{t, x[t] /. s, y[t] /. s}, {t, 0, 15}]]
```



Zadatok

Ljuljanje nevodenca

- nevodenec:



x - udaljenost; punak vrha nevodenca od ravnotežnog položaja

$$m\ddot{x} = F(x, \dot{x})$$

dionične sile

; $F(x, \dot{x})$ - sila koja opisuje elastične sile koje vlaže uognuti nevodenec u položaj ravnoteže; ^(P-delta učinak) opisuje utjecaj gravitacije koja je nevodenec uognutiji togačće može iz položaja ravnoteže; opisuje trenje tij gusičaju oscilaciju proporcionalno brzini

- varivijus $F(x, \dot{x})$ u Taylorov red oko ravnotežnog položaja

$$\text{u } x=0, \dot{x}=0:$$

$$F(x, \dot{x}) = F_0 + \frac{\partial F}{\partial x} \Big|_{x=0} \cdot x + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} \Big|_{x=0} x^2 + \frac{1}{3!} \frac{\partial^3 F}{\partial x^3} \Big|_{x=0} x^3 + \dots +$$

$$+ \frac{\partial F}{\partial \dot{x}} \Big|_{\dot{x}=0} \cdot \dot{x} + \frac{1}{2} \frac{\partial^2 F}{\partial \dot{x}^2} \Big|_{\dot{x}=0} \dot{x}^2 + \frac{1}{3!} \frac{\partial^3 F}{\partial \dot{x}^3} \Big|_{\dot{x}=0} \dot{x}^3 + \dots$$

1) sila $F(x, \dot{x})$ je neparna funkcija (mijenja predznak kada $x \rightarrow -x$)
 \Rightarrow nici parne potencije u x, \dot{x} isčeravaju!

2) u položaju ravnoteže sila je nula $\Rightarrow F_0 = 0$
 \curvearrowleft nečelična elastična sila

$$F(x, \dot{x}) \approx -kx + g_2 x^3 - \gamma \dot{x} + \dots$$

$(k > 0) \quad (g_2 > 0) \quad (\gamma > 0)$

Costatah, nici članovi - zanemarjivo

$$m\ddot{x} = -kx - \gamma \dot{x} + g_2 x^3 \quad | : m$$

$$\ddot{x} + \gamma \dot{x} + \omega^2 x - \xi x^3 = 0$$

$$; \quad \gamma = \frac{\gamma}{m}, \omega = \sqrt{\frac{k}{m}}, \xi = \frac{g_2}{\omega^2}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\gamma y - \omega^2 x + \xi x^3 \end{cases}$$

$$\stackrel{F_x}{\rightarrow} \text{F.T.} \quad y_c = 0 \\ x(-\omega^2 + \xi x^2) = 0 \rightarrow x_{c1} = 0$$

$$T_1(0, 0); T_{2,3}\left(\pm \frac{\omega}{\sqrt{\xi}}, 0\right)$$

$\omega < \xi < 0$,
 samo je $T_1(0, 0)$
 licna točka

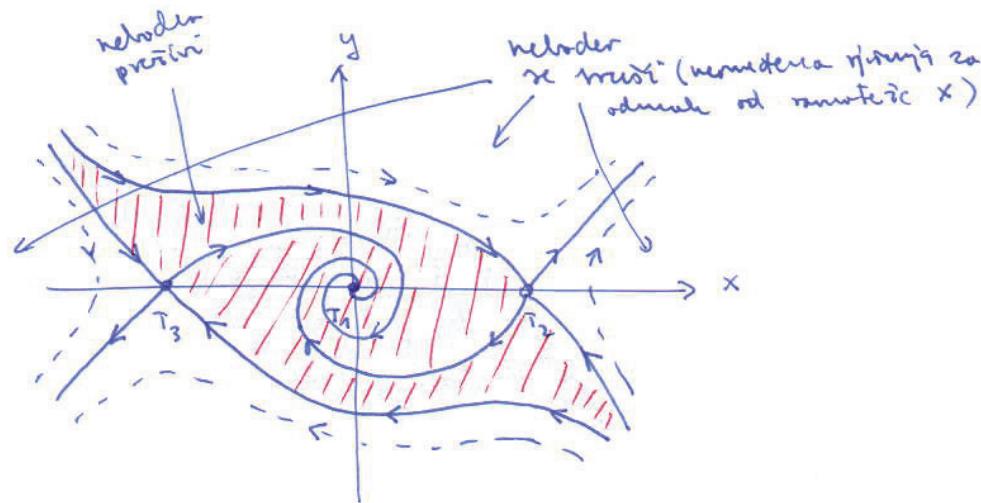
napomena:

- 1) ako je $\xi = 0 \Rightarrow \ddot{x} + \gamma \dot{x} + \omega^2 x = 0$ - obični gusični harmon. oscilator
- 2) ako se uključi P-delta učinak, inicira nečeličnu povratnu silu

$$A = \begin{bmatrix} 0 & \gamma \\ -\omega^2 + 3\beta\gamma^2 & \gamma \end{bmatrix}$$

$$T_1(0,0) : A = \begin{pmatrix} 0 & \gamma \\ -\omega^2 & \gamma \end{pmatrix} \rightarrow \begin{array}{l} T = -\gamma < 0 \\ \Delta = \omega^2 > 0 \\ \mathcal{D} = \gamma^2 - 4\omega^2 \end{array} \Rightarrow \begin{array}{l} \text{stabilni} \\ \text{poljes} (\mathcal{D} < 0) \\ \text{"ličnik" } (\mathcal{D} > 0) \end{array}$$

$$T_{1,3}\left(\pm\frac{\omega}{\sqrt{3}}, 0\right) : A = \begin{pmatrix} 0 & \gamma \\ 2\omega^2 & -\gamma \end{pmatrix} \rightarrow \begin{array}{l} T = -\gamma < 0 \\ \Delta = -2\omega^2 < 0 \\ \mathcal{D} = \gamma^2 + 8\omega^2 > 0 \end{array} \Rightarrow \text{sedlo!}$$

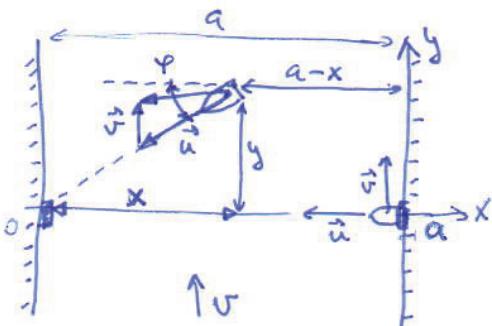


Zadatak Primjer katastrofe - "brod"

Tresmo vrijeku širine a koja teče jednolikom brzinom v . Brod zeli projekti na drugu stranu. On se gibati jednolikom spravom vode brzinom u . Kormilac upravlja unijkem takim da pravac unijecave ka dolje na suprotnoj obali usuprot ovog s kojeg se otinju. Jedini parametar je $d \equiv v/u$.

- a) Odredi putanje uniske $\circ d$ (da li će stići preko ili neće).
 b) Koliko je vrijeme potrebno da stigne preko veće od Fermatove?

a)



$$t_0 \varphi = \frac{y}{x}, \quad \varphi \in [0, \frac{\pi}{2}]$$

$$\sin^2 \varphi = \frac{t_0^2 \varphi}{1+t_0^2 \varphi} = \frac{y^2}{x^2+y^2}$$

$$\omega_0^2 \varphi = \frac{1}{1+t_0^2 \varphi} = \frac{x^2}{x^2+y^2}$$

Initialni uvjeti: $x(0) = a, y(0) = 0$

$$\dot{x} = -u \cos \varphi \quad | : u$$

$$\dot{y} = v - u \sin \varphi \quad | : u$$

$$x' = -\cos \varphi = -\frac{x}{\sqrt{x^2+y^2}}$$

$$y' = d - \sin \varphi = d - \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{d - \frac{y}{\sqrt{x^2+y^2}}}{-\frac{x}{\sqrt{x^2+y^2}}}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{d}{x} \sqrt{x^2+y^2}$$

$$= \frac{y}{x} - d \sqrt{1 + (\frac{y}{x})^2} \quad ; \quad z \equiv \frac{y}{x}$$

↓

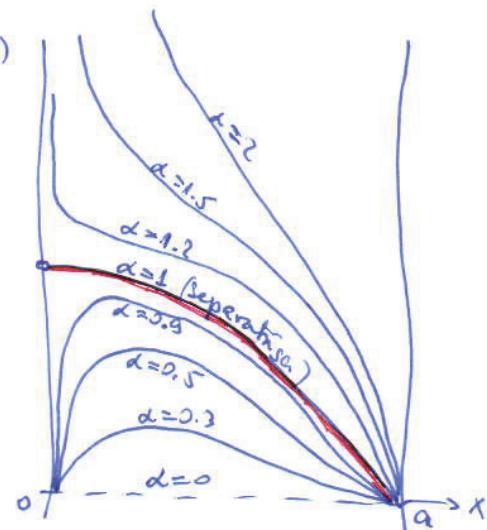
$$x + x \frac{dz}{dx} = x - d \sqrt{1+z^2} \quad | : x$$

$$\frac{dz}{dx} = -\frac{d}{x} \sqrt{1+z^2}$$

$$\frac{dz}{\sqrt{1+z^2}} = -\frac{d}{x} dx \quad | \int$$

$$; \quad \tilde{t} \equiv u \cdot t, \quad d \equiv \frac{v}{u}$$

$$\left. \begin{array}{l} \text{(momentum)} \\ x(0)=a \\ y(0)=0 \end{array} \right\} \rightarrow$$



$$\ln(z + \sqrt{1+z^2}) = -\alpha \ln x + c \rightarrow -\alpha \ln c$$

$$\ln(z + \sqrt{1+z^2}) = \ln(x \cdot c)^{-\alpha}$$

$$z + \sqrt{1+z^2} = (cx)^{-\alpha}$$

$$\frac{y}{x} + \sqrt{1+(\frac{y}{x})^2} = (cx)^{-\alpha} |^2$$

$$1 + (\frac{y}{x})^2 = (cx)^{-2\alpha} - 2(cx)^{-\alpha}(\frac{y}{x}) + (\cancel{\frac{y^2}{x}})$$

$$2(cx)^{-\alpha} \frac{y}{x} = -1 + (cx)^{-2\alpha} \quad | \cdot x \quad | : 2(cx)^{-\alpha}$$

$$\underline{y(x) = \frac{x}{2} [-(cx)^{+\alpha} + (cx)^{-\alpha}]}$$

r.u. $y(0)=0$
 $x(0)=a$: $0 = \frac{a}{2} [(ca)^{-\alpha} - (ca)^\alpha] \quad | \cdot (ca)^\alpha$

$$(ca)^{2\alpha} = 1 \Rightarrow c = \frac{1}{a}$$

$$\Rightarrow \underline{y(x) = \frac{x}{2} \left[\left(\frac{x}{a}\right)^{-\alpha} - \left(\frac{x}{a}\right)^\alpha \right]}$$

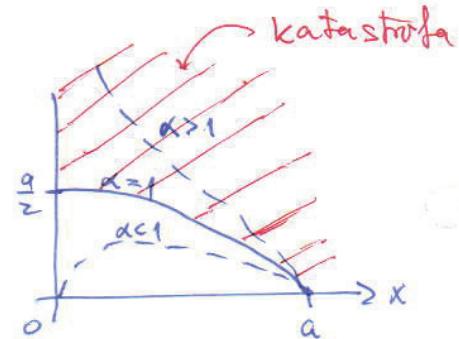
Asymptotické pouvazání závislosti y a x v okolí $x=0$:

$$\lim_{x \rightarrow 0} \frac{x}{2} \left[\left(\frac{x}{a}\right)^{-\alpha} - \left(\frac{x}{a}\right)^\alpha \right] = \lim_{x \rightarrow 0} \frac{x}{2} \frac{a^{2\alpha} - x^{2\alpha}}{x^\alpha a^\alpha} =$$

$$\left(\frac{a}{x}\right)^\alpha - \left(\frac{x}{a}\right)^\alpha = \frac{a^{2\alpha} - x^{2\alpha}}{x^\alpha a^\alpha}$$

$$= \frac{\lim_{x \rightarrow 0} (a^{2\alpha} - x^{2\alpha})}{\lim_{x \rightarrow 0} 2a^\alpha x^{\alpha-1}} = \frac{a^{2\alpha}}{2a^\alpha \lim_{x \rightarrow 0} x^{\alpha-1}} =$$

$$= \frac{a^\alpha}{2} \lim_{x \rightarrow 0} x^{1-\alpha} = \begin{cases} \infty, & \alpha > 1 \\ \frac{a}{2}, & \alpha = 1 \\ 0, & \alpha < 1 \end{cases} \rightarrow$$



Za $\alpha > 1$ dochází k tzv. "KATASTROFĚ" (když následuje nechtek a lze u něgo ga nijeho odnese). $\alpha=1$ je bifurkace čísla tažka.

b) "vrrijeme preleta"

$$x' = \frac{dx}{dt} = -\frac{x}{\sqrt{x^2+y^2}} = -\frac{1}{\sqrt{1+(\frac{y}{x})^2}} = -\frac{1}{\sqrt{1+\left[\frac{\frac{1}{2}\left[(\frac{x}{a})^{-\alpha}-(\frac{x}{a})^\alpha\right]}{x}\right]^2}}$$

$$\frac{dx}{dt} = -\frac{1}{\sqrt{1+\frac{1}{4}\left[\left(\frac{x}{a}\right)^{-\alpha}-\left(\frac{x}{a}\right)^\alpha\right]^2}}$$

$$\begin{aligned} \Rightarrow dt &= -\sqrt{1+\frac{1}{4}\left[\left(\frac{x}{a}\right)^{-\alpha}-\left(\frac{x}{a}\right)^\alpha\right]^2} dx = \left\{ \begin{array}{l} \text{subst.} \\ \frac{x}{a} = x \\ dx = adx, \int dx \rightarrow \int_1^0 dx \end{array} \right\} = \\ &= -a \sqrt{1+\frac{1}{4}\left[\frac{1}{x^\alpha}-x^\alpha\right]^2} dx \\ &= -a \sqrt{1+\frac{1}{4}\left(\frac{1-x^{2\alpha}}{x^\alpha}\right)^2} dx \\ &= -a \sqrt{\frac{4x^{2\alpha}+(1-x^{2\alpha})^2}{4x^{2\alpha}}} dx \\ &= -\frac{a}{2x^\alpha} \sqrt{4x^{2\alpha}+1-2x^{2\alpha}+x^{4\alpha}} dx \\ &= -\frac{a}{2x^\alpha} \sqrt{x^{4\alpha}+2x^{2\alpha}+1} dx \quad ; \quad |x^{2\alpha}+1| = x^{2\alpha}+1 \text{ für } x \geq 0 \\ &= -\frac{a}{2} \frac{x^{2\alpha}+1}{x^\alpha} dx \\ dt &= -\frac{a}{2} [x^\alpha + x^{-\alpha}] dx \quad \Rightarrow \tilde{t} = -\underbrace{\frac{a}{2} \int_1^0 [x^\alpha + x^{-\alpha}] dx}_{1} \end{aligned}$$

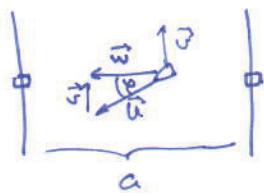
$$\begin{aligned} 1^\circ) \underline{\alpha < 1} : \quad \tilde{t} &= -\frac{a}{2} \left[\frac{1}{\alpha+1} x^{1+\alpha} \Big|_1^0 + \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^0 \right] = \frac{a}{2} \left[\frac{1}{1+\alpha} + \frac{1}{1-\alpha} \right] = \\ &= \frac{a}{2} \frac{1-\alpha+1+\alpha}{1-\alpha^2} = \frac{a}{2} \frac{2}{1-\alpha^2} = \frac{a}{1-\alpha^2} \quad ; \quad (\tilde{t} = u \cdot t) \end{aligned}$$

$$\Rightarrow t = \frac{a/u}{1-\alpha^2}$$

$$2^\circ) \underline{\alpha > 1} : \quad \tilde{t} = -\frac{a}{2} \left[\frac{1}{1+\alpha} x^{1+\alpha} \Big|_1^0 + \frac{1}{1-\alpha} x^{1-\alpha} \Big|_1^0 \right] = \frac{a}{2} \left[\frac{1}{1+\alpha} - \frac{1}{1-\alpha} \left(\frac{1}{x^\alpha} - \frac{1}{x^{1-\alpha}} \right) \right] = \infty$$

$$3^\circ) \underline{\alpha = 1} : \quad \tilde{t} = -\frac{a}{2} \left[\frac{x^2}{2} \Big|_1^0 + \ln x \Big|_1^\infty \right] = \infty$$

Fermatovo vrijeme:



$$\sin \varphi = \frac{v}{u} = \alpha$$

$$w = u \cos \varphi = u \sqrt{1 - \alpha^2}$$

$$t_F = \frac{a}{w} = \frac{a}{u \sqrt{1 - \alpha^2}}$$

$$\text{no; } t_F = \frac{a}{w} = \frac{a}{u \sqrt{1 - \alpha^2}}$$

$$\text{ali } t_F \rightarrow \tilde{t}_F \equiv u \cdot t_F \quad ; \quad \tilde{t}_F = \frac{a}{\sqrt{1 - \alpha^2}}$$

ovjerj:

$$\boxed{\frac{\tilde{t}}{\tilde{t}_F} = \frac{1}{\sqrt{1 - \alpha^2}}}$$

Love Affairs : Romeo & Julia

Romeo je zaljubljen u Juliju. Julija se tipične činjenice psihologije:

- što je Romeo zaljubljeni u nju, ona više ljubi od nje, ali kad se on obudi od nje i poine uručati, Julija se poche za nju više raznijavati. Romeo je pale suprotnog porečanja: što je Julija više zaljubljena u njega, on je više zaljubljen u nju i, obrnuto, što je ona bladerija prema njemu, on se više bladi od nje.

$$\begin{aligned} R(t) &= \text{Romeova ljubav/mišnja za Juliju u trenutku } t \\ J(t) &= \text{Julijeva ljubav/mišnja za Romu u trenutku } t \end{aligned}$$

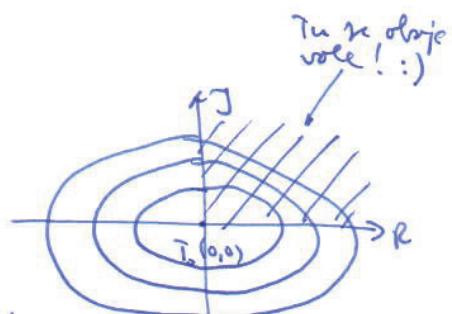
A ovom pomerajućem modelu jednačke luge opisuju dinamiku sustava glase:

$$\begin{aligned} \dot{R} &= aJ \\ \dot{J} &= -bR \end{aligned} \quad ; a, b > 0$$

$$\text{F.I. } T_0(0,0)$$

$$A = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$$

$$\begin{aligned} T &= 0 \\ \Delta &= ab > 0 \\ D &= -4ab < 0 \end{aligned} \quad \left. \begin{array}{l} \Rightarrow \text{centar!} \end{array} \right.$$



Ljubav Romeo i Julije je vjerojatno kroz ljubavlji/mišnji. Oboje se istovremeno voli $\frac{1}{4}$ vremena.

Modificirajući model u općeniti linearni sustav:

$$\begin{aligned} \dot{R} &= aR + bJ \\ \dot{J} &= cR + dJ \end{aligned} \quad \begin{array}{l} a, b, c, d \text{ mogu biti općenito} \\ \text{i pozitivni i negativni} \end{array}$$

Tzv. predznaka a, b, c, d uđi izvor "ljubavnog stila".

Npr. $a > 0$ i $b > 0$ tada de se Romeo više raznijava za Juliju što je ona zaljubljena u njega i što je on sam zaljubljeni u nju.

Npr. $a < 0$ i $b > 0$ označava "oprernog ljubavnika"; tali ga što je ona zaljubljena u njega, ali protagođi operacu i tako se što je on sam zaljubljeni.

itd.

(1') Testuojimo išsodas vere dviųjų "operuotių gylymų":

$$R = aR + bJ$$

$$J = bR + aJ$$

nguris "operuoti"

$a < 0, b > 0, |a| \neq |b|$

F.T.

$$aR + bJ = 0$$

$$bR + aJ = 0 \Rightarrow J = -\frac{b}{a}R$$

$$aR = \frac{b^2}{a}R$$

$$a^2R = b^2R$$

$$\Rightarrow R = 0$$

$$\Rightarrow J = 0$$

po metu

$T_0(0,0)$

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

\Rightarrow

$$T = 2a < 0$$

$$\Delta = a^2 - b^2$$

$$\mathcal{D} = 4a^2 - 4(a^2 - b^2) = 4b^2 > 0$$

$T_0(0,0)$ je
sedlo za $a^2 < b^2$
je stabilus
ovor za $a^2 > b^2$

sr. vijudinti: $(a-x)^2 - b^2 = 0$

$$x^2 - 2ax + a^2 - b^2 = 0$$

$$\Rightarrow x_1, x_2 = a \pm b$$

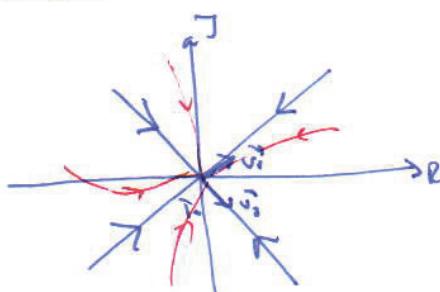
$\lambda_1, \lambda_2 < 0$ zu
 $a-b & a+b < 0$
 $\lambda_1 > 0, \lambda_2 < 0$ zu $a+b > 0$

sr. reikšmės:

$$\lambda_1 = a+b : \begin{pmatrix} a+x-b & b \\ b & a+x-b \end{pmatrix} = \begin{pmatrix} -b & b \\ b & -b \end{pmatrix} \left| \begin{pmatrix} A \\ B \end{pmatrix} \right. \Rightarrow \begin{array}{l} A=B \\ (B \neq 1) \end{array} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

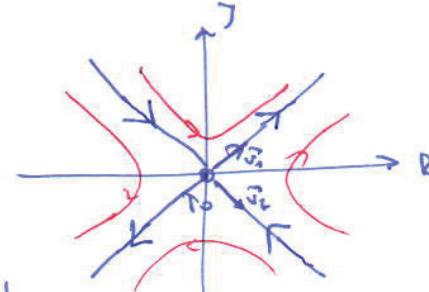
$$\lambda_2 = a-b : \begin{pmatrix} a-x+b & b \\ b & a-x+b \end{pmatrix} = \begin{pmatrix} b & b \\ b & b \end{pmatrix} \left| \begin{pmatrix} A \\ B \end{pmatrix} \right. \Rightarrow \begin{array}{l} A > -b \\ (B \neq -1) \end{array} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(a) $a+b < 0$



\hookrightarrow Vera evoluja į metuoblynę
indiferentnost ($R=0, J=0$).

(b) $a+b \geq 0$



\hookrightarrow Vera je izaromijsi; orinė o
iničiūnai operuojant, vera evoluja
iši u beskoniškai atvėranti iši u
totalni rat. Operacijų srogi gylymuk
na kogu sen isti.

4.5

Gusani (harmonički) oscilator

(a) LINEARNI

$$\ddot{x} + \alpha \dot{x} + \omega^2 x = 0$$

frije

$$\begin{cases} x_1 \equiv x \\ x_2 \equiv \dot{x} \end{cases}$$

$$\ddot{x} = -\alpha \dot{x} - \omega^2 x$$

$$\dot{x}_1 = x_2$$

$$\ddot{x} = -\alpha \dot{x} - \omega^2 x$$

$$\dot{x}_2 = -\alpha x_2 - \omega^2 x_1$$

$$\begin{cases} F_1 = x_2 \\ F_2 = -\alpha x_2 - \omega^2 x_1 \end{cases}$$

F.T.

$$\begin{cases} x_2 = 0 \\ -\alpha x_2 - \omega^2 x_1 = 0 \end{cases} \rightarrow x_1 = 0$$

$$\underline{T_c(0,0)}$$

$$\mathcal{L} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0 \quad -\alpha = -\alpha$$

→ lična derivacija konstanta
su u vijeku harmoničnog gibanja

$\alpha > 0$ (frije) \Rightarrow dissipacije ($\mathcal{L} < 0$)

$\alpha = 0$ (neutruzni) \Rightarrow konzervativan ($\mathcal{L} = 0$)

$(\alpha < 0)$ (dodavanje energije) \Rightarrow antidispipacija $\mathcal{L} > 0$,
uporabljajući energiju - otvoreni sustav

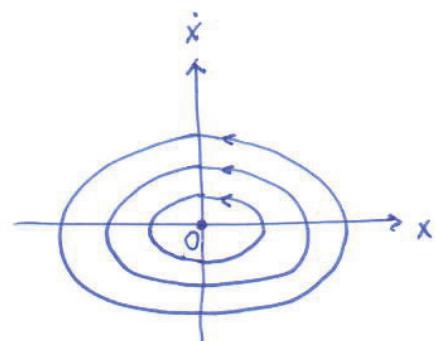
$$\rightarrow \lambda_{1,2} = \frac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4\omega^2})$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega & -\alpha \end{bmatrix} \quad ; \quad T = -\alpha \quad \Delta = \omega^2$$

$$\mathfrak{D} = \alpha^2 - 4\omega^2$$

(1°) $\alpha = 0$ (neutruzni)

$$\begin{cases} T = 0 \\ \Delta = \omega^2 > 0 \\ \mathfrak{D} = -4\omega^2 < 0 \end{cases} \Rightarrow \begin{array}{l} \text{center} \\ (\text{eliptično F.T.}) \end{array}$$



$$(\lambda_{1,2} = \pm i\omega); \text{ rj: } x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

(2°) $\alpha > 0$ (gusani)

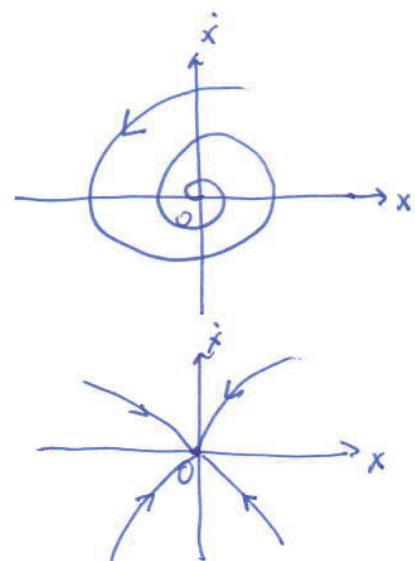
$$T = -\alpha < 0$$

$$\Delta = \omega^2 > 0$$

$$\mathfrak{D} = \alpha^2 - 4\omega^2 \Rightarrow$$

$$\left. \begin{array}{ll} \alpha^2 < 4\omega^2 & \rightarrow \text{stabilni} \\ & \text{polus} \\ & (\text{potencijalno} \\ & \text{gusani}) \\ \alpha^2 > 4\omega^2 & \rightarrow \text{otor} \\ & (\text{neutruzno} \\ & \text{gusani}) \end{array} \right.$$

$$\text{rj: } x(t) = e^{-\frac{\alpha t}{2}} (c_1 \cos \omega t + c_2 \sin \omega t)$$



Energija oscilatora:

$$\tilde{E}(x, \dot{x}) = \frac{1}{2} (\underbrace{\dot{x}^2}_{\text{kin.}} + \underbrace{\omega^2 x^2}_{\text{pot.}})$$

$$\begin{aligned}\frac{d\tilde{E}}{dt} &= \frac{1}{2} (2\ddot{x}\dot{x} + 2\omega^2 x\dot{x}) \\ &= \dot{x} (\underbrace{\ddot{x} + \omega^2 x}_{= -\alpha \dot{x}}) \quad ; \quad \ddot{x} + \alpha \dot{x} + \omega^2 x = 0\end{aligned}$$

$= -\alpha(\dot{x})^2 \rightarrow$ Dissipacija energije proporcionalna je kvadratu brzine \dot{x} ;

- $\alpha > 0$ - dissipacija ne var.
- $\alpha = 0$ - sačuvanje (konzervativni)
- $\alpha < 0$ - pumpanje energije u sistem

(b) NEULINEARNI (matematičko ujednošto)

$$\ddot{x} + \alpha \dot{x} + \omega^2 \sin x = 0$$

$$\begin{cases} x_1 \equiv x \\ x_2 \equiv \dot{x} \end{cases}$$

 \Rightarrow

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\alpha x_2 - \omega^2 \sin x_1 \end{cases}$$

$$F_1 = x_2$$

$$F_2 = -\alpha x_2 - \omega^2 \sin x_1$$

F.T. :

$$x_2 = 0$$

$$-\alpha x_2 - \omega^2 \sin x_1 = 0 \rightarrow \sin x_1 = 0 \rightarrow x_1 = k\pi, k \in \mathbb{Z}$$

$$T_c (k\pi, 0), k \in \mathbb{Z}$$

$$\mathcal{L} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} = 0 - \alpha = -\alpha$$

$$A: \quad a_{11} = 0 \quad a_{12} = 1 \quad \Rightarrow A = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x_1 & -\alpha \end{pmatrix}$$

$$a_{21} = -\omega^2 \cos x_1 \quad a_{22} = -\alpha$$

$$|A|_{T_c} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos k\pi & -\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^k \omega^2 & -\alpha \end{pmatrix}$$

$$\tau = -\alpha$$

$$\Delta = (-1)^k \omega^2$$

$$\mathfrak{D} = \alpha^2 - 4(-1)^k \omega^2$$

$$l \in \mathbb{Z} \quad l=2l' \quad (\text{parni})$$

$$\tau = -\alpha$$

$$\Delta = \omega^2 > 0$$

$$\mathfrak{D} = \alpha^2 - 4\omega^2$$

$$l=2l+1, l \in \mathbb{Z} \quad (\text{neparni})$$

$$\tau = -\alpha$$

$$\Delta = -\omega^2$$

$$\mathfrak{D} = \alpha^2 + 4\omega^2 > 0$$

stabilni fokus

(ili ovor ako je $\alpha^2 > 4\omega^2$)

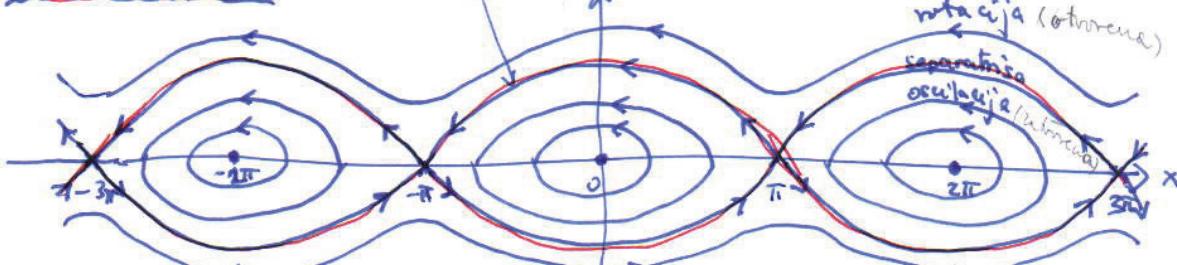
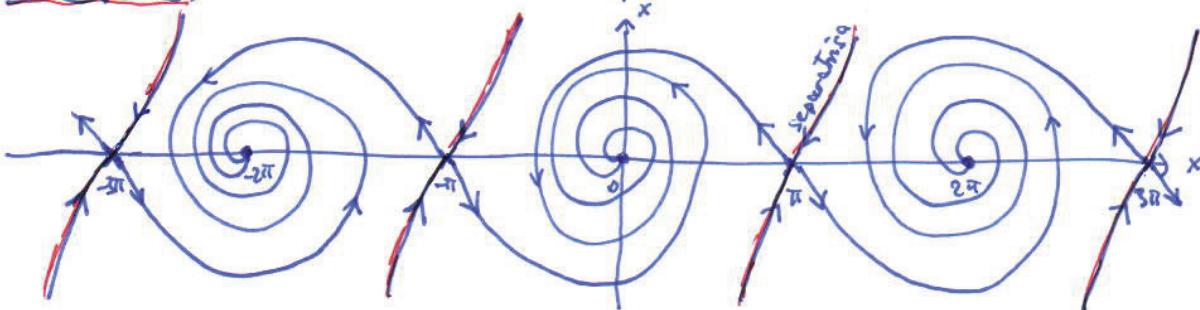
sedlo

1°) $\alpha = 0$ (negativni)

HETEROKLINIČKA

TRANSPORTIRJA

(saddle connection)

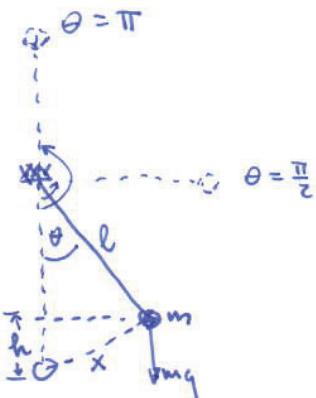
2°) $\alpha \neq 0$ (pozitivni)

6.6

SPLITATRICA \rightarrow SOLITON

Promatrazimo neuski svršaj ($\alpha=0$) gdje se rotaciona dijeli u dva tipa gibanja nelinearnog oscilatora: titanje i rotacija.

Hamiltonian matematičkog ujednake:



$$H = E_{\text{kin}} + E_{\text{pot}} \\ = \frac{mv^2}{2} + mgh$$

$$x = \theta \cdot l - \text{dužina luka}$$

$$v = \dot{x} = l\dot{\theta}$$

$$h = l - l \cos \theta = l(1 - \cos \theta)$$

zapomnja: linearizacija za $\theta \ll 1$:

$$\cos \theta \approx 1 - \frac{\theta^2}{2}$$

$$H = \frac{1}{2}ml^2\dot{\theta}^2 + mgl \frac{\theta^2}{2}$$

$$= \frac{1}{2}m(l\dot{\theta})^2 + m \frac{g}{l} \frac{(l\theta)^2}{2}$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2$$

$\omega^2 = \frac{g}{l}$

$$H = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos \theta) \quad | : ml^2$$

skalirajući energiju:

$$\tilde{H} = \frac{1}{2}\dot{\theta}^2 + \left(\frac{g}{l}\right)(1 - \cos \theta)$$

$$\tilde{H} = \frac{H}{ml^2}, \quad \omega \equiv \sqrt{\frac{g}{l}}$$

$$\tilde{x} \rightarrow \begin{bmatrix} \frac{1}{2} \ln \omega^2 \\ \frac{1}{2} \end{bmatrix} = \\ = \begin{bmatrix} N \text{ s} \\ \frac{1}{2} \ln \omega^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \ln \frac{N^2}{2} \\ \frac{1}{2} \ln \omega^2 \end{bmatrix} = \\ = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \Leftrightarrow \omega^2$$

① separativni iznos za $\theta = \pi$ tj. $\tilde{H} = \tilde{E}_{\text{pot}}^{(\text{max})}$:

$$\tilde{E}_{\text{pot}}^{(\text{max})} = \omega^2(1 - \cos \pi) = 2\omega^2$$

$$2\omega^2 = \frac{1}{2}\dot{\theta}^2 + \omega^2(1 - \cos \theta) \quad | \cdot 2$$

$$\dot{\theta}^2 = 2\omega^2(1 + \cos \theta)$$

$$\dot{\theta} = \sqrt{2\omega^2 \sqrt{1 + \cos \theta}}$$

$$\dot{\theta} = 2\omega \sqrt{\frac{1}{2}(1 + \cos \theta)}$$

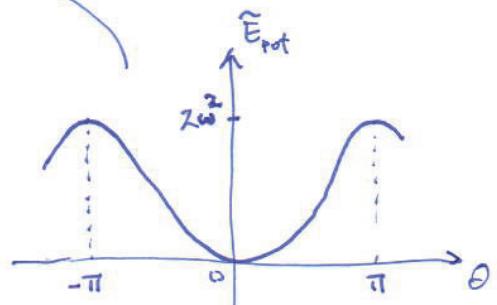
$$\dot{\theta} = 2\omega \cos \frac{\theta}{2} \quad \rightarrow \text{fiksno gibanje} \\ \text{za separativnu}$$

$$\frac{d\theta}{\cos \frac{\theta}{2}} = 2\omega dt \quad | \int \quad \text{Brojtegju int 325}$$

$$2 \ln \operatorname{Tg} \left(\frac{\theta}{4} + \frac{\pi}{4} \right) = 2\omega t + C \quad | : 2$$

$$\ln \operatorname{Tg} \left(\frac{\theta}{4} + \frac{\pi}{4} \right) = \omega t + \tilde{C}$$

$$\theta(t) = 4 \arctan \operatorname{Tg} e^{\omega t + \tilde{C}} - \pi \quad \rightarrow \text{optičke vjetraje}$$



78

asimetriju prati ujet da vremenski profil $\theta(t)$ u $t=0$
biti $\theta(0) = 0$:

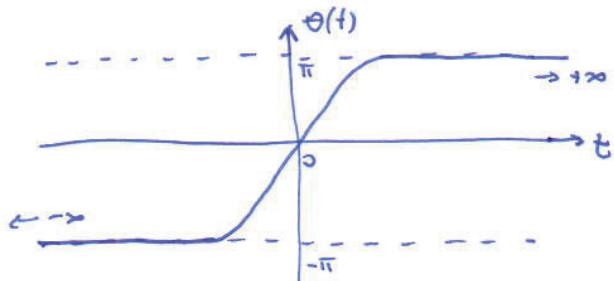
$$0 = \ln \operatorname{arc tg} \frac{e^{\tilde{C}}}{\tilde{C}} - \pi \Rightarrow \operatorname{arc tg} \tilde{C} = \frac{\pi}{4} \Rightarrow \tilde{C} = \tan \frac{\pi}{4} = 1 \Rightarrow \underline{\tilde{C} = 1}$$

$$\Rightarrow \boxed{\theta(t) = \ln \operatorname{arc tg} e^{wt} - \pi}$$

$t=0$: $\theta(0) = \ln \operatorname{arc tg} 1 - \pi = 0$

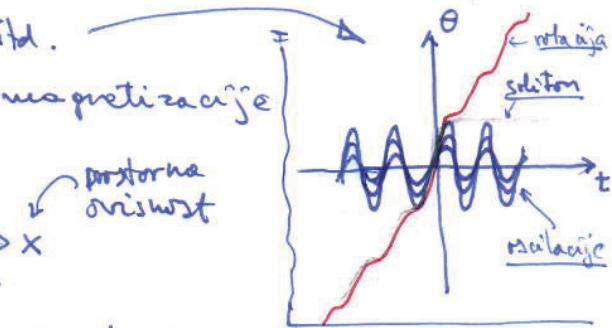
$t=-\pi$: $\theta(-\pi) = \ln \operatorname{arc tg} 0 - \pi = -\pi$

$t=\pi$: $\theta(\pi) = \ln \operatorname{arc tg} \infty - \pi = \pi$



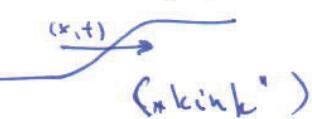
ovo rješenje zove se SOLITON!

Solitonsko rješenje jedva se može separirati između 2 tipa
gibanja, 2 tipa unutarnjih mreža itd.
Npr. dva dimenzionalna zid između 2 polova magnetizacije



Ako soliton ima prostorno i vremensku osinost, tad se arc tg - profil
giba prostorno u vremenu - solitarni val

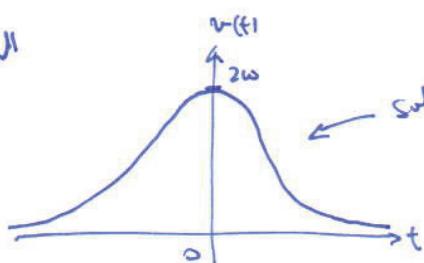
To je nelinearni val posebnih svojstava.



brzina:

$$v = \dot{\theta} = \ln \frac{w e^{wt}}{1 + e^{2wt}} = 2w \frac{1}{\frac{e^{wt} + e^{-wt}}{2}}$$

$$v(t) = \frac{2w}{\cosh wt}$$



soliton-like profil

cti za $t \rightarrow \pm \infty$, brzina pada
na nulu.

Valna jednadžba i nelinearnost

- linearna linearna valna jednadžba (d'Alembert)

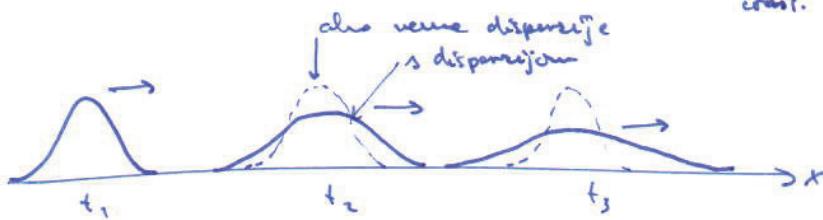
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0 ; v - brzina širenja vala$$

- rješenje: $u(x,t) = (A e^{i k x} + B e^{-i k x}) e^{-i \omega t} \rightarrow$ općeniti napis:

$$u(x,t) = f(x-vt) + g(x+vt)$$

postupak linearizacije operatore: $(\frac{\partial}{\partial t} \pm v \frac{\partial}{\partial x}) (\frac{\partial}{\partial t} \mp v \frac{\partial}{\partial x}) u = 0$
zadržimo samo to

$$\Rightarrow \boxed{u_t + vu_x = 0} \rightarrow \text{rješenje: } u(x,t) = f(x-vt) \xrightarrow{\text{const.}} \rightarrow \text{putajući val (linearni):}$$



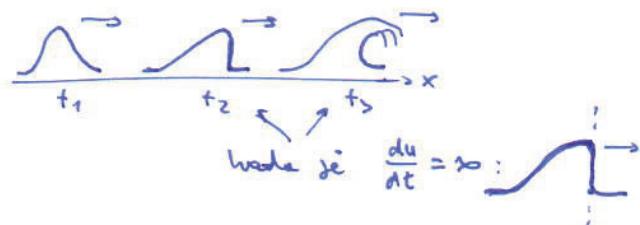
$\omega(k)$

$\omega_k = \frac{\partial \omega}{\partial k}$ - grupna brzina
(u velikom paketu varijante k - komponente imaju različite brzine širenja)

nelinearnost:

1) $v = v(u)$: - onjnost brzine ovisno o amplitudi vala

rješenje je implicitno dano s: $u(x,t) = f(x - v(u)^{1/2} \cdot t) \Rightarrow$ time se uvodi nelinearnost prelacišnja vala



2) nelinearne disperzija

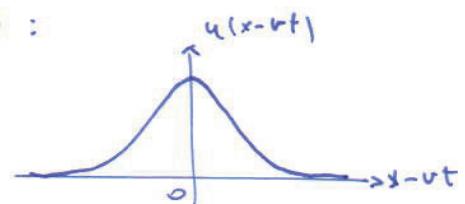
$$\text{npr. } \omega(k) = v \cdot k - \beta k^3$$

npr. Kortevég-de-Vriesova jednadžba (KdV):

$$\boxed{u_t + (v+u)u_x + \beta u_{xxx} = 0} \quad (\underline{\underline{KdV}})$$

→ rješenje (npr. načinom inverzne transformacije):

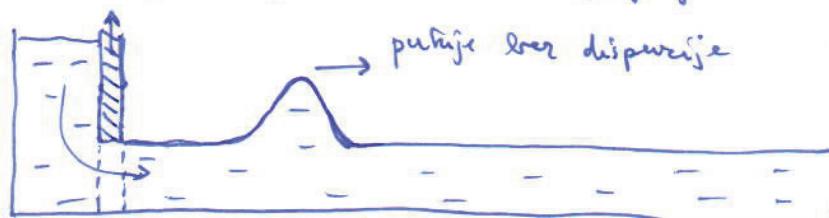
$$u(x-vt) = \frac{3\alpha}{ch^2 \left[\sqrt{\frac{\alpha}{4\beta}} (x-vt) \right]} \quad - \text{soliton!}$$

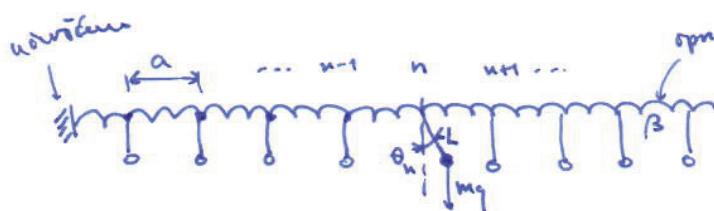


svojstva: - ne vrijedi princip superpozicije

- ne mijenja profil vala se propagira međijem

pohar:





male amplitude: obični disperzionalni val



soluton: kontinuiran početno ujednako re puni temelj; kink se svaki bes propagiraju profila.

moment inercije
ujednako

$$\Gamma_{1n} = -mgL \sin \theta \rightarrow \text{podesivoj ujednako}$$

$$\Gamma_{2n} = -\beta(\theta_n - \theta_{n-1}) - \beta(\theta_n - \theta_{n+1})$$

\hookrightarrow Hookejeva elastičnost varira ujednako s prelaskom (konst. elastičnosti β)

$$\text{def: } \omega_0^2 \equiv \frac{mgL}{I}$$

$$c_0^2 \equiv \frac{\alpha^2 \beta}{I}$$

$$\ddot{\theta}_n + \omega_0^2 \sin \theta_n = \frac{c_0^2}{a^2} (\theta_{n+1} + \theta_{n-1} - 2\theta_n), \quad n = 1 \dots N$$

approx. kontinuirane: $\theta_n(t) \rightarrow \theta(x, t)$ (θ je sprem mijenja u vremenu)

$$\theta_{n+1} = \theta \pm a \frac{\partial \theta}{\partial x} + \frac{a^2}{2} \frac{\partial^2 \theta}{\partial x^2} \pm \dots$$

za $\beta \gg mgl$ ili $\frac{c_0^2}{a^2} \gg \omega_0^2$, torsija je puno jača od gravitacije

$\Rightarrow \frac{c_0}{\omega_0} \equiv d \gg a$ - ujet na karakteristične duljine (za $d \approx a$ nem θ stvarno varira među susjedima pa kontinuiran je ujedn.)
parametri diskretizaci

$$\boxed{\frac{\partial^2 \theta}{\partial t^2} - c_0^2 \frac{\partial^2 \theta}{\partial x^2} + \omega_0^2 \sin \theta = 0}$$

~ inercija ~ torsija ~ gravitacija

- SINE-GORDON EQ.; ne-linearna valna jednadžba, cijekatko je integralna

rešenje:

(10) male amplitude \Rightarrow linearizacija \rightarrow KLEIN-GORDON EQ. (SG)

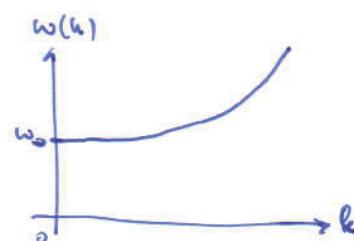
$$\ddot{\theta} - c_0^2 \frac{\partial^2 \theta}{\partial x^2} + \omega_0^2 \theta = 0 \rightarrow$$

$$\theta = \theta_0 \cos(kx - \omega t)$$

: (kontinuiran i periodičan)

$$\omega(k) = \sqrt{\omega_0^2 + c_0^2 k^2} \rightarrow \text{disperzionalna relacija}$$

cut-off: valovi s $\omega < \omega_0$ ne mogu se propagirati (imaginarni k)



(2^v) Velike amplitude - kink soliton

81

def. $T \equiv \omega_0 t$, $X_0 \equiv \frac{\omega_0}{c_0} x$

$$\Rightarrow \frac{\partial^2 \theta}{\partial T^2} - \frac{\partial^2 \theta}{\partial X^2} + \sin \theta = 0 \quad \text{||} \quad (\text{homogeni obliki jednačine})$$

- Tražimo lokalizirani val stavlja oblike profile

$$\theta = \theta(s) = \theta(X - uT) \quad (s \equiv X - uT)$$

↑ normalizirana varijabla; $u \in (-1, 1)$

lokalizirano rješenje: $\theta \rightarrow 0$ & $\frac{d\theta}{ds} \rightarrow 0$ kad $s \rightarrow \pm\infty$

- postupak:

$$d\theta = \frac{\partial \theta}{\partial X} dX + \frac{\partial \theta}{\partial T} dT = \frac{\partial \theta}{\partial s} ds - u \frac{\partial \theta}{\partial s} dt \quad \left(\frac{\partial}{\partial X} = \frac{\partial}{\partial s}, \frac{\partial}{\partial T} = -u \frac{\partial}{\partial s} \right)$$

$$\Rightarrow (1-u^2) \frac{d^2 \theta}{ds^2} = \sin \theta \quad | \frac{d\theta}{ds}$$

$$\frac{d^2 \theta}{ds^2} \cdot \frac{d\theta}{ds} - \frac{1}{1-u^2} \sin \theta \frac{d\theta}{ds} = 0$$

$$\frac{d}{ds} \left[\frac{1}{2} \left(\frac{d\theta}{ds} \right)^2 + \frac{1}{1-u^2} \cos \theta \right] = 0 \quad \text{||} \quad \int \int$$

check: $\frac{1}{2} \cdot 2 \frac{d\theta}{ds} \cdot \frac{d^2 \theta}{ds^2} - \frac{1}{1-u^2} \sin \theta \frac{d\theta}{ds} = 0$
 $\frac{d^2 \theta}{ds^2} - \frac{1}{1-u^2} \sin \theta = 0 \quad \checkmark$

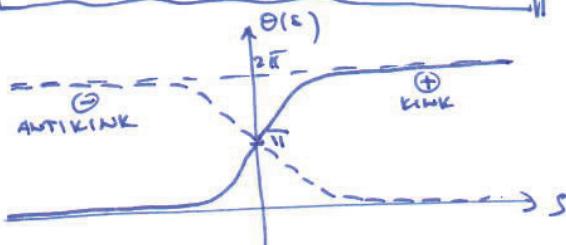
$$\frac{d\theta}{ds} = \pm \sqrt{\frac{2(C - \cos \theta)}{1-u^2}}, \quad \text{za } \theta \rightarrow 0 \text{ & } \frac{d\theta}{ds} \rightarrow 0 \quad \text{kad } s \rightarrow \pm\infty \quad \text{xi } \underline{C=1}$$

rij: $\frac{s-s_0}{\sqrt{1-u^2}} = \int_{\theta(s_0)}^{\theta(s)} \frac{d\theta}{\sqrt{2(1-\cos \theta)}} \quad , \quad s_0 = X_0 - uT_0$

$$\pm \frac{s-s_0}{\sqrt{1-u^2}} = \ln \left(\frac{\theta}{4} \right)$$

za $s_0 = \mp l$ stjedi:

$$\theta(s) = \text{atan} \left[e^{\pm \frac{s-s_0}{\sqrt{1-u^2}}} \right] \rightarrow \text{soliton!}$$



+ : KINK
- : ANTIKINK

Čestica magistru kink solitona:

Energija: $E = \sum_{n=1}^N \left[\frac{I}{2} \left(\frac{d\theta_n}{dt} \right)^2 + \frac{I}{2} \left(\theta_{n+1} - \theta_n \right)^2 + mgL(1 - \cos \theta_n) \right]$

- aproks. kontinuum: $\frac{\partial \theta}{\partial x} \approx \frac{1}{a} (\theta_{n+1} - \theta_n)$

$$E = \frac{I}{a} \omega_0 c_0 \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{d\theta}{dt} \right)^2 + \frac{1}{2} \left(\frac{d\theta}{dx} \right)^2 + (1 - \cos \theta) \right] dx =$$

kontinuum
varijabla $d\theta/ds$

$$\approx \hbar \frac{I}{a} \omega_0 c_0 \frac{1}{\sqrt{1-u^2}} \left(1 + \frac{1}{u^2} + \frac{1}{1-u^2} \right) = 8 \frac{I}{a} \frac{\omega_0 c_0}{\sqrt{1-u^2}} \quad \text{ili}$$

$$= \frac{\omega_0 c_0}{\sqrt{1 - \frac{v^2}{c_0^2}}} \rightarrow \text{"relativistička forma"} \text{ gdje je } \omega_0 = 8 \frac{I}{a} \frac{\omega_0}{c_0} = 8 \frac{I}{a} \frac{1}{d} \text{ "masa solitona"}$$

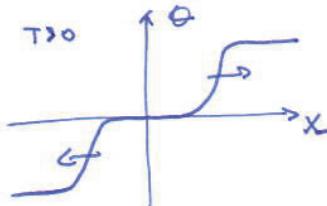
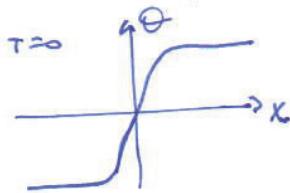
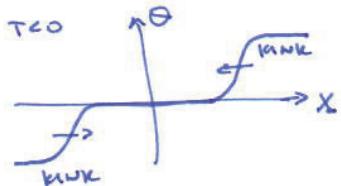
Sudari solitona - ilustracija

Potom se najpoznatijeg oblika rješenja $\Theta(x,t) = \text{arctg} \left[\frac{F(x)}{G(t)} \right]$: mostimo u SG jednadžbi (ima dosta za napraviti).

Nalazimo 3 tipa rješenja:

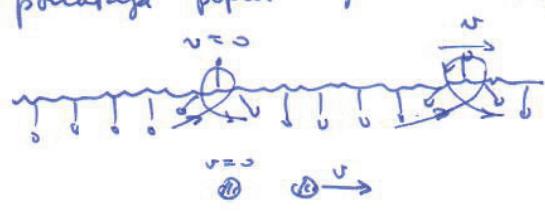
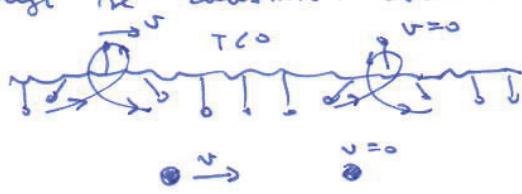
1) KINK-KINK sudar

$$\Theta(x,t) = \text{arctg} \left[a \sin \left(\frac{x}{\sqrt{1-u^2}} \right) \operatorname{sech} \left(\frac{ut}{\sqrt{1-u^2}} \right) \right]$$



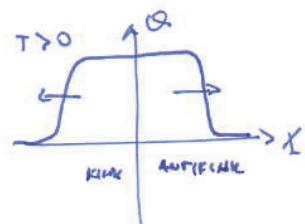
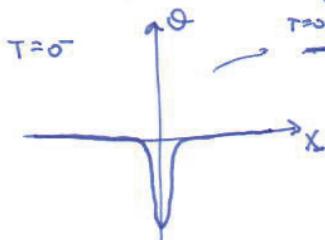
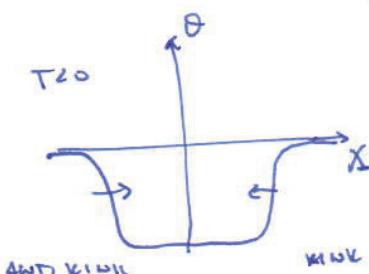
Experiment:

Na trivijalnoj liniji nepravim 2 kink solitona (istog signa rotacije) gdje jedan rotira, a drugi rotira na suprotnu bočinu v u $t=0$. Valov sudara onaj koji je mlađe nestane u gubitku energije v , a drugi se zaustavi. Soliton se u mlađem poravnava poput biljnih kugli.



2) KINK-ANTIKINK sudar

$$\Theta(x,t) = \text{arctg} \left[\frac{1}{u} \operatorname{sech} \left(\frac{x}{\sqrt{1-u^2}} \right) \sin \left(\frac{ut}{\sqrt{1-u^2}} \right) \right]$$



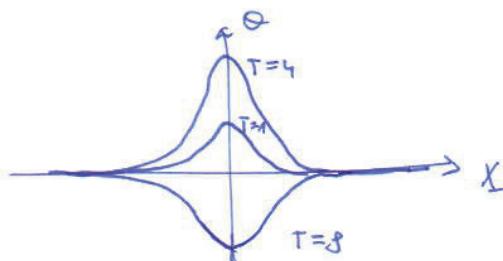
Uz to je njihova godišnja rotacija valovom bočinom i velikom
valovom sudaru se ne anihiliraju! (prot. do nene dissipacije)

3) BREATHER soliton

To je verouostno stvarje kink i antikink solitona koji se okreću za male
veline gibanja solitona.

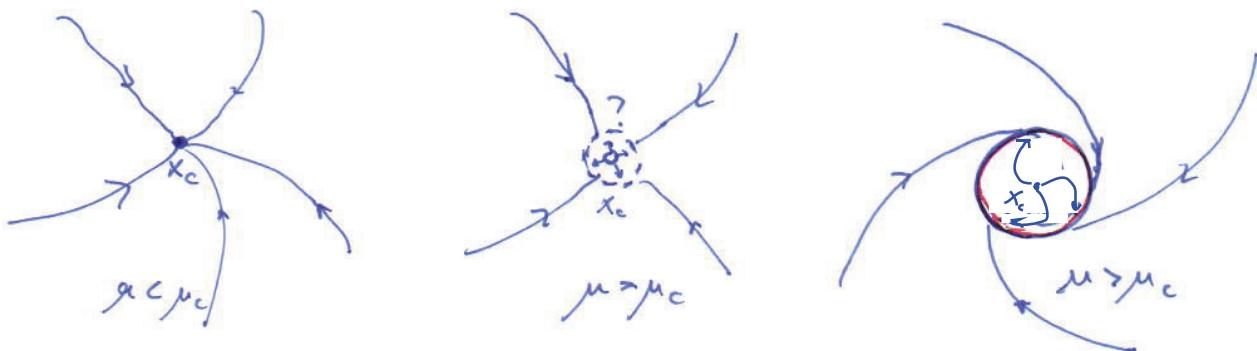
$$\Theta(x,t) = \text{arctg} \left[\frac{\sqrt{1-u^2}}{u} \operatorname{sech} \left(\pi \sqrt{u^2-1} x \right) \sin (\pi t) \right]$$

Karakteristične veličine
(sududi parametri modela)



Promatrymo ritmačju u kojoj je Jacobian singularan ($\det A = 0$, imamo isčeravajuću svojstvenu vrijednost $\lambda_1 = 0$). Tada nije samo fix. točka, nego je cijela kontinuirana linija singularnosti.

Promatrymo sljedeći scenarij: Imamo stabilnu fix. točku i mijenjanje parametra μ . F.T. za $\mu = \mu_c$ izgubi stabilnost. Pitajuće je: kakva struktura je u odnosu (veličina se u 2D svemiru portom)?



(*) Globalni formi tok je \rightarrow dolji ide u smjeru x_c koja bolje postane nestabilna. Jedini način da se to topološki ostvari je nastanak zatvorenih kontinuiranih linija okolo x_c koja je stabilna \rightarrow to je GRANIČNO KOLO (LIMIT CYCLE).

[primjer] (najjednostavniji) - u polarnom sustavu - primjer grančnog kola

$$\begin{aligned}\dot{\varphi} &= \omega & \rightarrow F_\varphi \\ \dot{r} &= \omega r (\tau_c - r) & \rightarrow F_r \\ &\quad \text{~\small KONTROLNI PARAMETAR}\end{aligned}$$

$$\underline{T_1}: \quad A_1 = \begin{pmatrix} 0 & \omega \\ 0 & \omega r_c \end{pmatrix} \Rightarrow \lambda_1(\omega r_c - \lambda) = 0 \\ \quad \begin{pmatrix} \lambda_1 = 0 \\ \lambda_2 = \omega r_c \end{pmatrix}$$

$$\text{F.T.} \quad \boxed{\begin{array}{l} \tau_1 = \phi \\ \tau_2 = r_c \end{array}}_{(\omega)} \quad , \quad A = \begin{bmatrix} 0 & 0 \\ 0 & \omega(r_c - 2r) \end{bmatrix}$$

$$\det A = 0 \Rightarrow \text{singulare!}$$

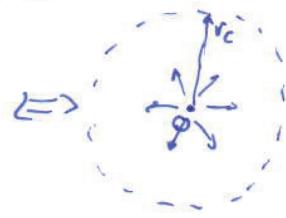
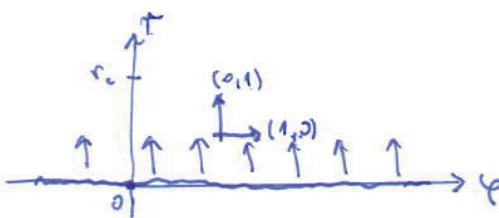
sv. vektori:

$$\lambda_1 = 0: \quad \begin{bmatrix} 0 & \omega \\ 0 & \omega r_c \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \quad \begin{array}{l} \omega A + \omega B = 0, \# A, B \\ \omega r_c B = 0 \Rightarrow B = 0, \# A \end{array} \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = \omega r_c: \quad \begin{bmatrix} -\omega r_c & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \quad \omega r_c A = 0 \rightarrow A = 0, \# B \rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{\omega r_c t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

exp. u vremenu t ne može biti
ak $r_c > 0$ duž $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ mjerila



$$\underline{\underline{J}_2} : \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -dr_c \end{pmatrix} \rightarrow \lambda_1(dr_c + \lambda) = 0$$

$\lambda_1 = 0$
 $\lambda_2 = -dr_c$

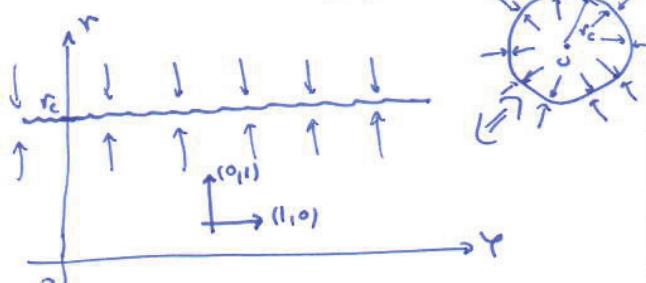
sv. režim:

$$\lambda_1 = 0: \begin{pmatrix} 0 & 0 \\ 0 & -dr_c \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \rightarrow dr_c B = 0 \rightarrow B = 0, A \neq 0 \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

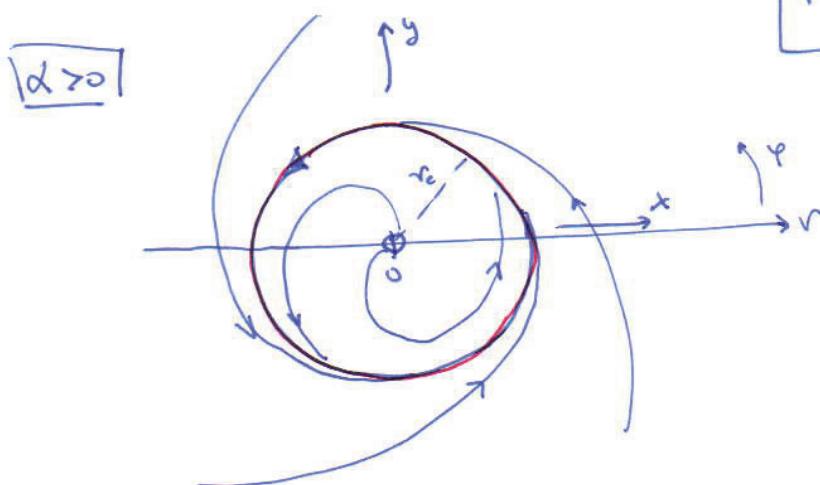
$$\lambda_2 = -dr_c: \begin{pmatrix} dr_c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \rightarrow dr_c A = 0 \rightarrow A = 0, B \neq 0 \rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x(t) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-dr_c t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

exponentijalno u vremenu konvergira ka $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ($r=r_c$) do $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ svima



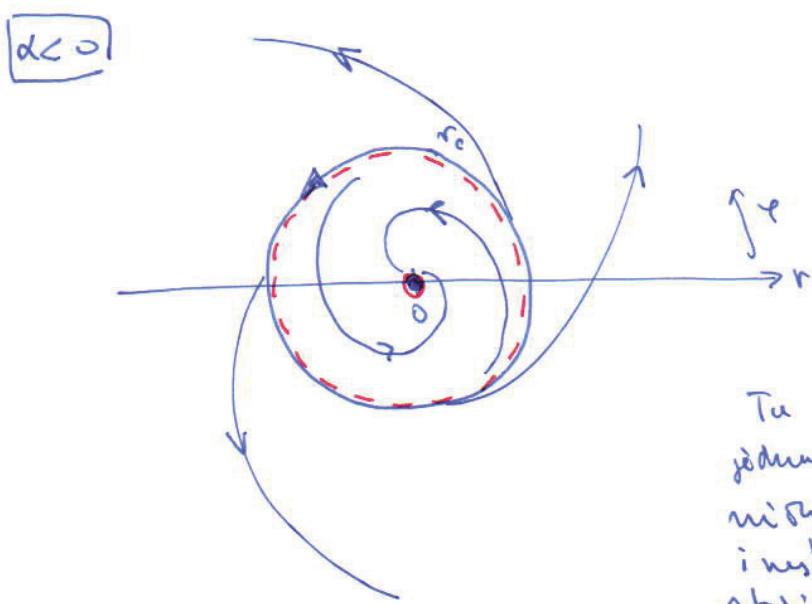
\Rightarrow stabilita u polarnom sustavu:



pretpostavka stabilita $r=r_c$:

$$r(\varphi) = r_c - (r_c - r(0)) e^{-\frac{-dr_c \varphi}{\omega}}$$

$\lim_{t \rightarrow \infty} r(t) = r_c, \forall \varphi = \omega t$
 $\hookrightarrow x(t) = r_c \cos \omega t = r_c \cos \omega t$
 $y(t) = r_c \sin \omega t = r_c \sin \omega t$



Tačka od prethodne mogle jednu fiz. točku i jednu građivu kolo (nisti nije nastalo i nestalo) mogu da zauzimaju stabilnost.

Ljekva derivacija:

$$\mathcal{L} = \frac{\partial F_p}{\partial \dot{r}} + \frac{\partial F_r}{\partial r} = \alpha(r_c - 2r) \rightarrow$$

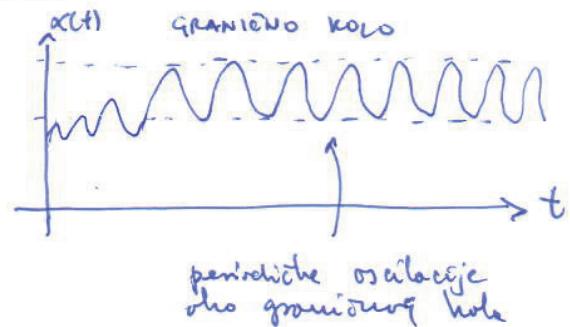
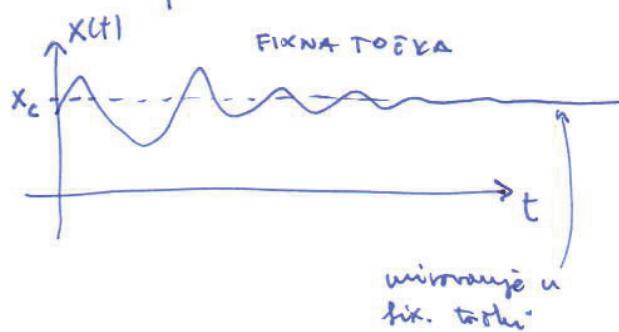
$\alpha > 0$	$\alpha < 0$
$\mathcal{L} > 0, r < \frac{r_c}{2}$	$\mathcal{L} > 0, r > \frac{r_c}{2}$
$\mathcal{L} < 0, r > \frac{r_c}{2}$	$\mathcal{L} < 0, r < \frac{r_c}{2}$

Sustav je, doble, dissipativan.

Građeno kolo ne može se pojaviti u konzervativnim linearnim sustavima.

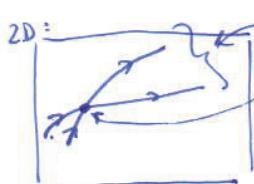
Sustav može biti bitno nekonzervativni i dissipativni. Građeno kolo putem je drugačiji objekt od periodične trajektorije u Hamiltonovom sustavu koji obavija centar (tako si $f=0$).

① Vrijnost građenog kola u fizici je jer za vektori od fixne točke dissipativnog sustava, u kojoj se sustav zastavlja, sustav mogi pada u građeno kolo nastavljajući gibanje periodičnim oscilacijama.



Građeno kolo modelira sustave koji zlog svoje konstrukcije, bez varijacije periodičnih sile tijekom, padaju u redinu spontano excitiranih oscilacija!

② U sustaru 2. reda građeno kolo je jedini topoloski mogući objekt ložetiji od točke da bude atraktor faraz toke. Ništa komplikovanije nije moguće jer li se trajektorije inače nisu mogle sjeti.



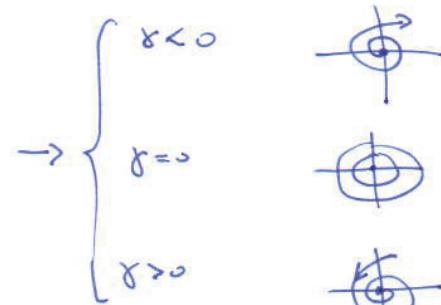
Druž mnoštvo ljudskosti dlovi od iste prostote (to nije F.T. nego spušte). To nemojte teorijski determinizam, ili žive teoreme o ogranicenju i jedinstvenosti rješenja d.-j.

(*) U sustaru dimenziji $d > 2$ trajektorije se mogu zaočiti. U vrijeme kad možete plnos ložetije strukture kao atraktori faraz toke (strani, oduševni atraktori).

Model sastav - van der Polov oscilator

- postavio od gusenog H.O.

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = 0$$



Kako postići crvanje kolo po
kjem će se sustav periodički gibati?

A ovakvom sustavu to nije moguće jer je linearan. Možemo sustav niniti nelinearnim i atvoerenim (za $\gamma < 0$ treba organizati dve energije da ne eksplodiraju, a za $\gamma > 0$ treba dodati energiju u sustav da se ne zagubi).

Jedan način da se to ostvari je nelinearni trenje (gumenje)

$$\gamma(x) = -\gamma_0 \cdot \left(1 - \frac{x^2}{x_0^2}\right) \quad \leftarrow \text{van der Polov model}$$

↑ nelinearnost (x^2 do ne mijenja predznak pri $x \rightarrow -x$)
dissipacija mijenja predznak ovisno o amplitudi;

$x > x_0 \Rightarrow \gamma(x) > 0$ - dissipacija

$x < x_0 \Rightarrow \gamma(x) < 0$ - antidisipacija

$$\ddot{x} - \gamma_0 \left(1 - \frac{x^2}{x_0^2}\right) \cdot \dot{x} + \omega^2 x = 0 \quad | : \omega^2 \quad | : x_0 \quad \leftarrow \text{skaliranje}$$

↑ autoumni sustav (nove sile tijekom)

$$\frac{d^2}{d(\omega t)^2} \left(\frac{x}{x_0} \right) - \left(\frac{\gamma_0}{\omega} \right) \cdot \left(1 - \left(\frac{x}{x_0} \right)^2 \right) \frac{d}{d(\omega t)} \frac{x}{x_0} + \frac{x}{x_0} = 0$$

$\equiv \ddot{t}$

$\equiv x$

$\equiv \mu$

$$\begin{aligned} \ddot{t} &\equiv \omega t && \rightarrow \text{frekvencija je stavljanje u shahu s vremenom} \\ x &\equiv \frac{x}{x_0} && \\ \mu &\equiv \frac{\gamma_0}{\omega} && \end{aligned}$$

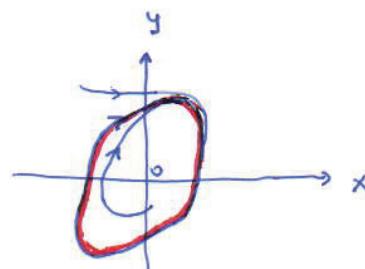
van der
Polova jednadžba

$$\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$$

kontrolni parametar $\mu \geq 0$
(nim skaliranjem imamo 1 kontro. parametar, a prije je bilo 3)

x_0, γ_0, ω

$$\begin{aligned} \dot{x} &= y && :: F_x \\ \ddot{y} &= \mu(1-x^2)y - x && :: F_y \end{aligned}$$



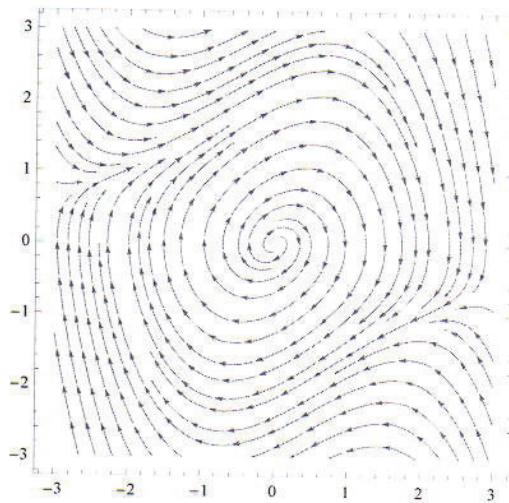
VAN DER POL OSCILLATOR

86-a

```
In[1]:= Fx[x_, y_] := y
```

```
Fy[x_, y_] := m * (1 - x^2) * y - x
```

```
StreamPlot[{y, .5 * y (1 - x^2) - x}, {x, -3, 3}, {y, -3, 3}]
```



```
In[3]:= m = .5;
```

```
h1 = NDSolve[{x'[t] == Fx[x[t], y[t]],
    y'[t] == Fy[x[t], y[t]], x[0] == 1, y[0] == 0}, {x, y}, {t, 0, 100}]
```

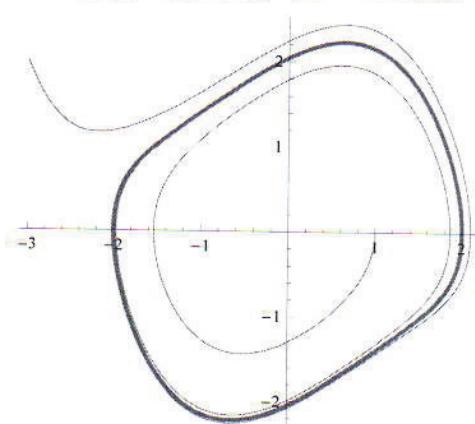
```
h2 = NDSolve[{x'[t] == Fx[x[t], y[t]], y'[t] == Fy[x[t], y[t]], x[0] == -3, y[0] == 2},
    {x, y}, {t, 0, 100}]
```

```
pp1 = ParametricPlot[Evaluate[{x[t], y[t]} /. h1], {t, 0, 100}]
```

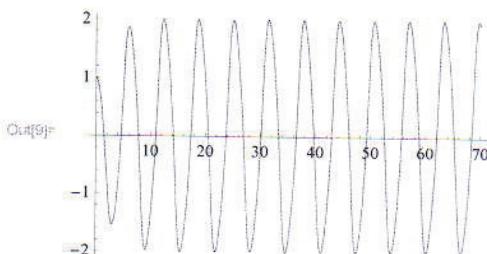
```
pp2 = ParametricPlot[Evaluate[{x[t], y[t]} /. h2], {t, 0, 100}]
```

```
pp3 = ParametricPlot[Evaluate[{x[t], y[t]} /. h1],
    {t, 90, 100}, PlotStyle -> Thickness[.01]]
```

```
Show[{pp1, pp2, pp3}, PlotRange -> All]
```



```
In[9]:= Plot[Evaluate[x[t] /. h1], {t, 0, 70}]
```



■ Kako građivac kolo može biti analitički?

87

- pogledajmo uvađenje u polarne koordinate:

$$\begin{aligned} F_x &= y \\ F_y &= \mu(1-x^2)y - x \end{aligned}$$

F.T.

$$\begin{array}{l} y=0 \\ x=0 \end{array} \quad \xrightarrow{\text{linearnica}} \quad T(0,0)$$

$$A = \begin{bmatrix} 0 & 1 \\ -2\mu xy - 1 & \mu(1-x^2) \end{bmatrix}$$

$$\underbrace{T(0,0)}_{\text{det } A = 1}; \quad A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$$

$$\det A = 1 \quad (\neq 0 !)$$

↓

Neka analiza veliki
znači jednu luku
točku koja je
nestabilna, a asymptotski
formi toh teče ka istodost.

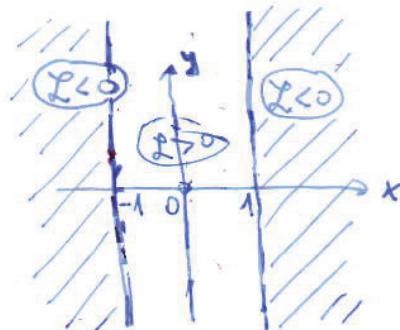
Linijska derivacija: (\dot{x}, \dot{y})

$$\begin{aligned} L &= \mu(1-x^2) \Rightarrow |x| > 1; L < 0 \\ &\quad |x| < 1; L > 0 \end{aligned}$$

Pogledajmo lići građivac kolo, ali
kolo ga odrediti?

sv. vrijednosti:

$$\begin{aligned} -\lambda(\mu - \lambda) + 1 &= 0 \\ \lambda^2 - \mu\lambda + 1 &= 0 \end{aligned} \quad \rightarrow \lambda_{1,2} = \frac{1}{2} \left[\mu \pm \sqrt{\mu^2 - 4} \right] \quad \begin{array}{l} \text{šta de imaju} \\ > 0 \Rightarrow \text{nestabilnost!} \end{array}$$



pokretajmo u polarnom sustavu:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu(1-x^2)y - x \end{aligned}$$

transformacija:

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \end{aligned}$$

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \varphi &= \arctan \frac{y}{x} \end{aligned}$$

$$\begin{aligned} x\dot{x} + y\dot{y} &= y\dot{x} + \mu(1-x^2)y^2 - x\dot{y} \\ \dot{r}r &= \mu(1-r^2 \cos^2 \varphi)r^2 \sin^2 \varphi \\ \dot{r} &= \mu r(1-r^2 \cos^2 \varphi) \sin^2 \varphi \end{aligned}$$

$$\begin{aligned} x\dot{y} - y\dot{x} &= \mu(1-x^2)(x\dot{x} - y\dot{y}) - (x^2 + y^2) \\ &= x^2 \cos \varphi \cdot \sin \varphi \end{aligned}$$

$$\begin{aligned} x^2 + y^2 &= r^2 \quad | \frac{d}{dt} | : 2 \\ r\dot{r} &= x\dot{x} + y\dot{y} \end{aligned} \quad (1)$$

$$\begin{aligned} x &= r \cos \varphi \quad | \frac{d}{dt} \\ \dot{x} &= \dot{r} \cos \varphi - r \sin \varphi \cdot \dot{\varphi} \quad | \cdot y \\ y &= r \sin \varphi \quad | \frac{d}{dt} \\ \dot{y} &= \dot{r} \sin \varphi + r \cos \varphi \cdot \dot{\varphi} \quad | \cdot x \end{aligned} \quad)$$

$$\begin{aligned} x\dot{y} - y\dot{x} &= \dot{r}r(\sin \varphi \cos \varphi - \sin \varphi \cos \varphi) + \\ &+ r^2 \dot{\varphi}(\sin^2 \varphi + \cos^2 \varphi) \end{aligned}$$

$$\dot{\varphi} = \frac{x\dot{y} - y\dot{x}}{r^2} \quad (2)$$

$$\dot{\varphi} = \mu(1-r^2 \cos^2 \varphi) \cos \varphi \sin \varphi - 1$$

→ Opet nista! Isto se vidi da je $r_{c1}=0$ fix. točka (prva), one druga
je neka funkcija od φ tj. $r_{c2} = f(\varphi)$. (Ne mora se dobiti jasno
misao u kojoj je jedan cijeli niz istražujućih mjeri reguliran kao u prethodnom
primjeru.)

Metoda 2 vremenske skale i uvedjivine jidnadizbe

88

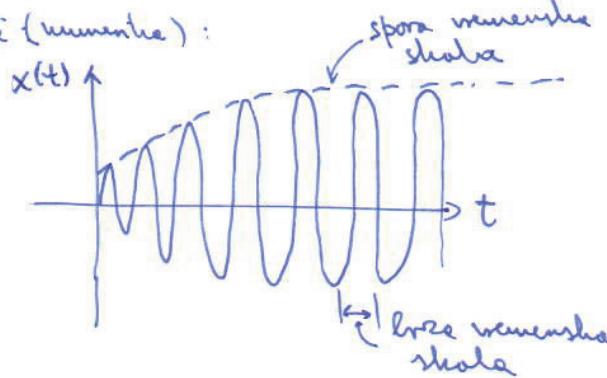
Potpisanih slabo nelinearni sustav \rightarrow parametar koji kontrolira nelinearnost je mali: $\ddot{x} + x + \mu \sin(x, \dot{x}) = 0$

$$\text{v.d.P.: } \ddot{x} - \mu(1-x^2)\dot{x} + x = 0$$

\rightarrow Standardna perturbacijska trijed po molidu μ u osnovnim sustavnim periodima u vredni!

$$x(t, \mu) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \dots \quad (\text{divergencija rezonancije})$$

rješenje (vremenske):



\rightarrow Vidimo da rješenje karakterizuje 2 vremenske skale; $\text{spora } (T) := \text{brza } (t)$.
asimptotsko konvergirajući rješenja ka granicnom vektoru

Apsoksimativne "metode 2 vremenske skale" nulti red aproksimacije pise kao:

$$x(t) = A(t) \cos(\omega t + \phi(t)) \quad ; \quad (\omega \text{ je wt je standardna mridzina sa spora vremenskom skalom } T \text{ i } \omega t \equiv \tilde{t} \text{ itd.})$$

(*) Ako sustav nema "elasticki sat" (varijaciju periodičnih vrednosti tijekom vremena t) izlazi po vidi i dozvoli približno konstantnu.

dalle vremenske:

$$\left\{ \begin{array}{l} x(t) = A(t) \cos t \\ \dot{x}(t) = \dot{A} \cos t - A \sin t \\ \ddot{x}(t) = \ddot{A} \cos t - \dot{A} \sin t - A \sin t \approx -2\dot{A} \sin t - A \sin t \end{array} \right. \quad \text{jednostavno je } A(t) = A \text{ funkcija od } t \quad (\dot{A} = 0)$$

$$-2\dot{A} \sin t - A \sin t - \mu(1-A^2 \cos^2 t)(\dot{A} \cos t - A \sin t) + A \cos t = 0$$

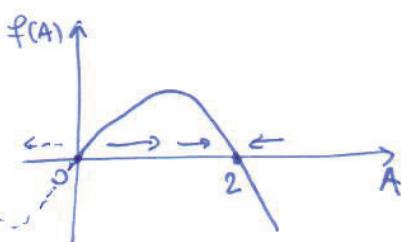
$$-2\dot{A} \sin t - \mu \dot{A} \cos t + \mu A \sin t + \mu A^2 \dot{A} \cos^3 t - \mu A^3 \cos^2 t \sin t = 0$$

$$-2\dot{A} \frac{1}{2} + \mu A \frac{1}{2} - \mu A^3 \cdot \frac{1}{8} = 0$$

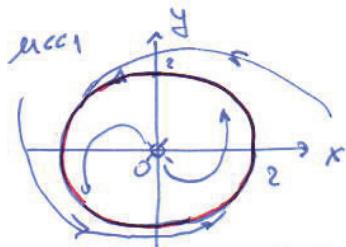
$$\dot{A} = \frac{1}{8} \mu A \cdot (4 - A^2) \quad \text{f(A)}$$

F.I.

$$\begin{cases} A_1 = 0 & \rightarrow \text{istodobni} \\ A_2 = 2 & \rightarrow \text{granicni vektor!} \end{cases}$$



\Rightarrow



(uvedjivo je gr. vektor - nema vektore momenti)

$\int_{-\pi}^{\pi} \sin^2 t dt = \frac{1}{2} \pi$
 \uparrow
uvedjivo je gr. vektor po brzoj vremenskoj skali!

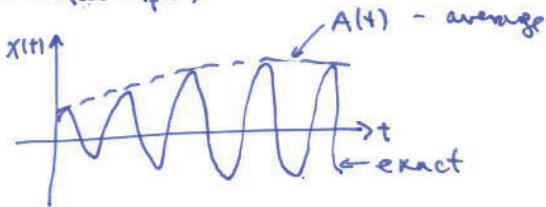
$$\begin{aligned} \langle \sin^2 t \rangle &= \langle \cos^2 t \rangle = \frac{1}{2} \\ \langle \sin t \cdot \cos t \rangle &= 0 \\ \langle \sin^3 t \cos t \rangle &= 0 \\ \langle \sin^2 t \cos^2 t \rangle &= \frac{1}{8} \\ \langle \cos^2 t \rangle &= \frac{3}{8} \end{aligned}$$

summary:

$$\text{Za } \ddot{x} + x + \mu h(x, \dot{x}) = 0 ; \quad x(t) \approx A_p(t) \cos(\omega t + \phi_p(t)) :$$

$$\boxed{\begin{aligned} \dot{A} &= \langle h \cdot \sin \varphi \rangle_p \\ A \cdot \dot{\varphi} &= \langle h \cos \varphi \rangle_p \end{aligned}}$$

($\varphi \equiv \omega t$)



Inpr. van der Pol:

$$\begin{aligned} h &= (x^2 - 1)\dot{x} = (A^2 \cos^2 \varphi - 1)(-A \sin \varphi) \Rightarrow \dot{A} = \frac{1}{2}A - \frac{1}{8}A^3 \\ &= -A^2 \cos^2 \varphi + A \sin \varphi \end{aligned}$$

$$A \ddot{\varphi} = \dot{\phi}$$

Tevemi koji pohrani eksistenciju granicnog kola

1 Teme Poincaré-Bendixon

R - zatvoreni i omotačni prostor sa svojim pozitivnim vektorom vez F.T.!



$$\dot{x} = \vec{F}(x) - \text{veličina polje koja sadrži } R$$

Ako postoji trajektorija omotačna u R (prolazi u R i ostaje u R do $t \rightarrow \infty$), tada R sadrži zatvorenu orbitu. (A one je u dissipativnom sustavu granicno kolo.)

2 Liénardov teorem

$$\ddot{x} + f(x) \cdot \dot{x} + g(x) = 0 \quad - \text{Liénardovu jednadžbu (sustav)}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x) - f(x) \cdot y \end{cases}$$

Liénardov sustav ima stabilno granicno kolo ako ishodišta ako $f(x)$ i $g(x)$ zadovljavaju nejče:

(1) f i g su derivabilne $\forall x$

restirujući $\left\{ \begin{array}{l} (2) g(-x) = -g(x) \quad (\text{neparna funkcija}) \\ (3) g(x) > 0 \quad za \quad x > 0 \end{array} \right.$

neki. $\left\{ \begin{array}{l} (4) f(-x) = f(x) \quad (\text{parne funkcije}) \\ (5) F(x) = \int f(x') dx' \quad \text{ima točku jednu pozitivnu nultučku } x=a > 0 \quad \& \quad \lim_{x \rightarrow \infty} F(x) = \infty. \end{array} \right.$

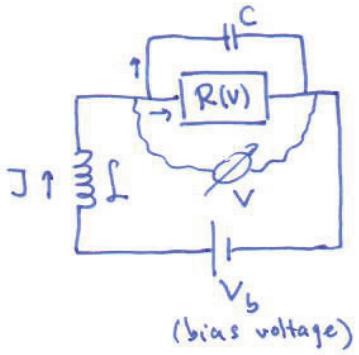
koristi se $F(x) < 0 \quad za \quad 0 < x < a \quad \& \quad F(x) > 0 \quad za \quad x > a$

Inpr. van der Pol: $g(x) = x$, $f(x) = -\mu(1-x^2)$

$$F(x) = -\mu \int_0^x (1-x^2) dx = \frac{1}{3}\mu x \left(\frac{x^2}{2} - 1 \right) \Rightarrow a = \sqrt{3} > 0 \Rightarrow \text{v.d.P. sustav je Liénardov sustav i sadrži stabilno granicno kolo!}$$

$$\cancel{x_F(x)} \quad \cancel{x_0} \quad x_0 > 0 \quad x_0 = \sqrt{3} > 0$$

LRC - kruž s negativnim diferencijalnim otporom



$$I = I_R(V) + I_C \quad \frac{dI}{dt} = C \cdot \frac{dV}{dt}$$

$$\left[\frac{dI}{dt} + V \right] = V_b$$

} 2 jednačine
(za napon i
napon)

$$V' = \frac{1}{C} (I - I_R(V))$$

$$I' = \frac{1}{L} (V_b - V)$$

$$\xrightarrow{\text{F.T.}} V_o = V_b$$

$$J(V_s) = J_R(V_s) \rightarrow J(V_b) = J(V_s)$$

$$\underline{J_R(V) = J(V)}$$

Linearizacija preko F.T.

$$V = V_b + v$$

$$I = I_0 + j$$

$$(*) \quad J_R(V) \approx J_R(V_b) + \frac{\partial J_R}{\partial V} (V - V_b) \quad \begin{matrix} = v \\ v_b \end{matrix}$$

$$= J(V_b) = I_0 \quad \begin{matrix} = g \\ = g - \text{diferencijalna vodljivost} \end{matrix} \quad (\bar{g}' = R_d - \text{dif. otpor})$$

u statickoj situaciji stvaraju kroz
otporak jednako je akupunč
statickoj strujnoj karakteristici ce
(naredno, jer je $\frac{dI}{dt} = 0$)

linearnizirane jednačine:

$$v' = \frac{1}{C} (J_0 + j - J_0 - g \cdot v)$$

$$j' = \frac{1}{L} (Y_b - Y_b - v)$$

$$a_{11} = \frac{\partial F_v}{\partial v} = -\frac{g}{C}, \quad a_{12} = \frac{\partial F_v}{\partial j} = \frac{1}{C}, \quad a_{21} = \frac{\partial F_j}{\partial v} = -\frac{1}{L}, \quad a_{22} = \frac{\partial F_j}{\partial j} = 0 \Rightarrow A = \begin{pmatrix} -\frac{g}{C} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{pmatrix}$$

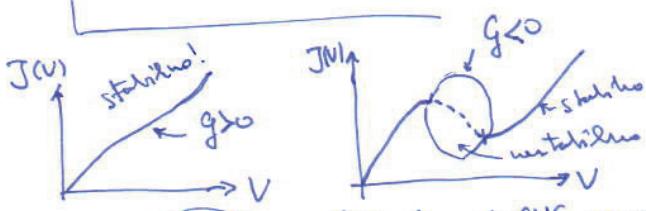
$$\text{so. mjestoverti: } \lambda \left(\frac{g}{C} + 1 \right) + \frac{1}{LC} = 0$$

$$\lambda^2 + \frac{g}{C} \lambda + \frac{1}{LC} = 0$$

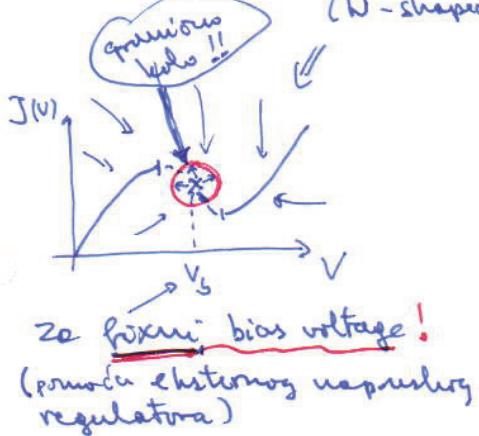
$$\rightarrow \lambda_{1,2} = \frac{1}{2LC} \left[-\frac{g}{C} \pm \sqrt{(\frac{g}{C})^2 - \frac{4}{LC}} \right]$$

ili $\lambda < 0$: realno,
ili imaginarno!

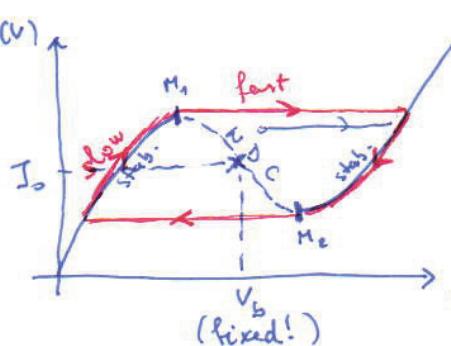
za $g < 0 \Rightarrow \text{NESTABILNOST}$
statische CVC



(N-shaped CVC \rightarrow negativni dif. otpor)



uloga $L \gg \Rightarrow M \approx \text{const}$ (spore promjene struje)



Nauještimo V_b na
NDC dio. On je nestabilan
i nestabilno je stabilni
dio CVC. Ali izvana funkc
nije V_b i nestabilno posti
tej V_b pa blizu po CVC (spore
zbroj $L \gg$), ne $M_{1,2}$ stvaraju
na drugu granu (CVC) i
nestabilno tako oscilanti
 $\Rightarrow \text{OSCILATOR!}$

■ polarizaciju ne kontinuiraju N-shaped CVC

Dalle;

$$\begin{aligned} V' &= \frac{1}{C} (J - J_R(V)) \quad | \quad \frac{d}{dt} \Rightarrow V'' = \frac{1}{C} (J' - J'_R(V)) \\ J' &= \frac{1}{L} (V_b - V) \end{aligned}$$

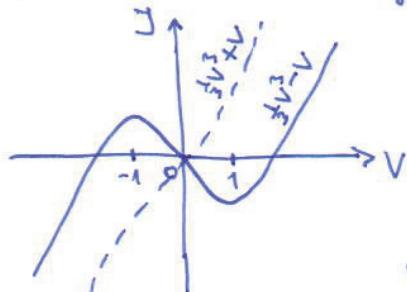
$$V'' + \left(\frac{1}{C} g(V) \right) V' + \frac{1}{LC} (V - V_b) = 0$$

transformiramo jednadžbu tako da joj ishodiste bude u fiksnoj točki određenoj s V_b (bias):

$$V'' + \frac{1}{C} g(V) \cdot V' + \frac{1}{LC} \cdot V = 0$$

\downarrow

Varijabne biderimonsione jedinice:



CVC model: $J(V) = \frac{1}{3} V^3 - \alpha V$; $\alpha > 0 \Rightarrow \text{NDC}$
 $\alpha < 0 \Rightarrow \text{PDC}$

$$g(V) = V^2 - \alpha$$

u biderim. jedinicama: $V'' + (V^2 - \alpha) V' + \frac{1}{\mu} V = 0$

uzimajući u obzir
ilustraciju NDC: $\alpha \geq 1$:

(1°) $V \equiv x$

$$\begin{cases} \dot{x} = y \\ \ddot{y} = (1-x^2)y - \frac{1}{\mu}x \end{cases}$$

$$(\ddot{x} - (1-x^2)\dot{x} + \frac{1}{\mu}x = 0)$$

$\mu \sim L$ kritičke
parametar
oblike
gr. kola

$|V| < 1 \Rightarrow \text{NDC}!$

(2°) (pozdrav
za u>
analizu) Lineardove transformacija $(V \equiv x)$

$$\begin{cases} y \equiv \dot{x} + \frac{x^3}{3} - x \end{cases} \rightarrow | \frac{d}{dt} \Rightarrow \begin{cases} \dot{y} = \ddot{x} + x^2 \dot{x} - \dot{x} \\ \dot{x} = x - \frac{x^3}{3} + y \end{cases}$$

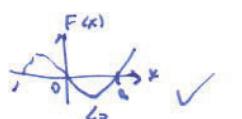
notice: $\ddot{x} - (1-x^2)\dot{x} = \frac{d}{dt} \left\{ x - \left(x - \frac{x^3}{3} \right) \right\} \equiv y$

jednadžba: $\ddot{x} - (1-x^2)\dot{x} + \frac{1}{\mu}x = 0$
 $\Rightarrow \ddot{x} = (1-x^2)\dot{x} - \frac{x}{\mu}$

$$\begin{aligned} \ddot{x} &= -(1-x^2)\dot{x} + (1-x^2)\dot{x} - \frac{x}{\mu} \\ &= -\frac{1}{\mu}x \end{aligned}$$

$$\Rightarrow \begin{cases} \dot{x} = x - \frac{x^3}{3} + y \\ \dot{y} = -\frac{1}{\mu}x \end{cases}$$

Liénardov th.: $f(x) = -(1-x^2)$, $g(x) = \frac{1}{\mu}x \Rightarrow F(x) = - \int_0^x (1-x^2) dx = \frac{1}{3}x(x^2-3) \Rightarrow \alpha = \sqrt{3} > 0$



\Rightarrow netačno i ne stabilno grančno kolo!

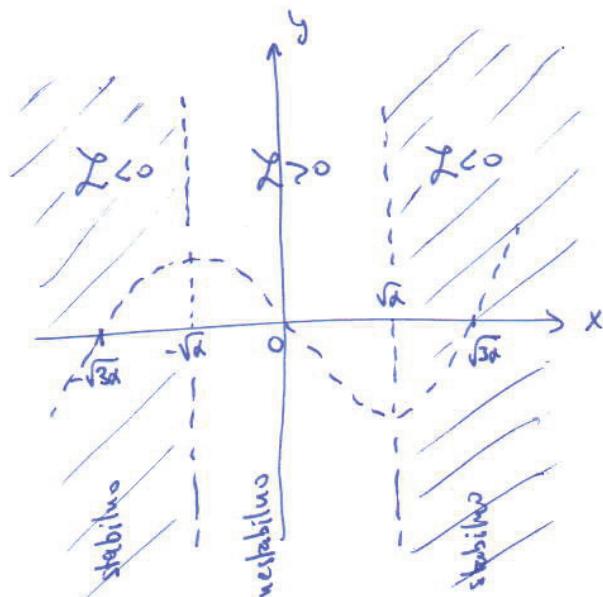
Lieye derivación:

$$F_x = \alpha x - \frac{x^3}{3} + y$$

$$F_y = -\frac{x}{\mu}$$

$$\Sigma = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \alpha - x^2$$

$$\Sigma : \begin{cases} > 0, |x| < \sqrt{\alpha}, \\ < 0, |x| > \sqrt{\alpha}, \end{cases} \begin{cases} \text{NDE } (\alpha > 0) \\ \text{PDC } (\alpha < 0) \end{cases}$$



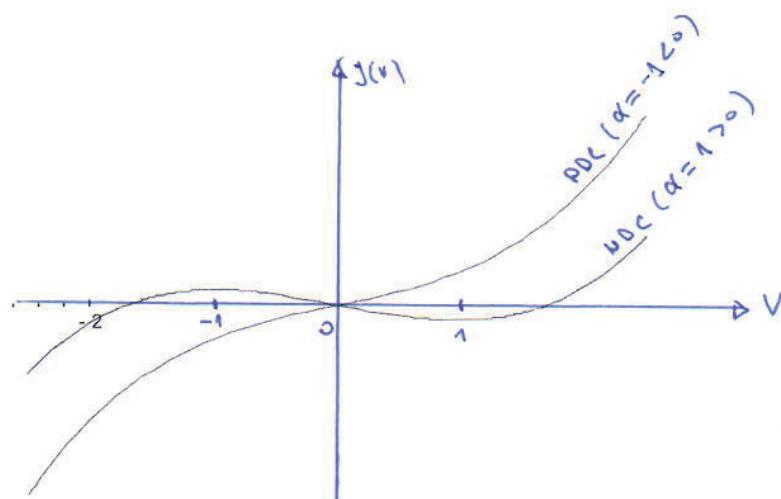
LRC kring \rightarrow positionell/negativum differenzialgleichung von v

g2-a

$$\ddot{x} - (\alpha - x^2)\dot{x} + \frac{1}{\mu}x = 0 ; v \sim x, \dot{v} \sim \dot{x}$$

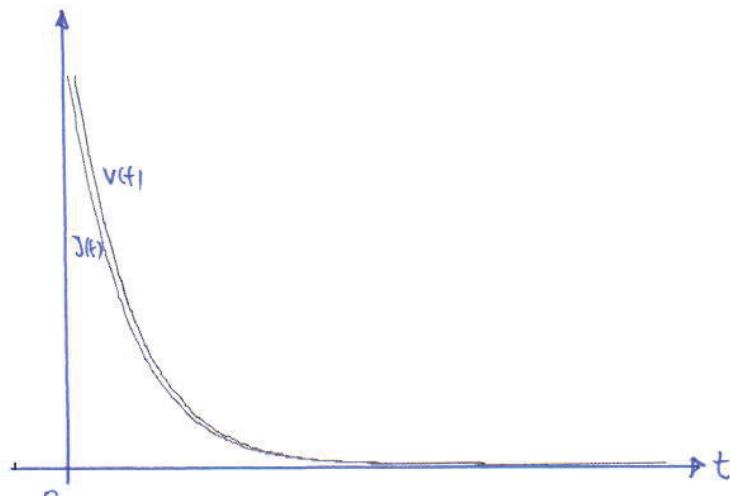
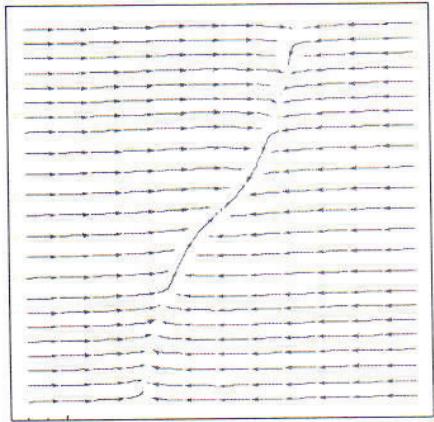
$$\dot{x} = \alpha x - \frac{x^3}{3} + y$$

$$\ddot{y} = -\frac{1}{\mu}x$$



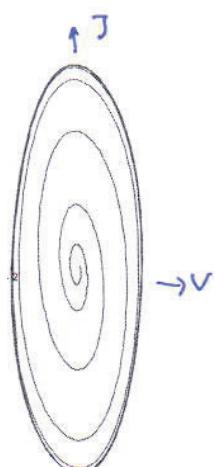
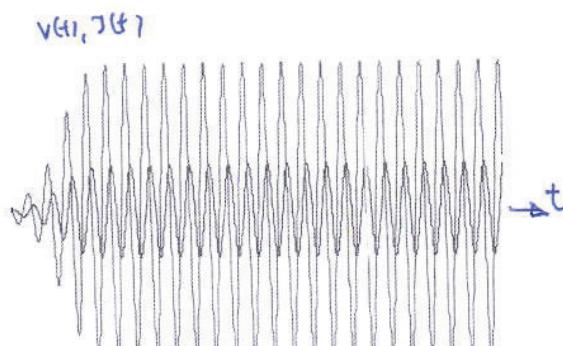
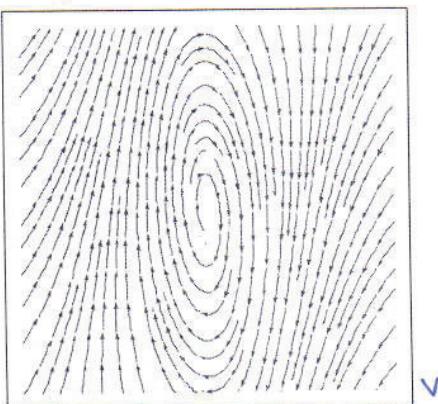
PDC
 $\alpha = -1 < 0$ - stabilität!

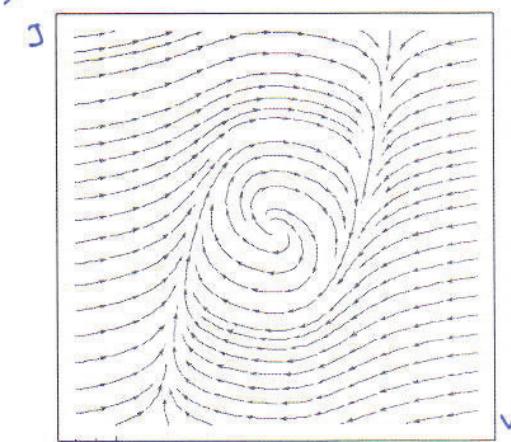
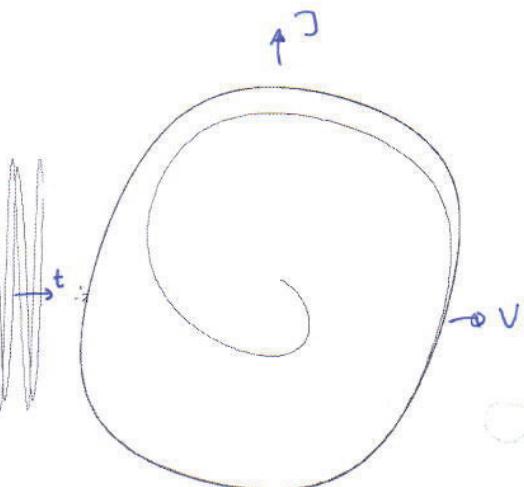
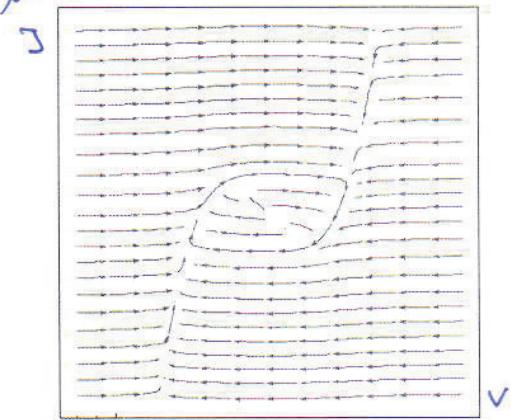
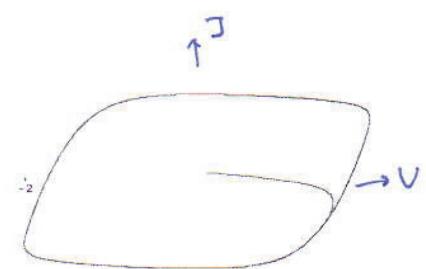
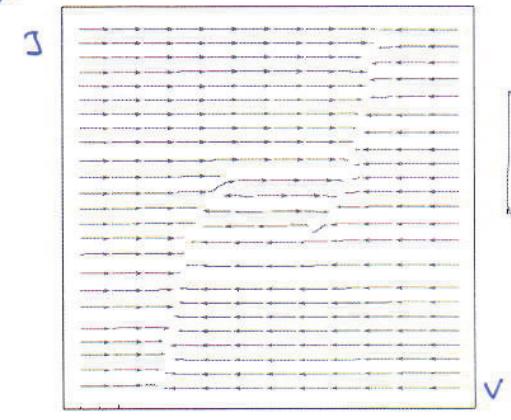
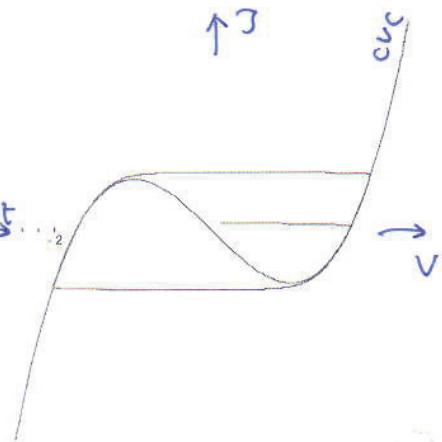
$$\mu = 1.0$$



NDCL - instabilität!
 $\alpha = 1 > 0$

$$\mu = 0.1$$

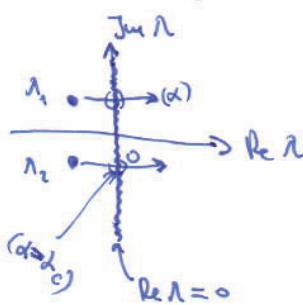


$\mu = 1$  $v(t_1, \cdot)(t)$  $\mu = 10$  $v(t)$  $\mu = 100$  $v(t)$ 

Vrijeme ne sada uvećava vrednost prve u kojoj stabilitet F.T. ičekati stacionarne primenom parametra i da je ustanove stabilitet granicno kolo.

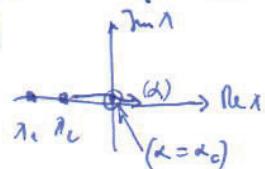


Istaknut scenario ostvariv je ako imamo 2 negativne vrijednosti dijeli realni dio je negativan za $d < d_c$ (da bi F.T. bila stabilna), za $d = d_c$ je $\text{Re}(\lambda_{1,2}) = 0$, i za $d > d_c$ je $\text{Re}(\lambda_{1,2}) > 0$ (nestabilnost). ($\lambda_{1,2} \in \mathbb{C}$)

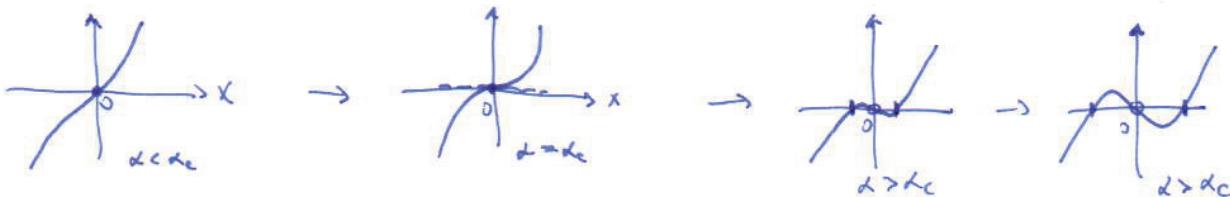


→ Objekti λ prelaze preko rečnika dijela u isti da; od stabilnog polura ustanove nestabilni fokus.

{ Slijedi se čvor-sredstvo bifurkacija; tamo je novi jedan λ prelazi preko (izmjeni novi 2 realne v. vrijednosti)



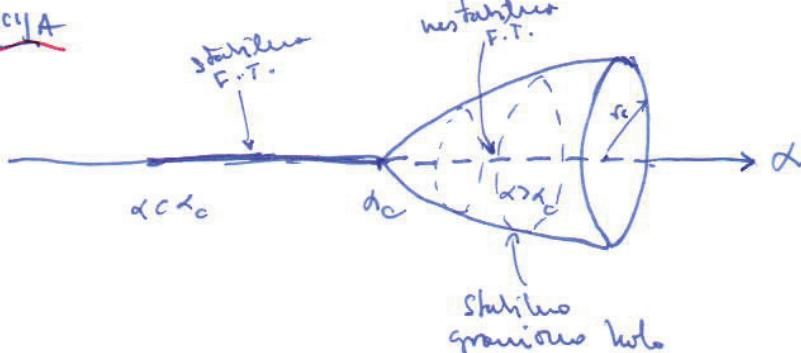
Ovo korespondira s prelazom u sustav:



U 1-D ovu je tite pitchfork bifurkaciju (1 st. F.T. pretvara se u 1 nestabilnu + 2 stabilne, tako da je formi tok je ≥ 3 otkao sačuvan).

U 2-D imamo 2 sujeza i sačuvaju asymptotickog formog toka implikuju formirajući kontinuirane zatvorene linije odu F.T. koja je regularna stabilitet \Rightarrow to je granicno kolo, a bifurkacija u kojoj ovu novu type ovih novih strukturalnih nestabilnosti nastava zove se:

HOPFOVA BIFURKACIJA



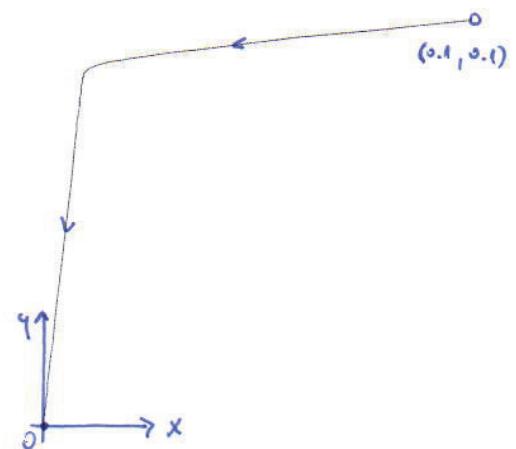
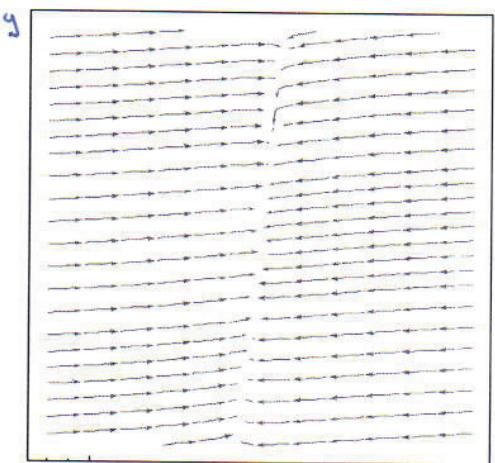
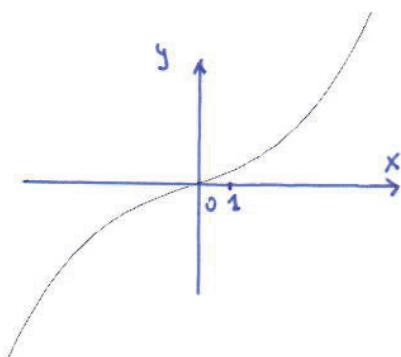
Hopfova bifurkacija : PDC \rightarrow UDC

93-a

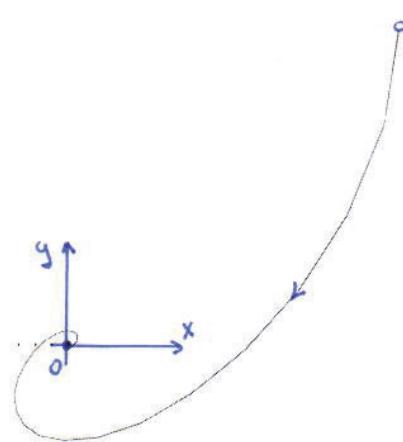
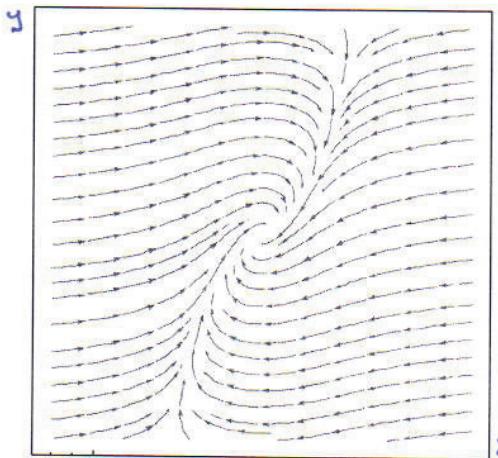
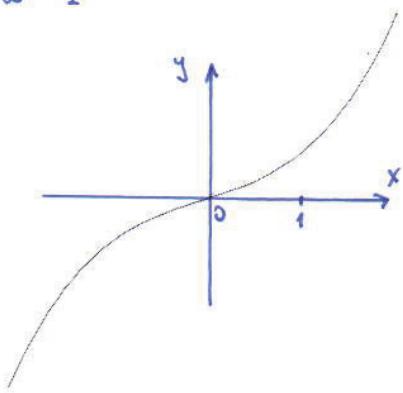
$$\begin{aligned} F_x &= \alpha x - \frac{x^3}{3} + y \\ F_y &= -\frac{x}{\mu}, \quad (\mu = 1) \end{aligned}$$

[PDC] ($\alpha < 0$)

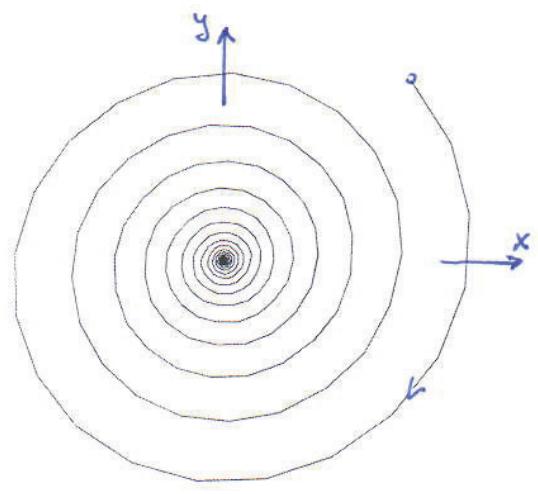
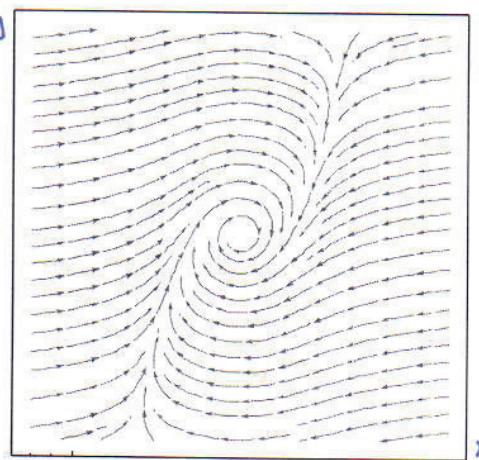
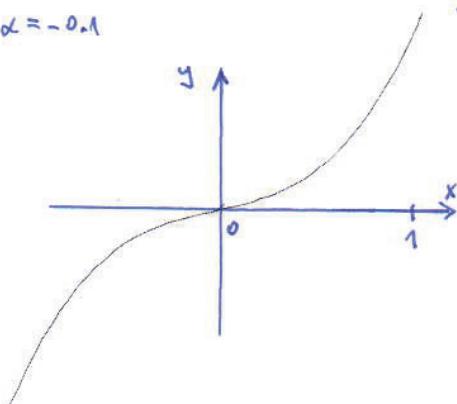
$$\alpha = -1.0$$



$$\alpha = -1$$

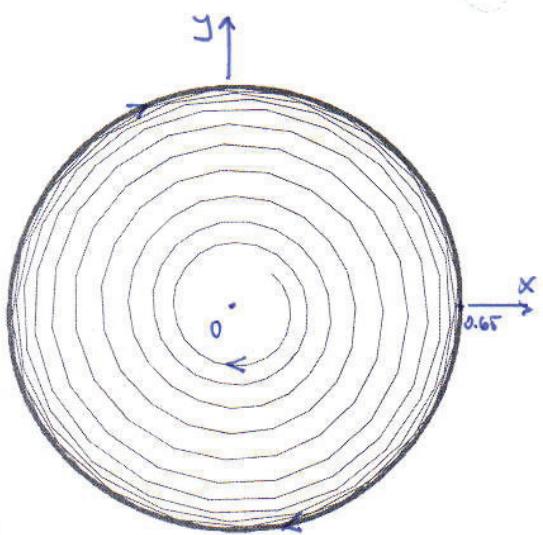
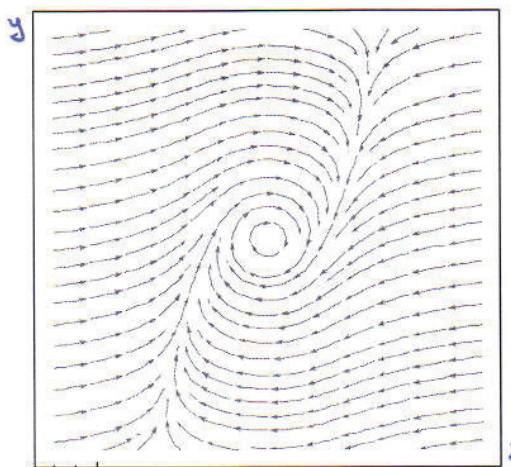
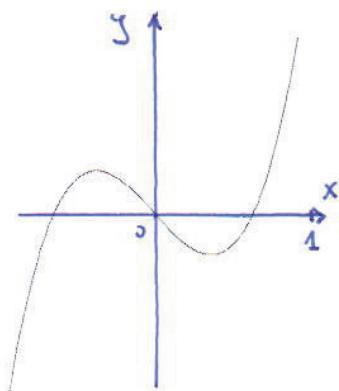


$$\alpha = -0.1$$

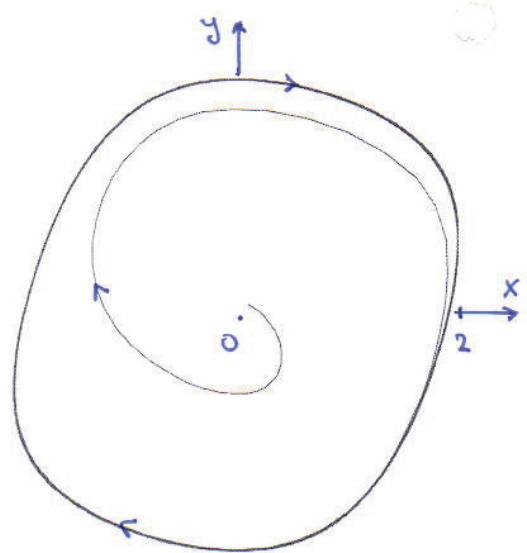
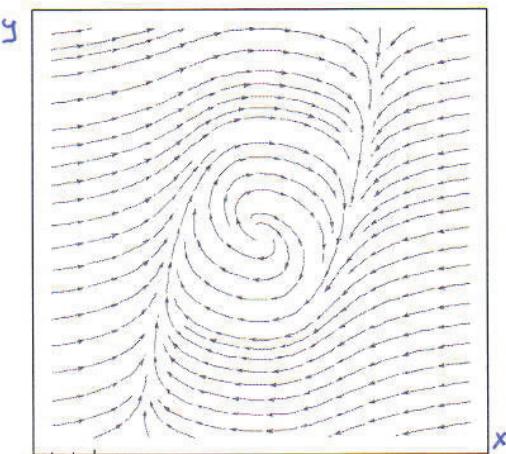
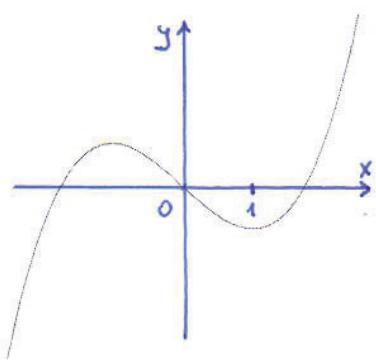


NDC ($\alpha > 0$)

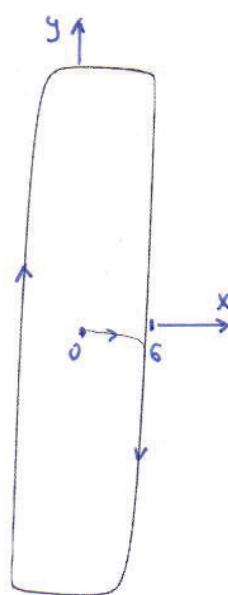
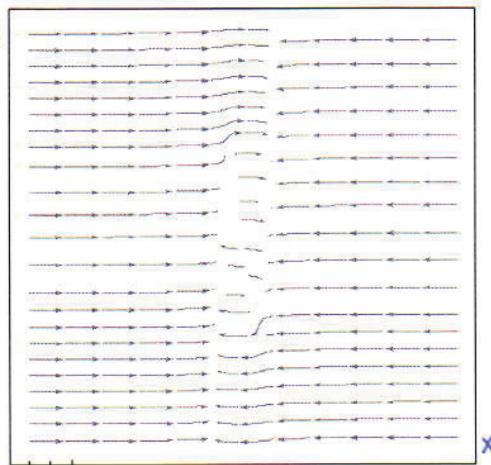
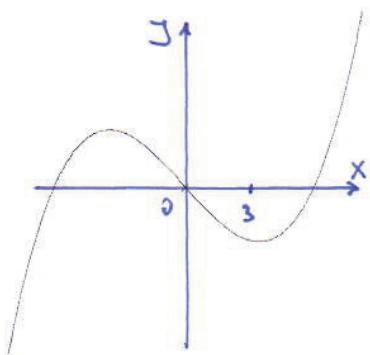
$\alpha = 0.1$



$\alpha = 1$



$\alpha = 10$



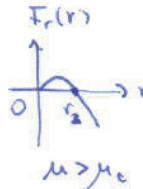
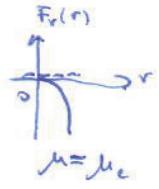
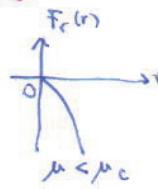
Potpjeće 2 glavne tipa kuglove bifurkacije:

(a) nadkritična - tvr. "stale nestabilnost"

$$\boxed{\text{upr}} \quad \dot{r} = \mu r - r^3 \rightarrow F_r = r(\mu - r^2)$$

$$\dot{\varphi} = \omega$$

$$r_1 = 0 \\ r_{2,3} = \pm \sqrt{\mu} \rightarrow \boxed{\mu_c = 0}$$



$$x = r \cos \varphi \\ y = r \sin \varphi$$

$$2 \text{ nultočke za } \mu > 0 \\ 1 \text{ nultočka za } \mu < 0$$

$$\dot{x} = \dot{r} \cos \varphi - r \dot{\varphi} \sin \varphi$$

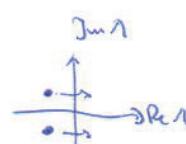
$$= \mu x - \omega y + \sigma_3 \quad \begin{matrix} \text{horizontal} \\ \text{stacionarni} \end{matrix}$$

$$\dot{y} = \dot{r} \sin \varphi + r \dot{\varphi} \cos \varphi \\ = \mu y + \omega x + \sigma_3$$

$$\Rightarrow A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

$$(\mu - \lambda)^2 + \omega^2 = 0$$

$$\lambda_{1,2} \approx \mu \pm i\omega$$



$$\boxed{T_c \sim \sqrt{\mu - \mu_c} \\ \omega_c \sim \Im(\lambda) \mid_{\mu=\mu_c}}$$

približni radijus i frekvencija nestabilog gr. kota blizu $\mu = \mu_c$

(b) podkritične - tvr. "jake nestabilnost"

$$\boxed{\text{upr}} \quad \dot{r} = \mu r + r^3 - r^5 \rightarrow F_r = r(\mu + r^2 - r^4)$$

$$\underline{r_1 = 0}$$

$$r_{2,3}^2 = \frac{1}{2}(1 \pm \sqrt{1+4\mu}) \rightarrow \text{unitarne vrijednosti } \mu \text{ su}$$

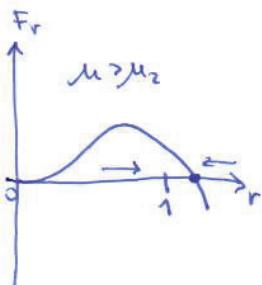
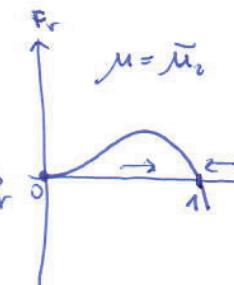
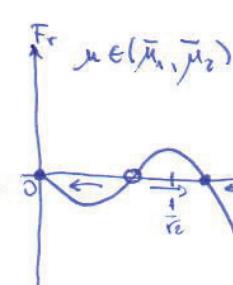
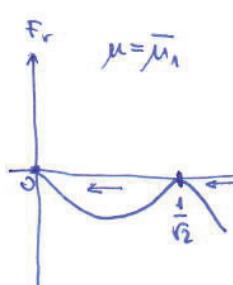
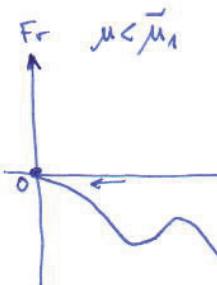
$$\bar{\mu}_1 = -\frac{1}{4} \quad i \quad \bar{\mu}_2 = 0$$

$$\rightarrow \text{za } \mu < \bar{\mu}_1 : \quad \begin{matrix} r_{2,3}^2 \in \mathbb{C} \\ r_{2,3} = \pm \sqrt{1/2} \end{matrix} \Rightarrow 1 \text{ nultočka (} r_1 = 0 \text{)}$$

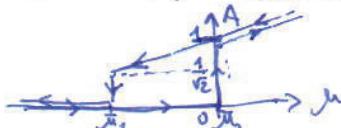
$$\rightarrow \text{za } \mu = \bar{\mu}_1 : \quad \begin{matrix} r_{2,3}^2 = \frac{1}{2} \\ r_{2,3} = \pm \sqrt{1/2} \end{matrix} \Rightarrow 2 \text{ nultočke (} r_1 = 0; r_2 = \pm \sqrt{1/2} \text{)}$$

$$\rightarrow \text{za } \mu \in (\bar{\mu}_1, \bar{\mu}_2) : \quad \begin{matrix} r_{2,3}^2 = \{a > 0 \\ b > 0\} \\ r_{2,3} = \{ \pm \sqrt{a} \\ \pm \sqrt{b} \} \end{matrix} \Rightarrow 3 \text{ nultočke (} r_1 = 0; r_{2,3} \text{)}$$

$$\rightarrow \text{za } \mu > \bar{\mu}_2 : \quad \begin{matrix} r_{2,3}^2 = \{c > 0 \\ d < 0\} \\ r_{2,3} = \pm \sqrt{c} \end{matrix} \Rightarrow 2 \text{ nultočke (} r_1 = 0, r_2 \text{)}$$



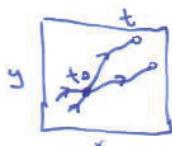
U ovom slučaju "jake nestabilnosti" dolazi do pojava histerezis. Ako oscilacija počinje s velikom amplitudom, ona se ne more isteznati smjerivo prema μ do $\mu = \bar{\mu}_2 = 0$ već se moraći sve do $\mu = \bar{\mu}_1 = -1/4$ da padne na nulu. Takođe ako krenemimo od malih amplituda, one se neće ukloniti od nule sve dok ne presegne $\mu = \bar{\mu}_2 = 0$:



Zadnje čak:

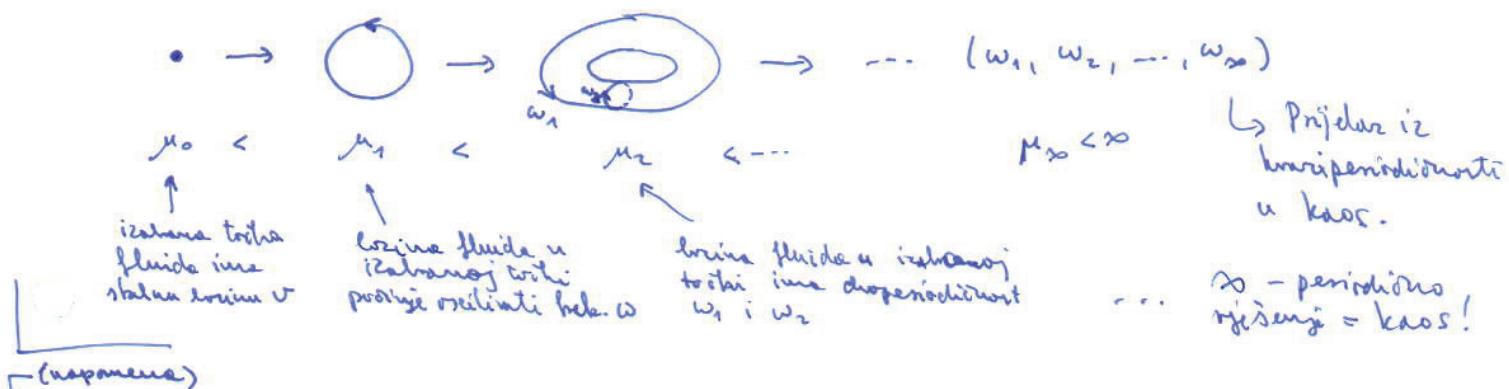
Hopfova bifurkacija uvedi nove fundamentalne teorije u sustav. Sustav koji se počne razvijati u fiksnoj točki, rado oscilira po granicnom kolu teorije u sustavu. Ne samo da je to model samo-excitirajuog oscilatora, već granicno kolo predstavlja najlošniji mogući formen ataktora u farnom prostoru sustava 2. reda.

Kako se trajektorije u farnom prostoru klasičnog sustava ne mijenjaju (namjeru klasičnog determinizma ili teorije o eksistenciji i jedinstvenosti vrijednosti dif. jednadžbe: jedna forma točka (punkt) imala bi 2 različite budućnosti tj. vrijednosti ne bi bilo jedinstveno), tada jedna stvarnija struktura nije moguća.



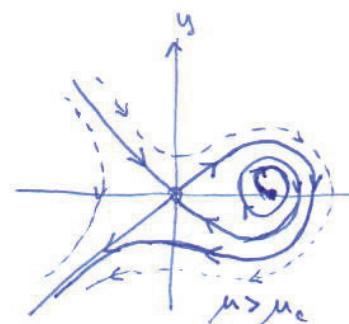
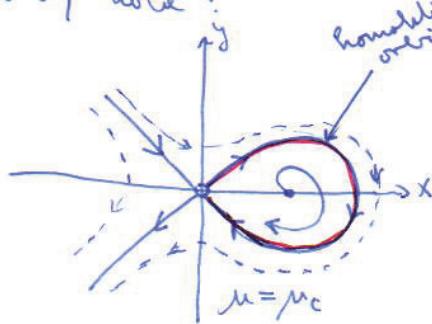
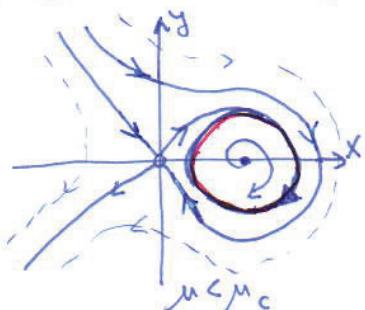
(U sustavima viših dimenzija moguće su deleže stvarijske strukture atraktora (struci atraktori) jer se u 3D ili više trouđeštoje mogu razlikovati bez sjećanja.)

- Napomenimo da je Hopfova bifurkacija tj. niz hrvava vektorova u Landauov scenarij uteske točke fluida u turbulenciju (fluid je sustav s sa stupnjem slobode).



(*) HOMOKLINIČKA BIFURKACIJA

Spojivati novu bifurkaciju čor-sedlo. Homoklinička bifurkacija nastaje spajanjem sedla i granicnog kola:



zad. $\dot{x} = y$
 $\dot{y} = \mu y + x - x^2 + xy$; $\mu_c \approx -0.8645$

§5

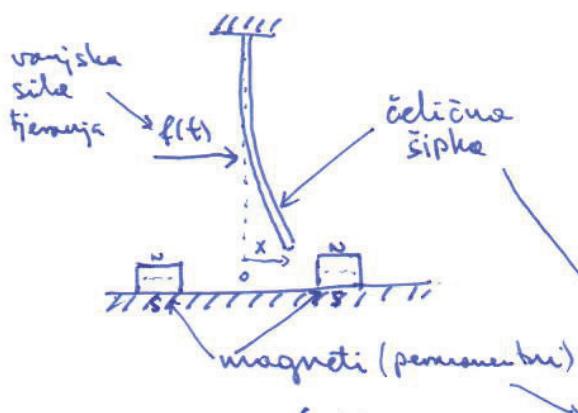
SUSTAVI REDA VIŠEG OD II - KADS

96

5.1

DUFFINGOV OSCILATOR

nemotivni sustav:

(1*) autonomni sustav ($f(t) = 0$):

$$m \ddot{x} = F(x, \dot{x})$$

sila
(na šipku) elastična sila sila treća

 $F(x, \dot{x})$:

- magnetski privlače šipku i djeluju silom u smjeru posmaka x
- elastičitet šipke nastoji girovati u položaj ravnoteže 0 (djeluje suprotno posmaku x)

- računaju se $F(x, \dot{x})$ u Taylorov red oko ravnotežnog položaja $x=0$ i $\dot{x}=0$:

$$F(x, \dot{x}) = F_0 + \left(\frac{\partial F}{\partial x}\right)_0 x + \frac{1}{2} \left(\frac{\partial^2 F}{\partial x^2}\right)_0 x^2 + \frac{1}{3!} \left(\frac{\partial^3 F}{\partial x^3}\right)_0 x^3 + \dots +$$

$$+ \left(\frac{\partial F}{\partial \dot{x}}\right)_0 \dot{x} + \frac{1}{2} \left(\frac{\partial^2 F}{\partial \dot{x}^2}\right)_0 \dot{x}^2 + \frac{1}{3!} \left(\frac{\partial^3 F}{\partial \dot{x}^3}\right)_0 \dot{x}^3 + \dots$$

- 1) sila $F(x, \dot{x})$ je neparni funkcija (mjenja predznak kada $x \rightarrow -x$)
 \Rightarrow ne parne potencije od x i \dot{x} isčeravaju!
- 2) u položaju ravnoteže sila je nula $\Rightarrow F_0 = 0$

$$F(x, \dot{x}) \approx kx + qx^3 + \gamma \dot{x} + \cancel{\dots}$$

$$\underline{k > 0}, \underline{q < 0}, \underline{\gamma < 0}$$

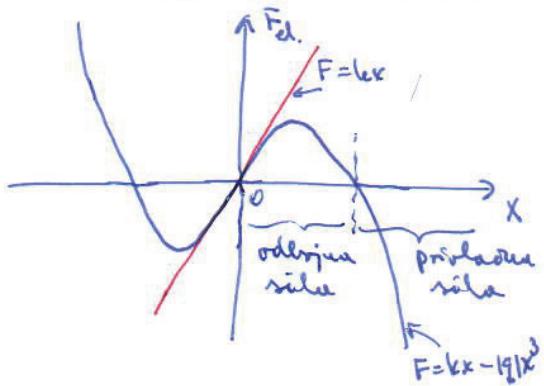
U ovom modelu:

- za male posmoke predomina sila od magneta (u smjeru x) $\Rightarrow k > 0$
- za veće posmoke dominantna sila je u smjeru suprotnom od x $\Rightarrow q = -|q| < 0$
- sila treća mijenja djelovanje u smjeru suprotnom od x $\Rightarrow \gamma = -|\gamma| < 0$

$$F(x, \dot{x}) = kx - |q|x^3 - |\gamma|\dot{x}$$

Ljostivali (viši članovi) - zanemarjivo!

(*) nelinearna "elastična sila":



- jednadžba
gibanja:

$$m\ddot{x} = kx - |\zeta| x^3 - \gamma_1 \dot{x} \quad | : m$$

$$\ddot{x} + \gamma \dot{x} - kx + \zeta x^3 = 0$$

- dinamički
sistav:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\gamma y + kx - \zeta x^3 \end{aligned} \quad \begin{array}{l} : F_x \\ : F_y \end{array}$$

$$\gamma = \frac{|\gamma_1|}{m} > 0$$

$$k = \frac{k}{m} > 0$$

$$\zeta = \frac{|\zeta|}{m} > 0$$

Potencijal koji odgovara gornjemu sustavu (tj. Hamiltoniju):

Hamiltonov
jednačine:

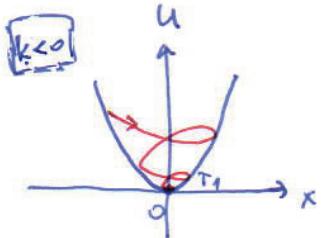
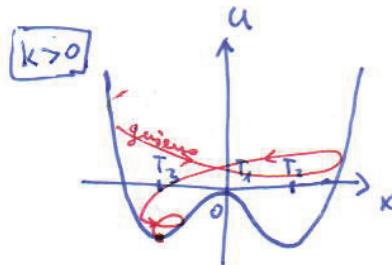
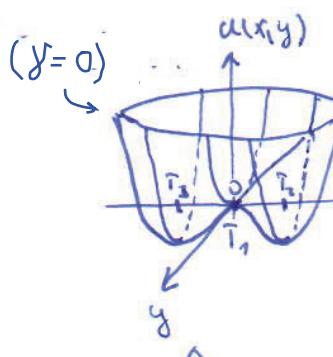
$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned} \Rightarrow \begin{aligned} \dot{x} &= \frac{\partial U}{\partial y} = y \quad \rightarrow U_y = \int y dy = \frac{1}{2} y^2 \\ \dot{y} &= -\frac{\partial U}{\partial x} = kx - \zeta x^3 - \gamma y \quad \rightarrow U_x = - \int (kx - \zeta x^3 - \gamma y) dx \\ &\quad = -\frac{k}{2} x^2 + \frac{\zeta}{4} x^4 + \gamma x y \end{aligned}$$

$$U(x, y) = \underbrace{\frac{1}{2} y^2}_{\text{kinetička energija}} - \underbrace{\frac{k}{2} x^2}_{\frac{1}{2} \dot{x}^2} + \underbrace{\frac{\zeta}{4} x^4}_{-\frac{k}{2} x^2 + \frac{\zeta}{4} x^4} + \gamma x y$$

$$W = F_p \cdot S = \gamma \dot{x} \cdot x$$

potencijal koji odgovara "sili":

$$\vec{F} = -\vec{\nabla}U \Rightarrow U(x,y) = -\frac{k}{2}x^2 + \frac{\xi}{4}x^4 + \frac{1}{2}y^2 + \gamma xy$$



double-well potential

single-well potential
(standardna Hookeova situacija $F = -k(x - x_0)$)

- jednadžba:

$$m\ddot{x} = kx - 1/2(x^3 - ly/x) \quad | : m$$

$$\ddot{x} + \gamma \dot{x} - kx + \xi x^3 = 0$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\gamma y + kx - \xi x^3 \end{cases}$$

$$\begin{aligned} \gamma &\equiv \frac{|ly|}{m} > 0 \\ k &\equiv \frac{k}{m} > 0 \\ \xi &\equiv \frac{|l^2|}{m} > 0 \end{aligned}$$

F.T.

$$\dot{y}_c = 0$$

$$x(K - \xi x^2) = 0 \Rightarrow x_{c1} = 0 \\ x_{c2,3} = \pm \sqrt{\frac{K}{\xi}}$$

$$\begin{cases} T_1(0,0) \\ T_{2,3}(\pm \sqrt{\frac{K}{\xi}}, 0) \end{cases}$$

(*) uočimo da za standardnu situaciju $k < 0$ F.T. $T_{2,3}$ isčekavaju (imaginarni) i ostaje samo $T_1(0,0)$ (single-well pot.)

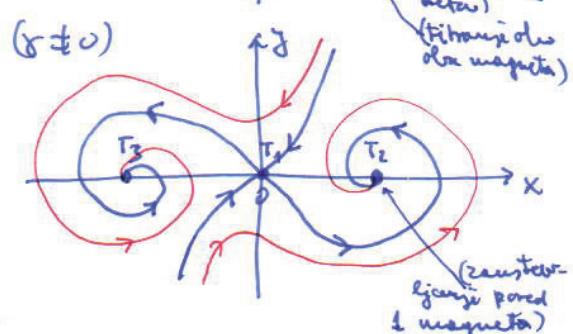
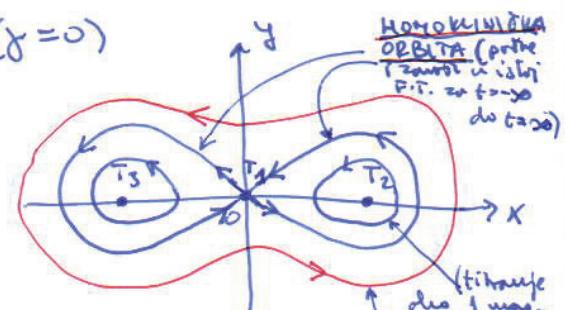
$$A = \begin{bmatrix} 0 & 1 \\ K - 3\xi x^2 & -\gamma \end{bmatrix}$$

$$T_1: A = \begin{bmatrix} 0 & 1 \\ K & -\gamma \end{bmatrix} \rightarrow \begin{aligned} T &= -\gamma < 0 \\ \Delta &= -K < 0 \\ \mathcal{D} &= \gamma^2 + 4K > 0 \\ \Rightarrow &\text{ sedlo! } (\text{za } \gamma = 0) \end{aligned}$$

$$T_{2,3}: A = \begin{bmatrix} 0 & 1 \\ -2K & -\gamma \end{bmatrix} \rightarrow \begin{aligned} T &= -\gamma \\ \Delta &= 2K > 0 \\ \mathcal{D} &= \gamma^2 - 8K \end{aligned}$$

$$1) \gamma = 0: \quad \begin{cases} T = 0 \\ \Delta = 2K > 0 \\ \mathcal{D} = -8K < 0 \end{cases} \quad \text{center!}$$

$$2) \gamma \neq 0: \quad \begin{cases} T < 0 \\ \Delta > 0 \\ \mathcal{D} = \gamma^2 - 8K \rightarrow > 0 \rightarrow \text{st. čvor!} \\ \mathcal{D} = \gamma^2 - 8K \rightarrow < 0 \rightarrow \text{st. fokus!} \end{cases}$$



(2°) periodický systém (neautonomický) ($f(t) \neq 0$)

99

$$\ddot{x} = F(x, \dot{x}) + f(t)$$

$$; f(t) = m\varphi_0 \cos \omega t$$

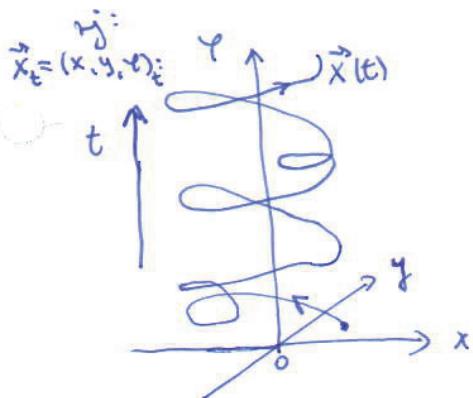
$$\ddot{x} + \gamma \dot{x} - \kappa x + \xi x^3 = f_0 \cos \omega t$$

$$\varphi = \omega t$$

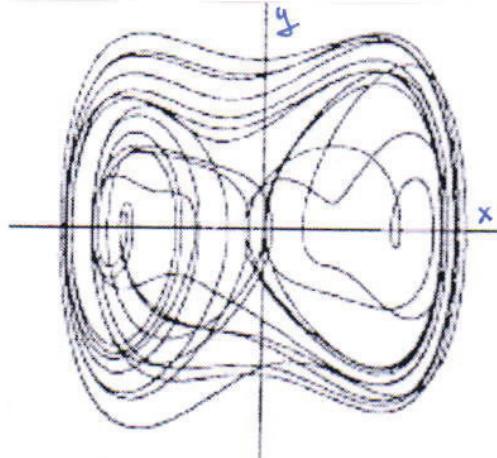
→ norm. variabilé
(stupně svobody)

$$\begin{aligned}\dot{x} &= y & \rightarrow F_x \\ \dot{y} &= -\gamma y + \kappa x - \xi x^3 + f_0 \cos \varphi & \rightarrow F_y \\ \dot{\varphi} &= \omega & \rightarrow F_\varphi\end{aligned}$$

→ neautonomický systém
2. reda ručičky je pravěká
která má autonome
systém všechny (třetí)
reduci!



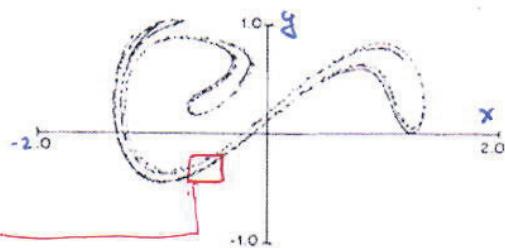
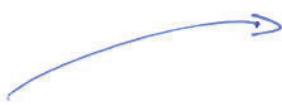
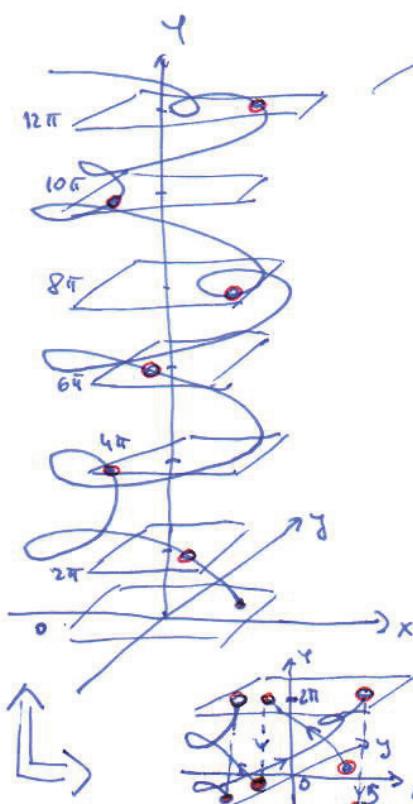
projekce na (x,y)
rovinu



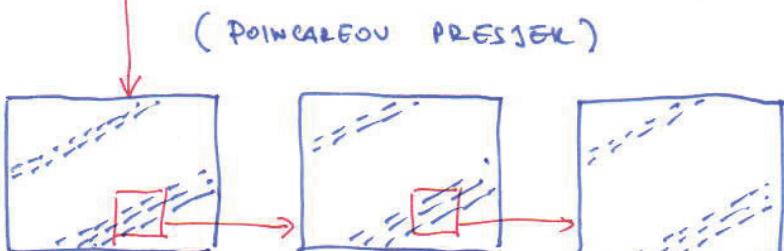
$\vec{F}(x, y, \varphi)$ je periodické funkce v φ -místu.

Definujeme diskretní přeslahaře

$$\vec{x}_n = (x(n), y(n), \varphi(n) \bmod 2\pi), n \in \mathbb{N}$$



(POINCARÉOVÝ PRESLEK)



Sloupce jsou SANSKŘÍNU (FRAKTÁLU) struktury!

To je STRANÍ (ČUDÉSNÍ) ATRAKTOR.

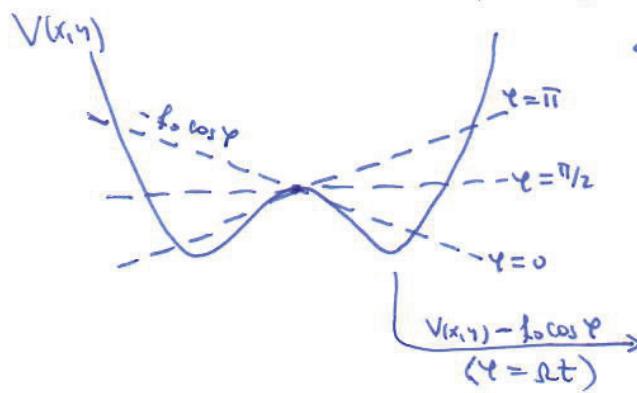
\Rightarrow u systému je prisutan KAOS!

(neprediktivitnost ne početné možnosti)

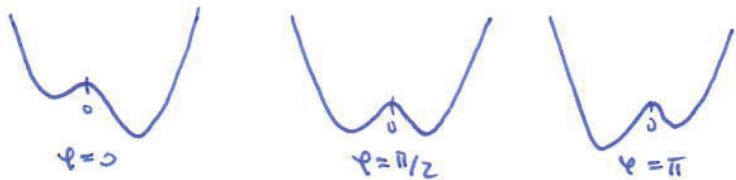
Koji je vrakove ove popave?

$$U(x,y) = \frac{1}{2}y^2 - \frac{k_2}{2}x^2 + \frac{3}{4}x^4 + 8xy - f_0 \cos \varphi \cdot x$$

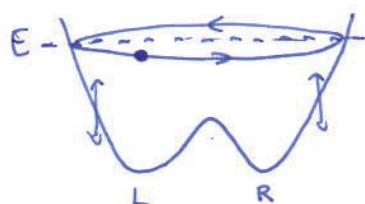
$V(x,y)$ gusiće red sile tjeranja



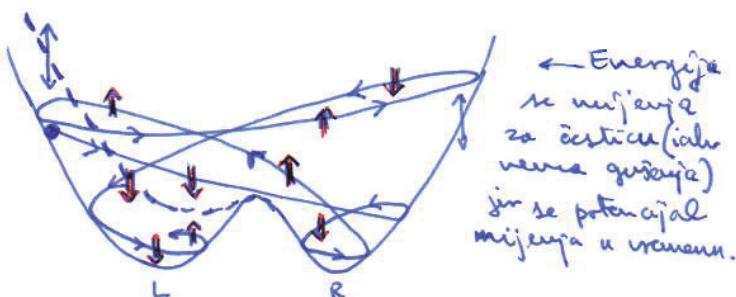
Uvodenje sile tjeranja omogućuje periodične oscilacije mijenjajući se amplitudom f_0 i frekvencijom ω .



Rečimo da nema gusiće ($E_i = \text{const.}$):



$\Leftarrow \Rightarrow$



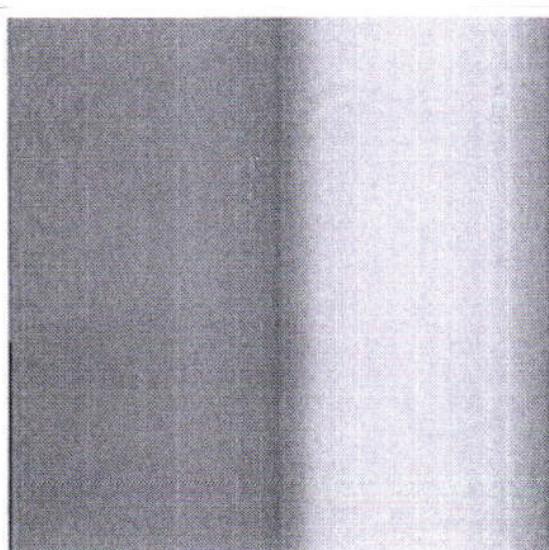
Tačka komplikovanija ("kaoštovanja") je situacija kada je gusiće uključeno.

Nesuvjedljivost frekvencije ophodne čestice u potencijalu je jasno u frekvenciji sile tjeranje tj. titrajući potencijal je kon-dolje, zbog prisustva 2 jama (nelinearnost!), uobičajeno "loptanje" između dva jama potpuno stohastički u horizontici. Sadržim time i hiperosjetljivost na početne uvjete.

(*) (Osim linearnih tenuki oscilator nema kosa.)

(*) Basen atrakcije:

Istovremeno točku s uvjetom (pri. uvjet) i gledam $x(t)$ u toku puno ciklusa ($t \gg$) da li je zadržala u L ili R jamu. Granice između L i R basena atrakcije je FRAKTALNA.



$f(t) = 0$

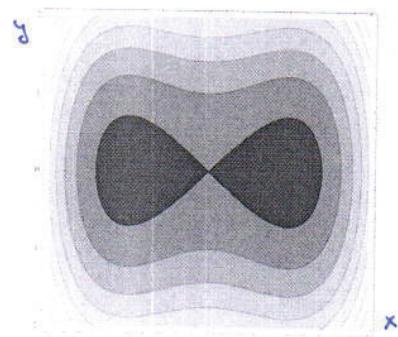
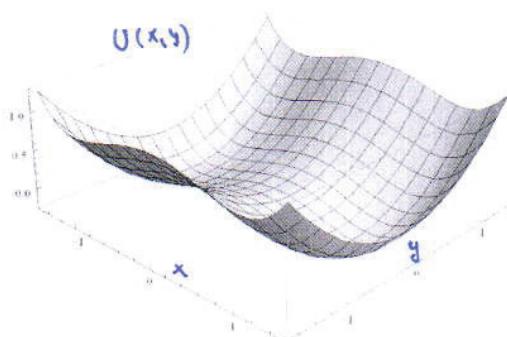


$f(t) \neq 0$

Duffingov oscillator - potencijal

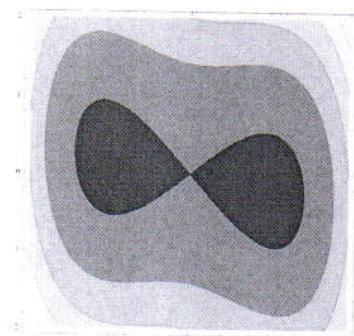
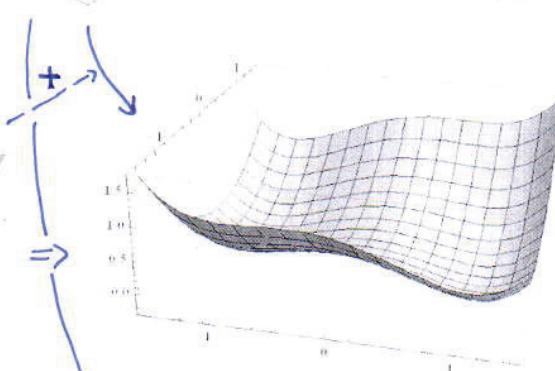
$$U(x,y) = \frac{1}{2}y^2 - \frac{\kappa}{2}x^2 + \frac{\gamma}{4}x^4 + \gamma xy - f_0 \cos \varphi \cdot x \quad ; \quad (\kappa=1, \gamma=1)$$

- guseni (γ=0, f₀=0)

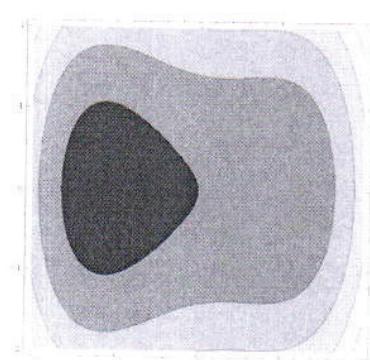
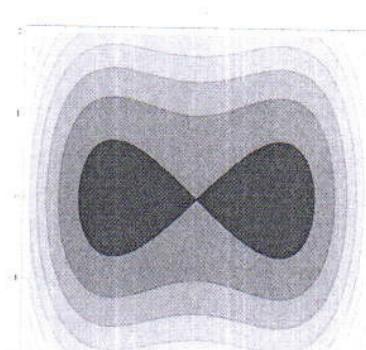
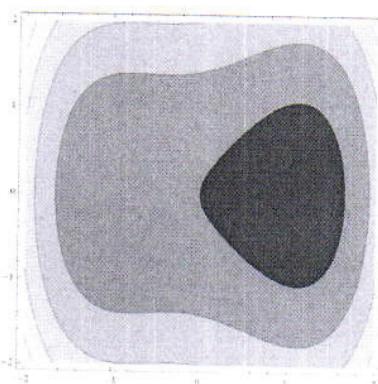
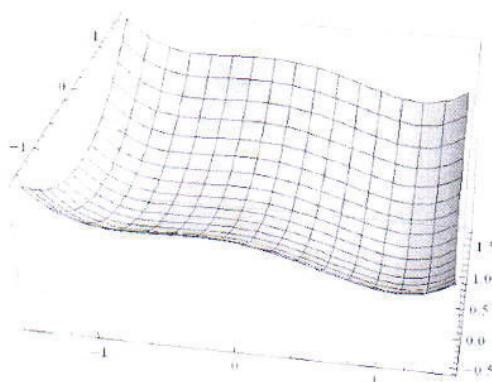
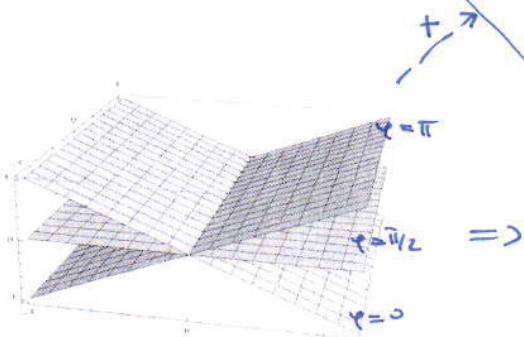


- guseni (γ=0.2, f₀=0)

$$v = x \cdot y$$



- tyčani (γ=0, f₀=0.3)

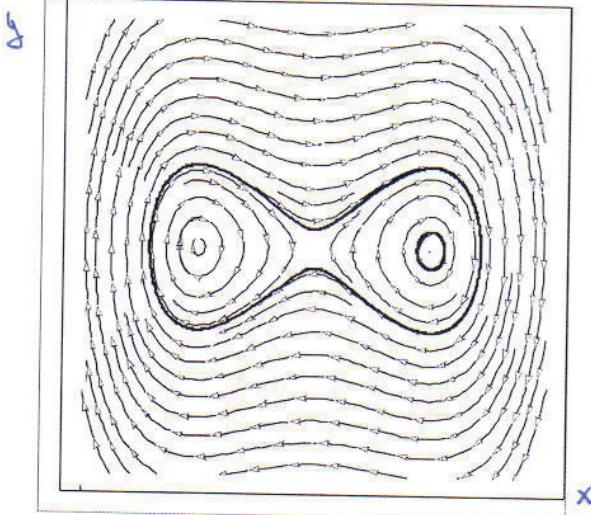


DUFFING OSCILATOR

100-b

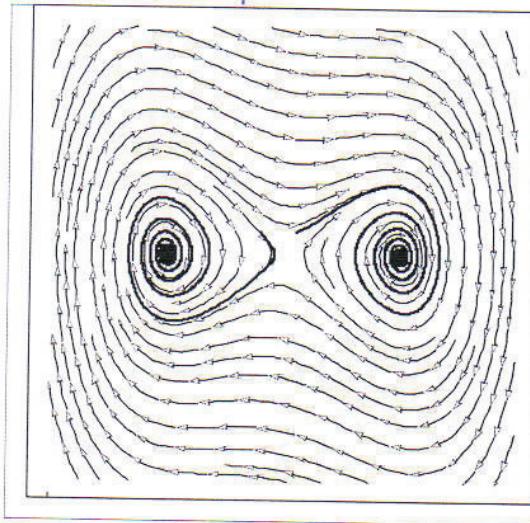
- netjerani slučaj ($f(t) = 0$)

ber gύenja ($\gamma = 0$)



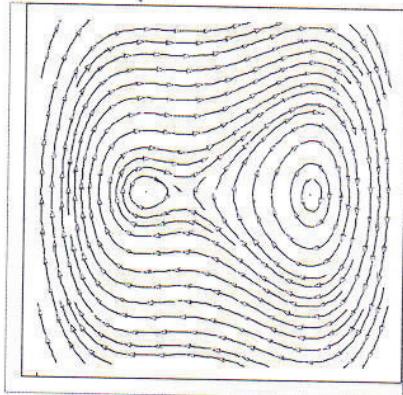
$$\begin{cases} K = 1 \\ \xi = 1 \\ \omega = 1 \\ f_0 = 0.3 \end{cases}$$

o gύenja ($\gamma \neq 0$) ($\gamma = 0.2$)

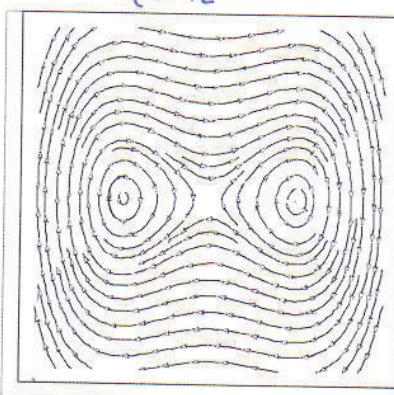


- tjerani slučaj ber gύenja ($f(t) \neq 0, \gamma = 0$)

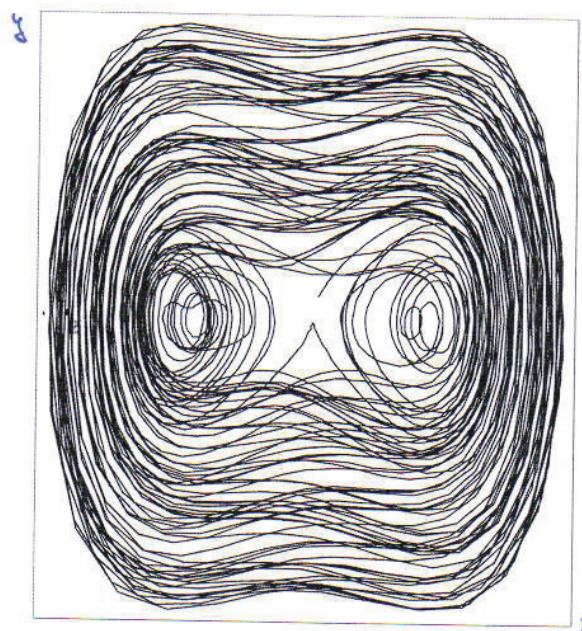
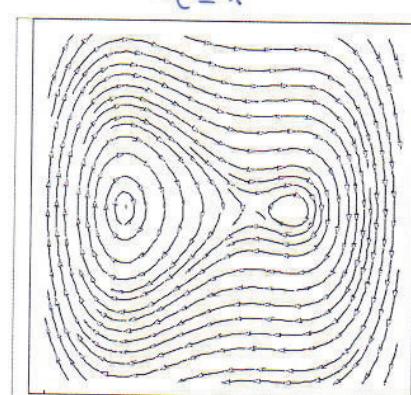
$\varphi = 0$



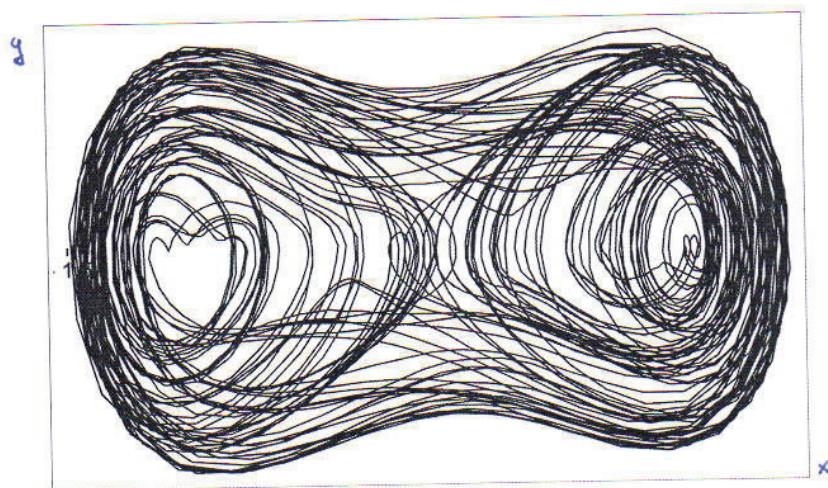
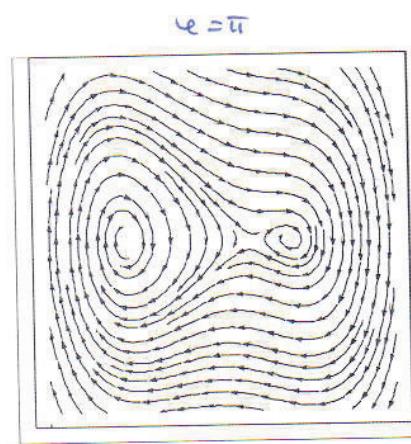
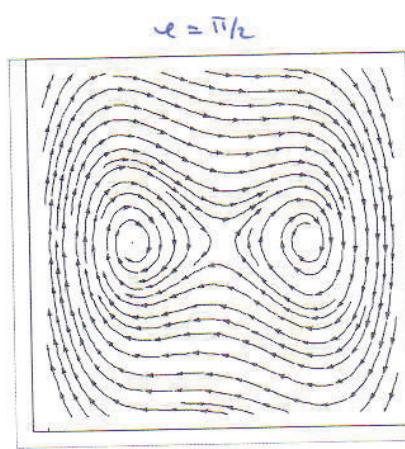
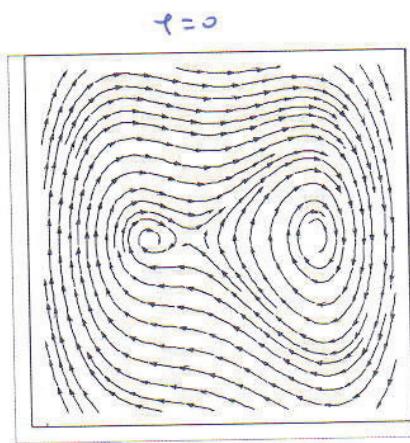
$\varphi = \pi/2$



$\varphi = \pi$



- tijevni stvrtaj s gubnjem ($f(+)\neq 0$, $\gamma\neq 0$) ($\gamma=0.2$)



```
In[78]:= fx[x_, y_, z_] := y
fy[x_, y_, z_] := -g*y + k*x - b*x^3 + f0 * Cos[z]
fz[x_, y_, z_] := w

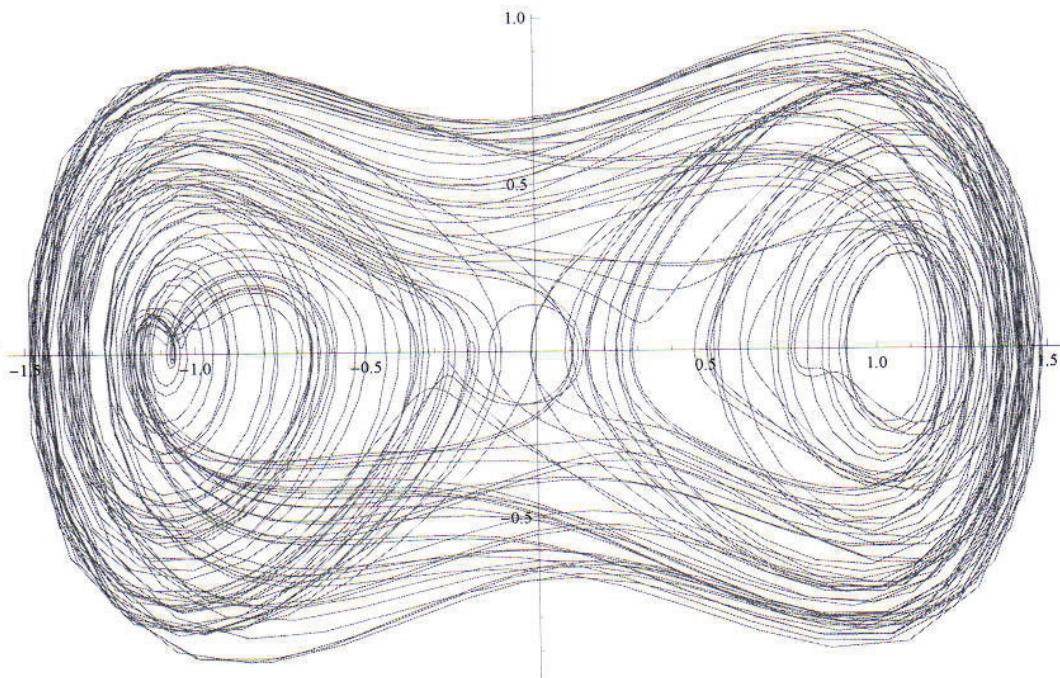
In[81]:= g = .2;
k = 1;
b = 1;
w = 1;
f0 = .3;

In[84]:= s = NDSolve[{x'[t] == fx[x[t], y[t], z[t]],
y'[t] == fy[x[t], y[t], z[t]], z'[t] == fz[x[t], y[t], z[t]],
x[0] == 0., y[0] == 0., z[0] == Pi/2}, {x, y, z}, {t, 0, 1000}]

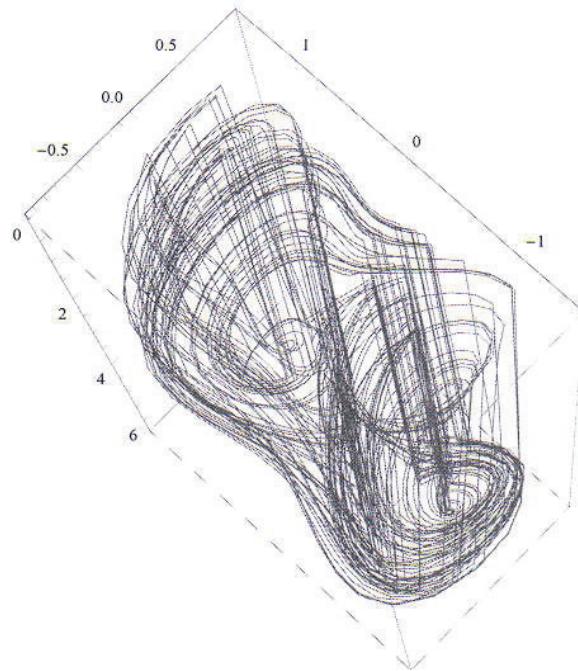
NDSolve::mxst : Maximum number of 10000 steps reached at the point t == 762.3034183297834`.
```

```
Out[84]= { {x → InterpolatingFunction[{{0., 762.303}}, <>],
y → InterpolatingFunction[{{0., 762.303}}, <>],
z → InterpolatingFunction[{{0., 762.303}}, <>]} }
```

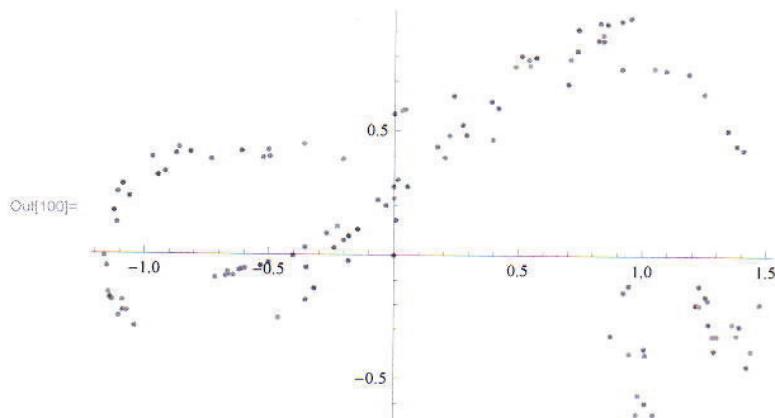
```
In[86]:= ParametricPlot[Evaluate[{x[t], y[t]} /. s], {t, 0, 762}, PlotRange → All]
```



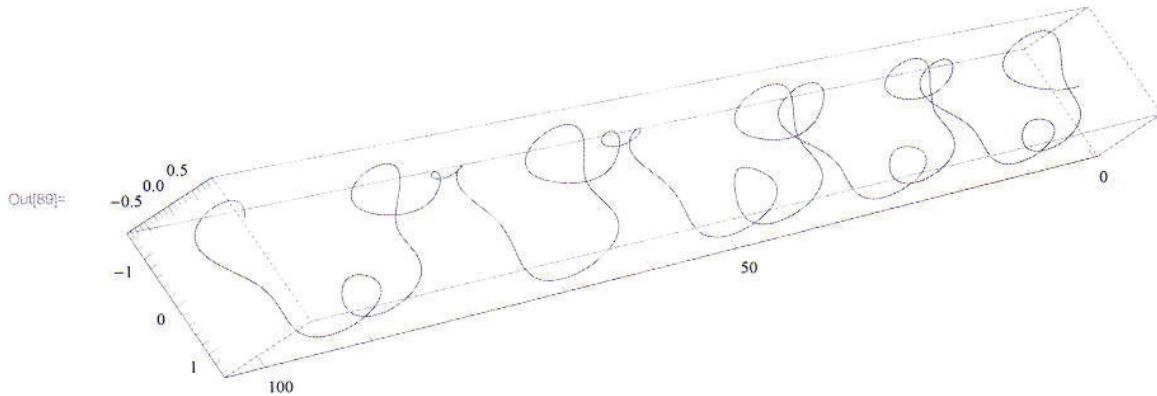
```
ParametricPlot3D[Evaluate[{x[t], y[t], Mod[z[t], 2 Pi]} /. s],  
{t, 0, 762}, PlotRange -> All]
```



```
In[99]:= T = Table[Evaluate[{x[2 Pi * i], y[2 Pi * i]} /. s], {i, 0, 120}];  
In[100]:= ListPlot[T]
```



```
In[89]= ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. s], {t, 0, 100}, PlotRange -> All]
```



```
In[94]= s1 = NDSolve[{x'[t] == fx[x[t], y[t], z[t]],  
y'[t] == fy[x[t], y[t], z[t]], z'[t] == fz[x[t], y[t], z[t]],  
x[0] == 0., y[0] == 0., z[0] == 0.}, {x, y, z}, {t, 0, 1000}];  
s2 = NDSolve[{x'[t] == fx[x[t], y[t], z[t]], y'[t] == fy[x[t], y[t], z[t]],  
z'[t] == fz[x[t], y[t], z[t]], x[0] == 0.1,  
y[0] == 0.1, z[0] == 0.}, {x, y, z}, {t, 0, 1000}]
```

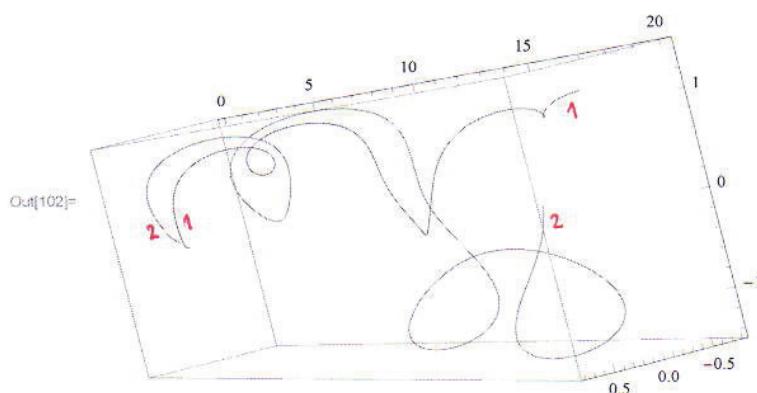
NDSolve::mxst : Maximum number of 10000 steps reached at the point t == 725.8877374358585`.

```
Out[94]= {{x -> InterpolatingFunction[{{0., 725.888}}, <>],  
y -> InterpolatingFunction[{{0., 725.888}}, <>],  
z -> InterpolatingFunction[{{0., 725.888}}, <>]}}
```

NDSolve::mxst : Maximum number of 10000 steps reached at the point t == 733.1184220706826`.

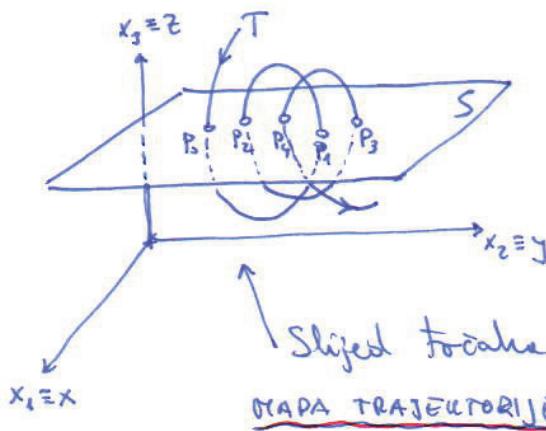
```
Out[95]= {{x -> InterpolatingFunction[{{0., 733.118}}, <>],  
y -> InterpolatingFunction[{{0., 733.118}}, <>],  
z -> InterpolatingFunction[{{0., 733.118}}, <>]}}
```

```
In[102]= ParametricPlot3D[Evaluate[{{x[t], y[t], z[t]} /. s1, {x[t], y[t], z[t]} /. s2}],  
{t, 0, 20}, PlotRange -> All]
```



U sustavima reda većeg od 2. često je vrlo komplikiranu pomicati kompletnu dinamiku. Uvodimo metodu (H. Poincaré) kojom putem glavne površine sustava (fj. KLASU POWAĆANJA MODELA (stabilnost, bifurkacije, periodičke orbite itd.)).

U formu prostom dimenziji N pogodno postavimo hiperplan S dimenzije $N-1$ i prema tome kada noša trajektorija T probade taj plan (npr. u 3D F.P. hiperplan je liti ravni).



(S može biti pogodno odabrano da ga trajektorija T dobro isprobade.)

(Ovo odgovara strukturalnom pogledu na sustav.)

- definisani preslikavanje: $P_{k+1} = T(P_k) = \underbrace{T[T(P_{k-1})]}_{= T^2[P_{k-1}]} = \dots = T^{k+1}[P_0]$
- Preslikavanje T određeno je dimenzijom sistema i ima svojstva:
 - determiniran (kao 1 točka pravci 1 trajektorija)
 - kontinuiravost
 - svakom slijedu $\{P_i\}$ pripada jedna trajektorija, a svaki slijed $\{P_i\}$ predstavlja (odnosno) jednu trajektoriju (bijektiv, tj. postoji inverz preslikavanja)

Slike $\{P_i\}$ ne moraju se povezivati periodički u vremenu!

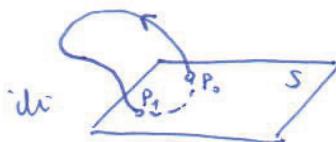
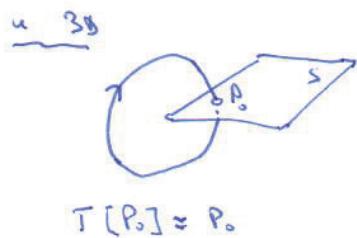
Zadnjom sustav jednorađen je primjer preslikavanje T , ali obrat ne vrijedi (može postojati drugačiji sustav s istim preslikavanjem T).

Najčešće se slijedi P.P. ne mogu obuhvatiti svi aspekti dinamičke sustava (npr. imo trajektorija koji ga uopće ne mijenja ili premašuju sime tröcken hiperplanom itd.). Najčešće može zavirati prijelaz iz baze u atraktor pa je za to dovoljno hiperplanom presjeci atraktor.

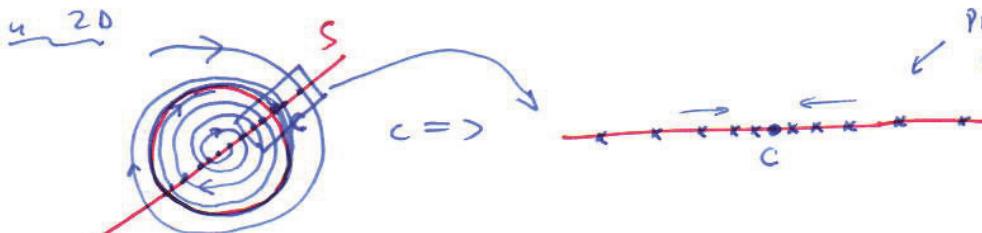
Ako je dinamičke sustave dissipativne, tada se: P.D. tako kontinuiraju vremenu (i za P.P. vrijedi Liouvilleov teorem).

Prelaskom na P.P. modelimo kontinuirani sustav reda N na diskretni sustav reda $N-1$.

upr. gravitacije u hodu i periodične trajektorija:



$$\begin{aligned} T[P_0] &= P_1 \\ T[P_1] &= P_0 \\ T^2[P_0] &\equiv \tilde{T}[P_0] = P_0 \quad (\text{opet isto}) \end{aligned}$$



Određivanje stabilnosti trajektorije na P.P.

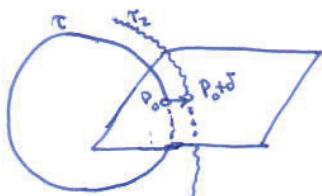
Ako je trajektorija T u hodu u hodu svijetle stabiliteti, tada ga ima i u hodu P.P.

$$P_0(x_0, y_0)$$

$$P_{n+1} = \tilde{T}[P_n]$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}$$

Promatramos u hodu trajektorije T i njenu prethodovanju:



$$\begin{aligned} T(\vec{P}_0 + \vec{\delta}) &= \begin{pmatrix} f(x_0 + \delta x, y_0 + \delta y) \\ g(x_0 + \delta x, y_0 + \delta y) \end{pmatrix} \\ &= \tilde{T}(\vec{P}_0) + \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}|_{(x_0, y_0)} & \frac{\partial f}{\partial y}|_{(x_0, y_0)} \\ \frac{\partial g}{\partial x}|_{(x_0, y_0)} & \frac{\partial g}{\partial y}|_{(x_0, y_0)} \end{bmatrix}}_{\text{linearanizacija}} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} + \vec{\delta} \\ &\quad M - \text{Eloguetova matrica} \\ &\quad \text{prethodovanje u reki (F.T.!)} \end{aligned}$$

$$\begin{aligned} T(\vec{P}_0 + \vec{\delta}) &\approx \vec{P}_0 + M \cdot \vec{\delta} \\ T^n(\vec{P}_0 + \vec{\delta}) &= \vec{P}_0 + M^n \vec{\delta} \end{aligned}$$

$$\Leftrightarrow T(\vec{x}^*) = \vec{x}^*$$

$\lambda_{1,2}$:

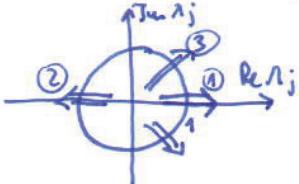
~ sv. vrijednost $M < 1 \Rightarrow$ trajektorija je STABILNA (u tom smislu)

~ sv. vrijednost $M > 1 \Rightarrow$ trajektorija je NESTABILNA (u tom smislu)

Brojčanito: $(\lambda_1 = 1 \text{ nije}) \Rightarrow$ trajektorija je nestabilna

Ako je λ_1 unutar jedinice, trajektorija je stabilna. Ako nije λ_1 izvan jed. kružnice, trajektorija postaje nestabilna!

Scenarij nestabilnosti:

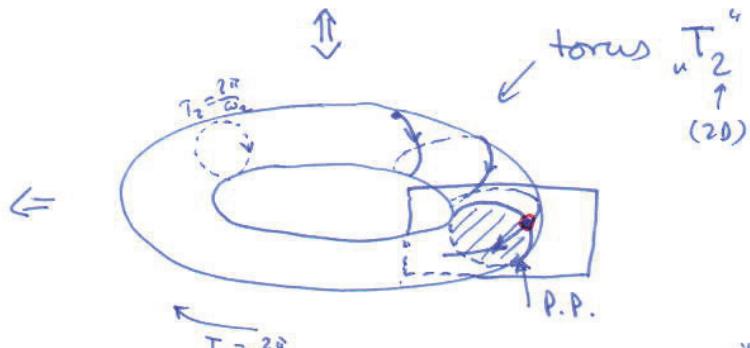
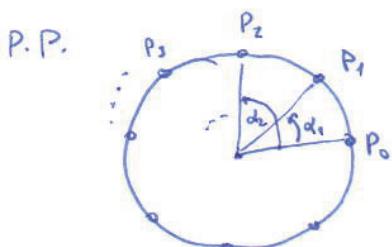
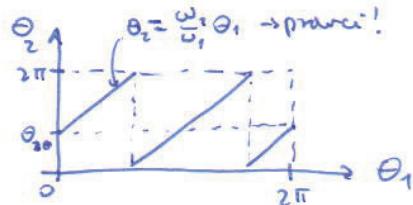


- ①: tangencijalne bifurkacije
- ②: odvajajuće periode (doubling period) bifurkacije
- ③: Hopfove bifurkacije

Demonstrirano kvariperiodičko gibanje u $N=2$ sustavu \Rightarrow imamo 2 karakteristike periodicitet (u varijablemne ko-akcije): $\omega_1 : \omega_2$.

$$\theta_1 = \omega_1 t \\ \theta_2 = \omega_2 t$$

$$\rightarrow \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{\omega_2}{\omega_1} \Rightarrow$$



$$\lambda_n = n \cdot \omega_2 T_1 = 2\pi n \frac{\omega_2}{\omega_1}$$

rijeme potrebu da
trajektorija n puta obide
veliki torus

$$\lambda_{km} = \lambda_k + 2\pi \frac{\omega_2}{\omega_1} \equiv T(\lambda_k)$$

intervall kada u svakom sucesivnom prelasku P.P.-a

Poincaréov presjek je
kružni je.

Ako su $\omega_1 : \omega_2$ razjedljive, trajektorija će se zatvoriti same u sebi (tj. će zadržati broj dan broj točaka).

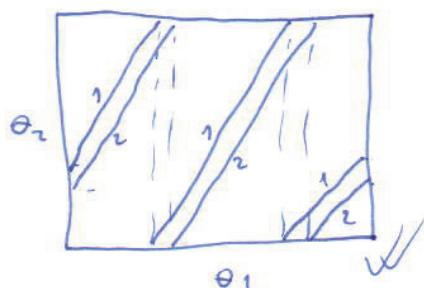
Sho je $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$, imamo kvariperiodičnost tj. P.P. se po volji gusto poljiva točkama (P.P.). [Ovo je usred i jedini moguci prijelaz u 2D sustav; bifurkacija u kvariperiodičko gibanju ($\mathbb{Q} \leftrightarrow \mathbb{R}$) - na tomu nema kaoticitetu!]

Pitanje:

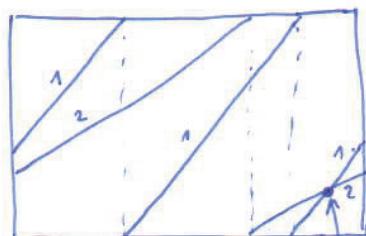
Da li je moguce gibanje takvo da 2 početno bliske trajektorije ustanovljuju dugi vremena bruke udaljenih jednu od druge za veličinu atraktora?

-na T_2 :

(1) trajektorije se puste



(2) trajektorije se rastvaraju



Kad li se na T_2
trajektorije rastvaraju
morale li se speti!
To je dobro moguce!

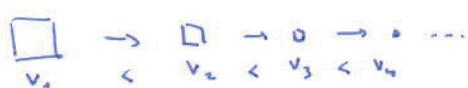
sijecu se!
 $\Rightarrow C$

Da biq imati nestabilnost:

- (1) atraktor mora imati dimenziju > 2 (jer ne T_3 "znači kružka")
- (2) atraktor mora imati dimenziju < 3 (jer je atraktor po mrezi nestabilni fiksni prostor koji je $D=3$)

Takođe dale mogućnost samo atraktoru čija topološka dimenzija nije cijeli broj tj. $D \in \mathbb{Q}$ (između 2 i 3) na kojem je moguća nestabilnost.

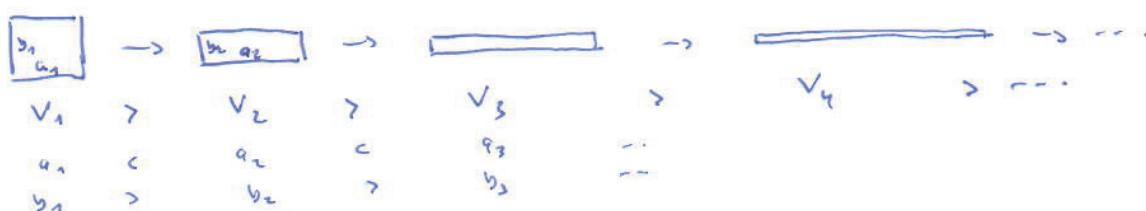
Kako konstruirati takav objekt?



Ako je razimanje u oba smjera, dobivamo kontinuitet u formi trokut (atraktor je F.I.).



Bar u jednom smjeru treba imeti skupinu! Tj. barem jednu sv. mrežu prethodnog (Ly-exponent) mora biti > 1 , ostale mogu biti < 1 .



(napomena: postupimo da se ovakvo nestabilno dogadati ako redlenu likne trokut u odnosu u vrijeme.) simeye
sazimajući

Na, zbog dimenijske sustava oblik F.P. ne mijenja se takođe u njim a i b već se i svira [to je pr. STRETCH & FOLD scenario]

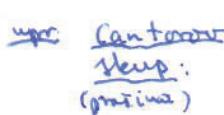


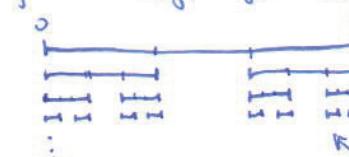
ad infinitum

dolije se SIMALE HORSESHOE (smolevine potkova).

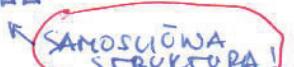
Smolevine potkove su primjer objekta čija je dimenzija između 1 i 2.

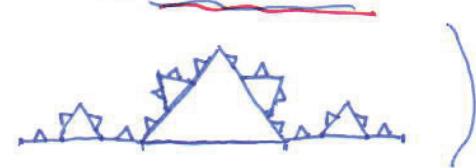
Takvi objekti čija je dimenzija $D \in \mathbb{Q}$ zovu se FRAKTALI.



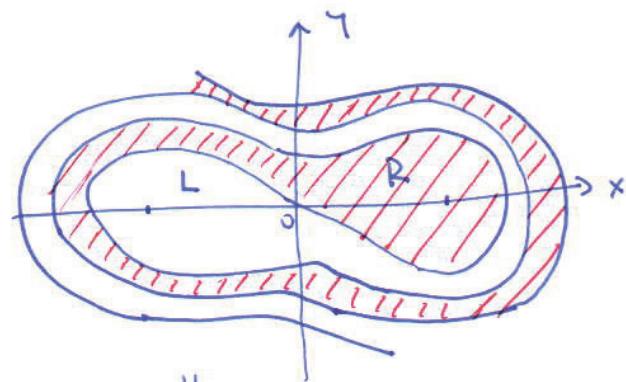
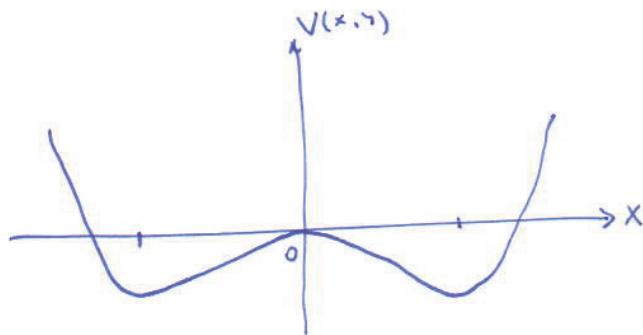


Kochova pakuljice:

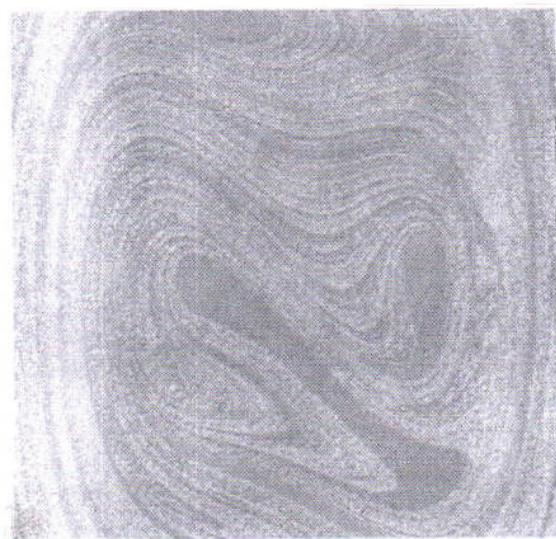




(*) p.s: sistema se Duffingovog oscilatoru (četvrtog) i kvara ustrukcije:

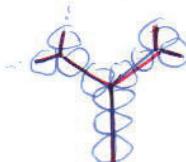


stretch & fold
into :



Kako izraziti dimenziju tj. karakterizirati fraktalni skup?
Tada nadim je je H-B dimenzija.

- Neka je p dimenzija "nedprostori" (tj. hiperplanske u kojoj leži fraktalni skup)
- Uzmemo male p -dimensionalne kugle radijusa ε i pokrivamo njima cijeli fraktalni objekt. Neka nam za to treba $N(\varepsilon)$ kugala. (upr.)



H-B dimenzija (D):

$$N(\varepsilon) \sim \varepsilon^{-D} \Rightarrow \ln N(\varepsilon) \sim \ln \left(\frac{1}{\varepsilon}\right)^D$$

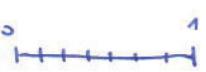
$$D \equiv \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln \left(\frac{1}{\varepsilon}\right)}$$

Ovde definisane dimenzije odgovaraju topoloskim dimenzijama objekta $\in \mathbb{N}$. H-B dimenzija je sredina od neke interpolacije za nećelulozne dimenzije.

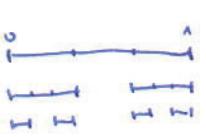
primjeri:

0) točka: \circ $\varepsilon = 1; N=1$
 $\varepsilon = \frac{1}{m}; N=1$

 $\Rightarrow D = \lim_{m \rightarrow \infty} \frac{\ln 1}{\ln m} = \frac{0}{\infty} = 0 \quad \checkmark$

1) dužina: 
 $\varepsilon = 1; W=1$
 $\varepsilon = \frac{1}{2}; W=2$
 $\varepsilon = \frac{1}{3}; W=3$
 \vdots
 $\varepsilon = \frac{1}{m}; W=m$

 $\Rightarrow D = \lim_{m \rightarrow \infty} \frac{\ln m}{\ln m} = 1 \quad \checkmark$

2) Cantorov skup 
 $\varepsilon = 1; W=1$
 $\varepsilon = \frac{1}{3}; W=2$
 $\varepsilon = \frac{1}{3} \cdot \frac{1}{3}; W=2^2$
 \vdots
 $\varepsilon = \frac{1}{3^m}; W=2^m$

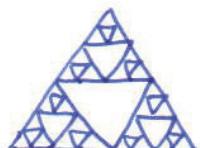
 $\Rightarrow D = \lim_{m \rightarrow \infty} \frac{\ln 2^m}{\ln 3^m} = \frac{\ln 2}{\ln 3} \approx 0.6309 < 1 !$

↑
objekt dimenzije između
 0 (točka) i 1 (dužina)

3) Kochova zaključica 

$$\begin{aligned} \varepsilon = \frac{1}{3}; N=4 \\ \varepsilon = \frac{1}{3} \cdot \frac{1}{3}; N=4 \cdot 4 \\ \vdots \\ \varepsilon = \frac{1}{3^m}; N=4^m \end{aligned} \Rightarrow D = \lim_{m \rightarrow \infty} \frac{\ln 4^m}{\ln 3^m} = \frac{\ln 4}{\ln 3} \approx 1.2619 \in [1, 2] !$$

4) Fraktal Sierpiński



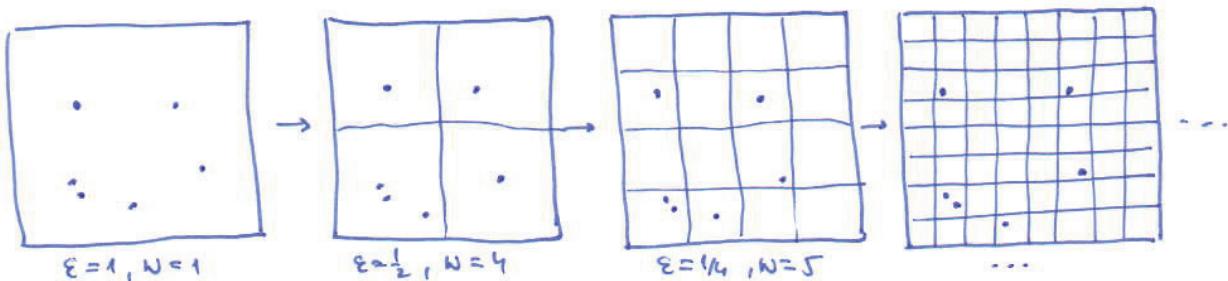
$$\varepsilon = \frac{1}{2}; N=3 \quad \dots \quad \varepsilon = \frac{1}{2^m}; N=3^m$$

$$D = \frac{\ln 3}{\ln 2} \approx 1.5847$$

↑
objekt dimenzije između
dužine (1) i plohe (2)

Kako numerički odrediti H-B dimenziju? Racionalom generisanih stepenih očekujući da je konstanta i ne ide do skale razmaka, a ne znaće niti eksplicitno zapisati taj koeficijent koje ga generira.

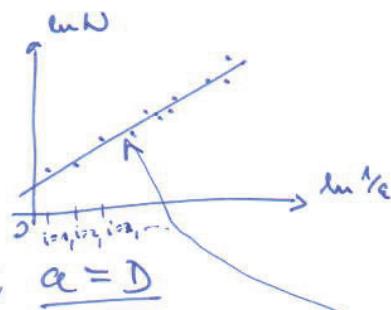
Za to koristimo approximaciju, tзв. Box-counting dimension (approximacija H-B dimenzije za konstantu stepen).



$$D = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln \frac{1}{\varepsilon}} \rightarrow \ln N = D \ln \frac{1}{\varepsilon}$$

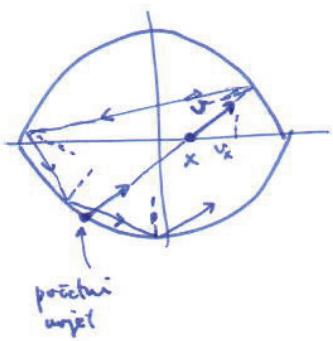
$$y = ax + b$$

↳ linearne regresija; $a = D$

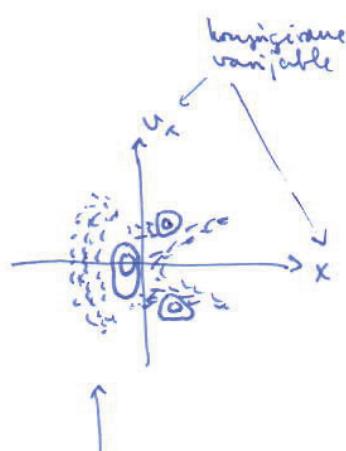


(*) Sustavljajući mrežu u mrežu kroz kružnik na polu i brojimo "kvadrice" koje su sastavne učlanice dijelom stepena (trotkom ili kvadratu). Za svaku iteraciju imamo $\varepsilon : N$ koje učitavamo na graf i u njega uđinemo lin. regresiju (parovi $(\ln \frac{1}{\varepsilon_i}, \ln N_i)$)

primjer: 2D Mandelbrot "biljka"



Svaku učanici od svih generacija par (x, v_x) . Taj stepen točaka očekuje P.P. u formi prostorne biljke.



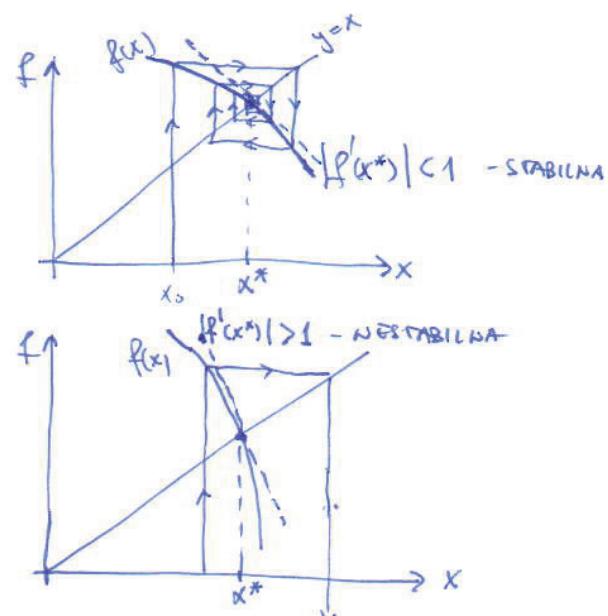
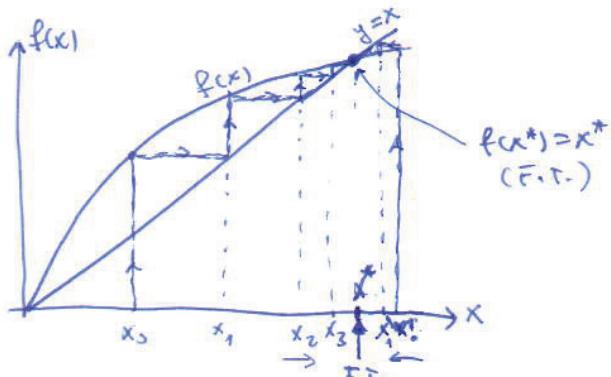
Vidite se Mandelbrot regije u hertzovu polarnoj.

$$\boxed{x_{n+1} = f(x_n)}$$

Želimo uverti preslikavanje $f(x_n) \rightarrow x_{n+1}$ za nejednoznačne:

- 1') kontinuiravaju
- 2') domene $0 < x < \infty$
- 3') $f(x^*) = x^* \Rightarrow x^*$ je fixna točka preslikavanja $f(x)$
- 4') $|f'(x^*)| < 1 \Rightarrow x^*$ je STABILNA
 $|f'(x^*)| > 1 \Rightarrow x^*$ je NESTABILNA

metoda slijecanih stika:



stabilnost fixne točke:

$$x_{n+1} = f(x_n)$$

$$x^* = f(x^*)$$

$$x_0 = x^* + \delta \quad ; \quad x_1 = f(x^* + \delta)$$

$$x_2 = f(x_1) \approx f[f(x^* + \delta)]$$

$$x_n = \underbrace{f[f[\dots f(x^* + \delta)]\dots]}_n$$

Točka x^* je stabilna ako skup $\{x_i\}$ konvergira ka x^*

$$\text{def: } \delta_n = x_n - x^* \Rightarrow x_n = x^* + \delta_n$$

$$\delta_{n+1} = x_{n+1} - x^* = f(x^* + \delta_n) - x^* \underset{\text{lin.}}{\approx} f(x^*) + \frac{\partial f}{\partial x} \Big|_{x^*} \cdot \delta_n - x^*$$

$$\delta_{n+1} = \frac{\partial f}{\partial x} \Big|_{x^*} \cdot \delta_n$$

$$\underset{\text{M (Floquet)}}{\approx} M \cdot \delta_n$$

$$M = \left| \frac{\partial f}{\partial x} \Big|_{x^*} \right| < 1 \Rightarrow x^* \text{ je stabilna} \quad (f'(x^*) \in (-1, 1))$$

$$M = \left| \frac{\partial f}{\partial x} \Big|_{x^*} \right| > 1 \Rightarrow x^* \text{ je nestabilna}$$

$$M = \left| \frac{\partial f}{\partial x} \Big|_{x^*} \right| = 1 \Rightarrow \text{granicne stabilitosti}$$

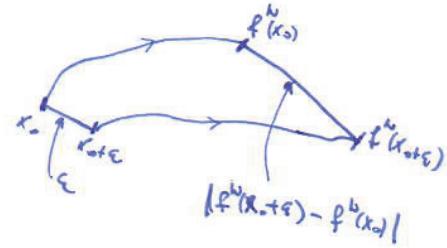
Ljapunovov exponent: $\lambda(x_0)$

Gledamo stabilnost prelikavanja u okolini x_0 .

$$x_0, x_0 + \varepsilon$$

$$f^N(x_0) = f[f[\dots f(x_0)]\dots]$$

$$f^N(x_0 + \varepsilon) - f^N(x_0), \quad \varepsilon \rightarrow 0, \quad N \rightarrow \infty$$



$$\lambda(x_0) = \lim_{N \rightarrow \infty} \left(\lim_{\varepsilon \rightarrow 0} \frac{1}{N} \ln \left| \frac{f^N(x_0 + \varepsilon) - f^N(x_0)}{\varepsilon} \right| \right) \stackrel{\text{diferencija!}}{=} \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left| \frac{Df^N(x_0)}{dx} \right|$$

$$\underbrace{\varepsilon e^{N\lambda(x_0)}}_{\lambda(x_0)} = \left| f^N(x_0 + \varepsilon) - f^N(x_0) \right| \rightarrow \text{udaljenost } x_0 \text{ i } x_0 + \varepsilon \text{ nakon } N \text{ iteracija}$$

$\lambda(x_0)$ gornji koliko je normalne izmedu slika poveća ($\operatorname{Re}(\lambda) > 0$) ili manji ($\operatorname{Re}(\lambda) < 0$) po sedmoj iteraciji.

pogledajmo:

$$\frac{d}{dx} f^N(x) \Big|_{x_0} = \frac{d}{dx} f[f(x_1)] \Big|_{x_0} = \frac{d f[f(x_1)]}{d f(x)} \cdot \frac{d f(x)}{dx} \Big|_{x=x_0} = \frac{d f(x_1)}{d x_1} \frac{d f(x_0)}{d x_0}$$

u jedi proporcije:

$$\frac{d}{dx} f^N(x_0) = f'(x_0) \cdot f'(x_1) \cdots f'(x_{N-1})$$

pa slijedi:

$$\lambda(x_0) = \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left| \prod_{i=0}^{N-1} f'(x_i) \right| = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|$$

↳ ly-exponent

LOGISTIČKA MAPA

110

Sjetimo se logističke jednadžbe: $\frac{dx}{dt} = bx - cx^2$; $b, c > 0$

Vremenske je mjerene diskretno (npr. u svakojih mjesecu inventura stoga mjeseca): $t = n \cdot \Delta t$, $n = 0, 1, 2, \dots \in \mathbb{N}$.

$$x(n \Delta t) \equiv y_n$$

$$x((n+1) \Delta t) \equiv y_{n+1}$$

$$y_{n+1} - y_n = b y_n - c y_n^2$$

$$y_{n+1} = (b+1) \left[y_n - \frac{c}{b+1} y_n^2 \right]$$

pođeđuje definicije $0 \leq y_n \leq \frac{b+1}{c}$

c stavljamo u skalu; supst. $y_n \equiv \frac{1+b}{c} x_n$; $0 \leq x_n \leq 1$

$$\text{def } b+1 \equiv 4r$$

=>

$$x_{n+1} = 4r x_n (1 - x_n)$$

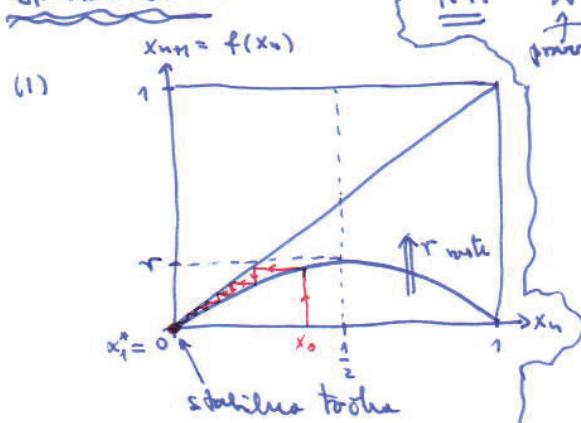
kontrolni parametar

- FIRGBAUMOVU PRESLIKAVANJE

(najjednostavniji populacijski model)

$$x_n \in (0, 1]$$

Stabilnost:



F.T.

$$x^* = 4r x^* (1 - x^*)$$

ponore

parabola (na dolje)

$$x_1^* = 0$$

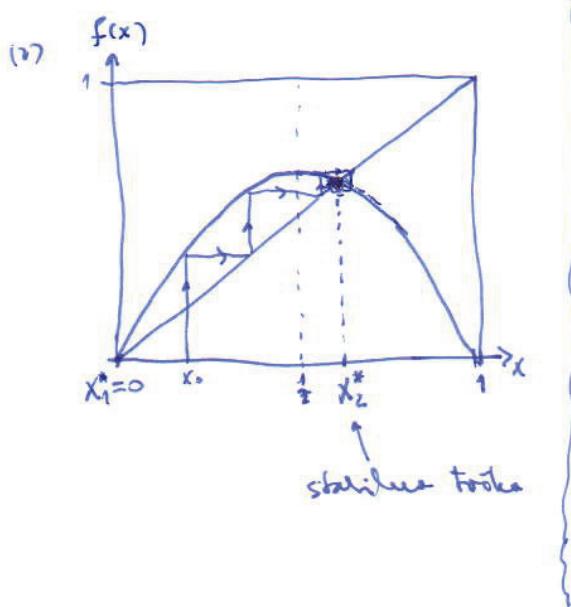
$$x_2^*: 1 = 4r (1 - x^*) \Rightarrow$$

$$x_2^* = 1 - \frac{1}{4r}$$

prijelazne re²zne
 $r > \frac{1}{4}$ (tada je
 $x_2^* > 0$)

$$f(x) = 4r x (1 - x)$$

$$f'(x) = 4r (1 - 2x)$$



Kada je $|f'(x^*)| < 1$, tj. kada je x^* stabila?

$$x^* = 0: f'(0) = 4r; |4r| < 1$$

$$-1 < 4r < 1 \quad |:4$$

$$-\frac{1}{4} < r < \frac{1}{4} \rightarrow r \in [0, \frac{1}{4})$$

STABILNA!
 $x^* = 0$

$$x^* = 1 - \frac{1}{4r}$$

$$f'(1 - \frac{1}{4r}) = 4r(1 - 2(1 - \frac{1}{4r})) = 2 - 4r$$

$$|2 - 4r| < 1$$

$$-1 < 2 - 4r < 1$$

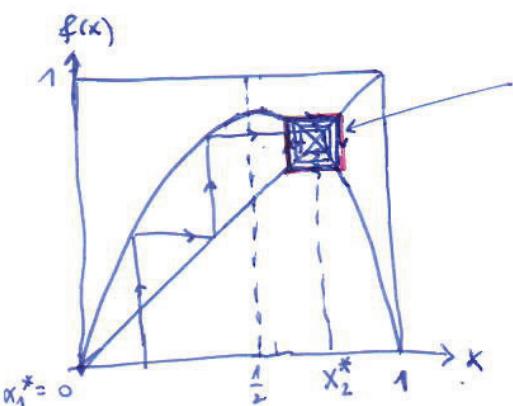
$$-3 < -4r < -1 \quad |:(-4)$$

$$\frac{1}{4} < r < \frac{3}{4}$$

$$\rightarrow r \in (\frac{1}{4}, \frac{3}{4})$$

$x^* = 1 - \frac{1}{4r}$
STABILNA!

za $r > \frac{3}{4}$ točka x_2^* postaje liti stabilna:

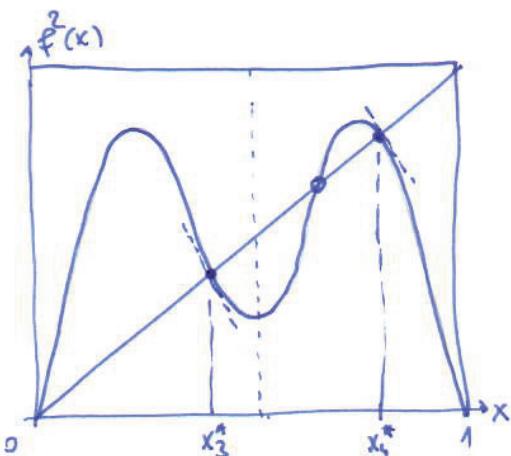


udaljeni smo se od x_2^* i razvili su se novi stabilni koordinati koji su određeni točkama u kojima je $f' = -1$ (tj. $|f'| = 1$)

x_1^* i x_2^* postale su nestabilne za $r > \frac{3}{4}$.

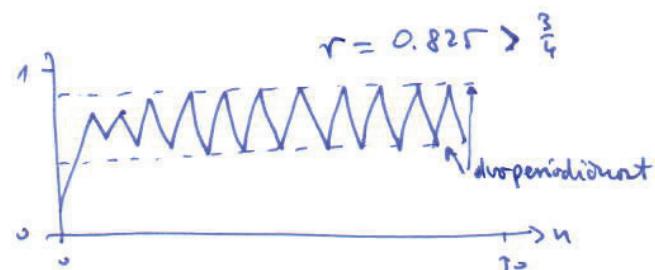
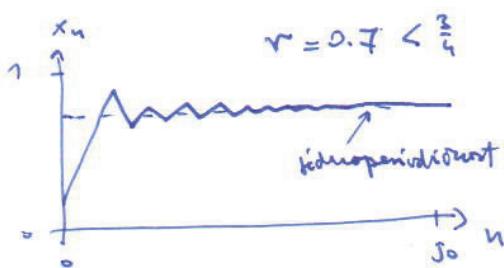
Šta delje? → Razmatramo 2. iteraciju u mreži; $f^2(x_n)$:

$$f^2(x_n) = f[f(x_n)] = r^2 [rx_n(1-x_n) [1 - rx_n(1-x_n)]]$$



→ Dobili smo 2 nove fixne točke x_3^* i x_4^* , može se pojaviti da je derivacija u obje točke ista tj. da obje stvaraju dojnjih i izgubuju stabilitet.

u području $r \in (\frac{3}{4}, ?)$ obje x_3^* i x_4^* su stabilne tj. imaju 2 atraktora \Rightarrow DVOPERIODIČNOST! (bifurkacija)



④ Prikaz povećavajućih delja za $f^3(x)$, $f^4(x)$ itd. Bifurkacije se događaju za:

$$r_0 = \frac{1}{4} = 0.25$$

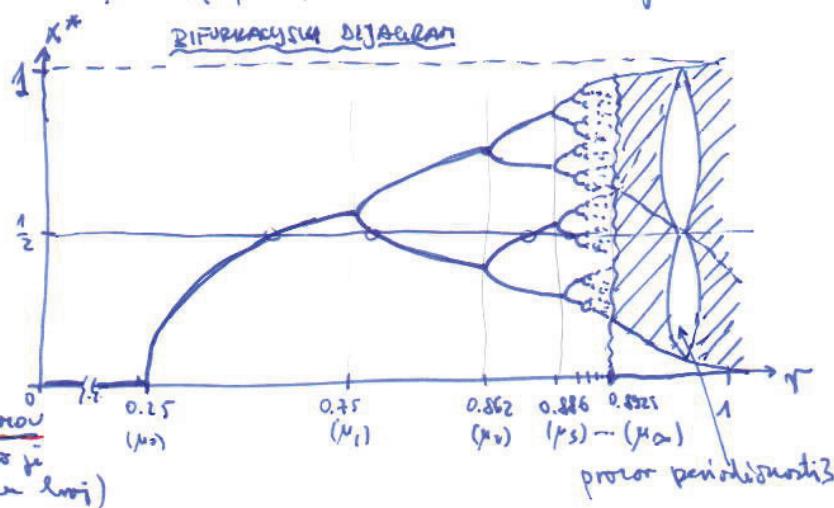
$$\begin{aligned} &\text{periodicitet} \\ &2^0 = 1 \\ &r_1 = \frac{3}{4} = 0.75 \\ &2^1 = 2 \\ &r_2 = \frac{1+\sqrt{5}}{4} = 0.862 \\ &2^2 = 4 \\ &r_3 = 0.886 \\ &2^3 = 8 \\ &r_4 \approx 0.8925 \end{aligned}$$

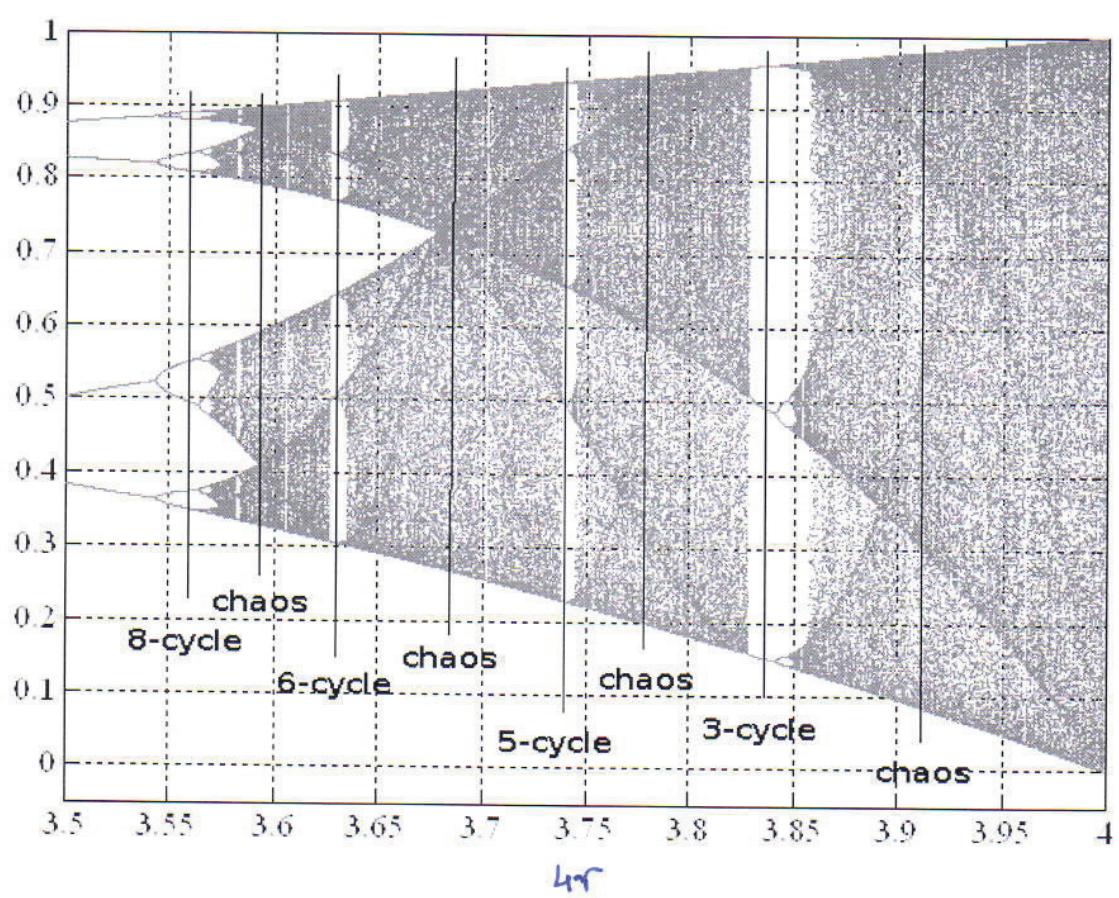
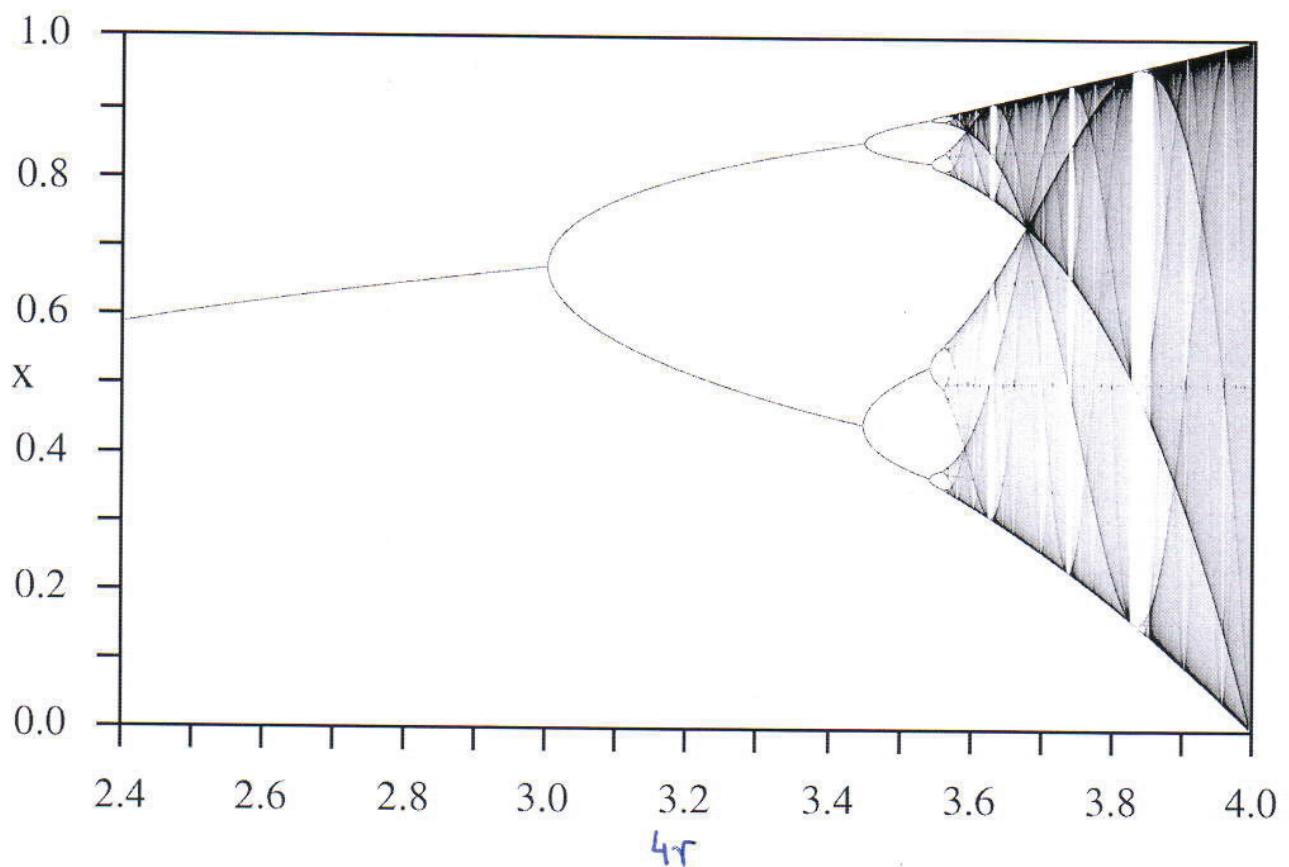
KAO!

Period atraktora

$$\delta = \lim_{n \rightarrow \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \approx 4.669 \dots$$

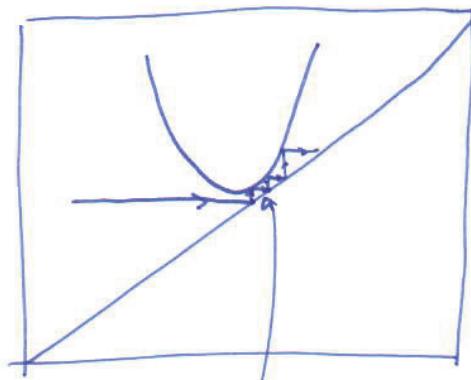
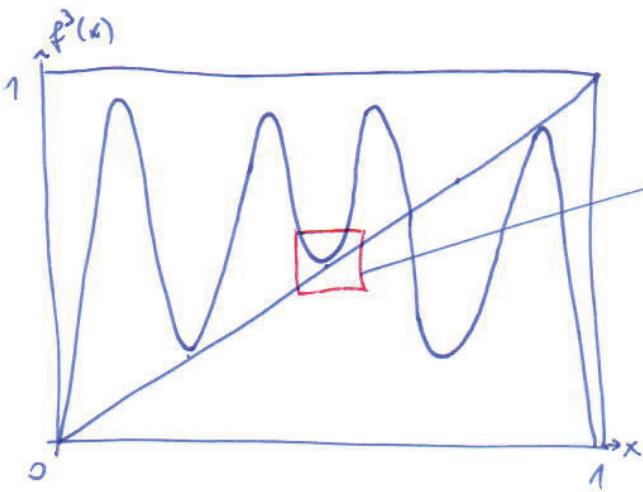
FEGEWBAMON
BEDO] (to je
univerzalan broj)





Prvotni periodicitet u chaosu - INTERMITENCIJA

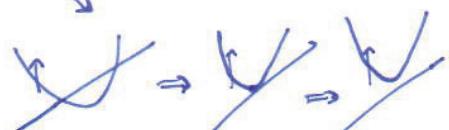
- za pravac perioda 3 : $f^3(x)$



puno malih kvalita \Rightarrow "primivarijantni" sustava (intermitencija)

Sustav se "zagleni" u izhom kvalitu i tuc provode puno iteracija (primiviti se). Tadim provete kvar kvaliteta na mala udaljenja po mapi (jer je "kvar") ali onda opet neiste me kvalitete (isti ili drugi) i opet se "zagleni". Tice kvalitete je ovaj 3 pa zaglavljivanje u najmanji degi pravac periodicitati 3 (ta "prva trojka").

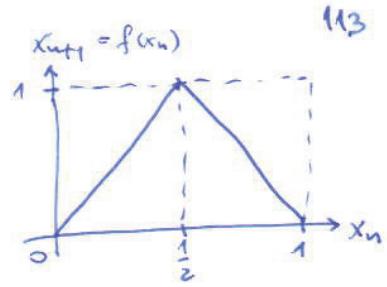
Ovaj se radi u kontekstu tangencijskoj bifurkaciji i tako scenariju kvalita u chaosu intermitencijom.



- ④ Primijetimo da "Feigenbaumov drob" ima samosličnu strukturu. Nekoliko razloga pravca periodiciti, opet se odnosi isti scenarij dupljenja perioda bez ne početku.

Trotkevna mopa ("tent map")

$$x_{n+1} = 1 - 2|x_n - \frac{1}{2}| = \begin{cases} 2x_n & x_n < \frac{1}{2} \\ 2(1-x_n) & x_n > \frac{1}{2} \end{cases}$$

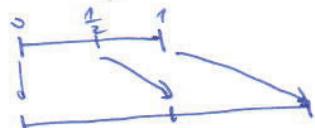


113

- poč. vijet: $x_n \in [0, 1] \Rightarrow x_{n+1} = 1 - 2|x_n - \frac{1}{2}| \in [0, 1]$ tj. orbita su zatvorene za $x \in [0, 1]$

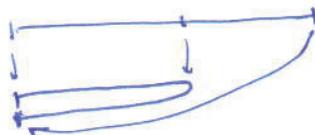
- različiti načini:

(i) stretching (uniformno razdvajanje u dobroteku diviziju)



⇒ eksponentijalna divergencija bliskih trajektorija.

(ii) folding (zavijanje na pole) \Rightarrow dvoji orbiti veruće u $[0, 1]$ i osni mopa neinvazibilnosti.



Konečni potovaj trake $x_0 \in [0, 1]$ u sklopu (i) i (ii) dan je neprom.

- Da bi se imeli chaos, mopa se mora u posledici razširiti.

- Da bi trajektorija bila ograničena (na atraktor), moraju imati zavijanje.

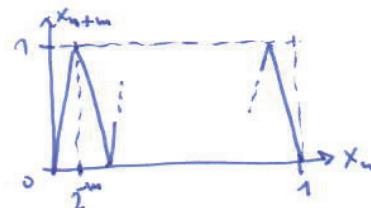
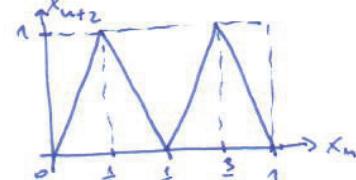
Dakle:

$$x_{n+1} = f(x_n)$$

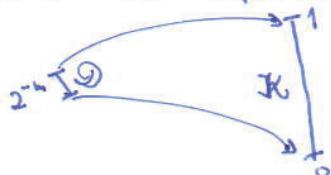
$$x_{n+2} = f^2(x_n) = f(f(x_n))$$

⋮

$$x_{n+m} = f^m(x_n)$$



Ako je x_0 u intervalu veličine $\pm 2^{-m}$ do neke točke, tada će x_m biti liko geje u $[0, 1]$ \Rightarrow eksponentijalna raspširjivost ne male projekcije poč. vijeta x_0 .



Lj-exponent

Osmislio je ne malo područje gdje vrijedi $x_{n+1} = 2x_n$ ($x_n < \frac{1}{2}$) radi jednostavnosti.

$$x_n = f^n(x_0) = 2 \cdot 2 \cdot 2 \cdots (2x_0) = 2^n x_0$$

$$\lambda(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{df^n}{dx} \right|_{x_0} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln 2^n = \ln 2$$

$$(f^n(x_0) = 2^n \cdot x_0; \frac{d f^n}{dx} = 2^n)$$

$$\lambda = \ln 2 > 0$$

\Rightarrow ekspon. divergencija trajektorija

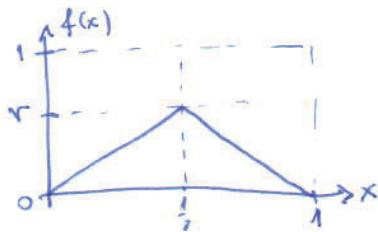
ne ovisi o x_0 !

Weljū ovis kontroli parameter (r) u mape:

14

$$f(x) = r(1 - 2|x - \frac{1}{2}|)$$

$$\begin{aligned} r &\in [0, 1] \\ x &\in [0, 1] \end{aligned}$$



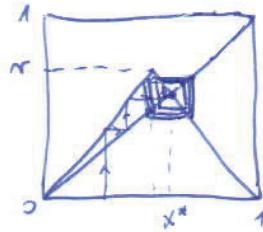
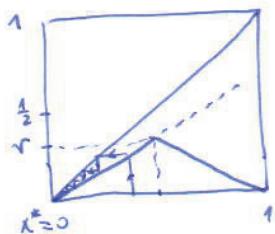
U kiti vidimo da r direktno određuje negit funkcije $f(x)$, a time i $M = f'(x_0)$, dokle stabilnost preslikavanja des x₀.

- opet promotriju $f(x) = 2^r x$, $x < \frac{1}{2}$

$$f'(x) = (2^r)^n \cdot x$$

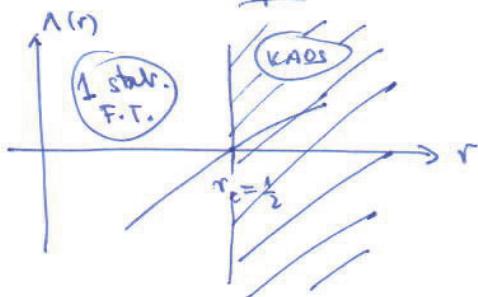
$$\lambda(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{df^n}{dx} \right|_{x_0} = \ln 2^r = \begin{cases} < 0, r < \frac{1}{2} \\ > 0, r > \frac{1}{2} \end{cases}$$

Dokle $r_c = \frac{1}{2}$ je kritična vrijednost kontrolnog parametra. Za $r < \frac{1}{2}$ imamo stabilnu fix. točku, a za $r > \frac{1}{2}$ nestabilnu:



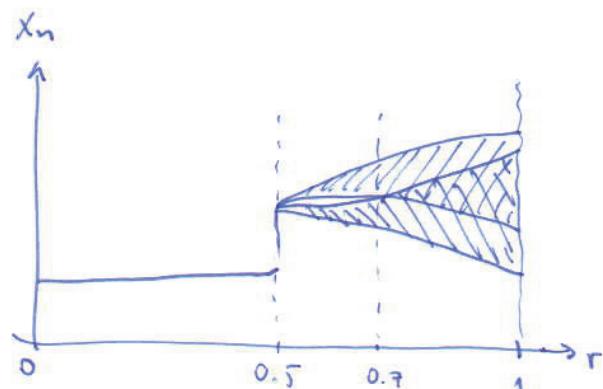
$$\begin{aligned} \leftarrow \text{jer je } f'(x) = & \begin{cases} 2^r, x < \frac{1}{2} \\ -2^r, x > \frac{1}{2} \end{cases} \\ \Rightarrow \text{za } r > \frac{1}{2} \text{ imamo} & \text{nestabilnu mape + F.T.} \end{aligned}$$

λ ima ulogu parametra uređenja sustava. (Analogni kriterijevi eksponenta (PG), parametar uređenja $\lambda = \ln 2^r$ skoline se kao potencijal u blizini r_c ; $\lambda \approx r - r_c$).



(D2)
Numerika:

Iterativi formula $x_{n+1} = r(1 - 2|x_n - \frac{1}{2}|)$ do 100-ite iteracije (npr. $N = 500$), odbrati prve 80% iteracija i računati zadnjih 20% (da je mogući sistem da nade u fixu točku).

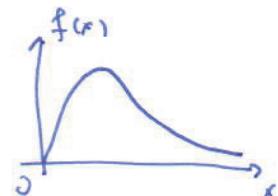


Duge subcesione generacije biolische populacije povrane su jednako
dlobom

$$x_{n+1} = x_n e^{r(1-x_n)}, \quad r > 0$$

↓

- exp. prirost za male gustoci
- epidemijiski pad za velike gustoci



F.T. :

$$x^* = x^* e^{r(1-x^*)} \quad | : x^*$$

$$1 = e^{r(1-x^*)} \quad | \ln$$

$$0 = r(1-x^*)$$

$$\Leftrightarrow \boxed{x^* = 1}$$

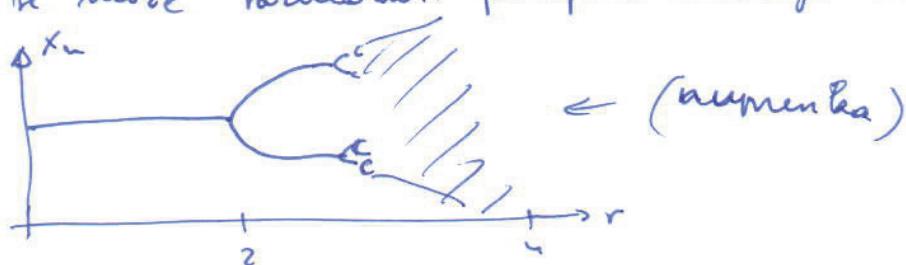
Unijednoost r dada F.T. postane nestabilna (BIFURKACIJA) :

$$f'(x) = e^{r(1-x)} + x \cdot (-r) \cdot e^{r(1-x)} \\ = e^{r(1-x)} (1 - rx)$$

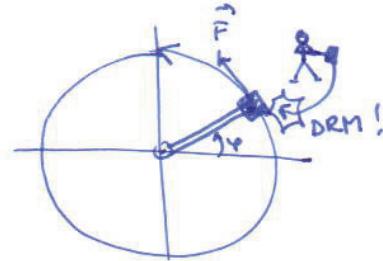
$$f'(x^*) = \underbrace{e^{r(1-x^*)}}_{=1} (1 - r) = 1 - r$$

nestabilnost: $|1-r| > 1 \iff r > 2 \Rightarrow \text{BIFURKACIJA}!$

(Dolje se može vremenuti putem analogne logistickog mape:



Rotator s tijenjem tijemom nizom po volji jake ih, ali kratkikh periodičnih udaraca.



F - sila tijenja;
periodični (T) djeluje
na rotator izvan
 $K \cdot f(\varphi)$ u trenutku
udarca

$$\ddot{\varphi} + \Gamma \dot{\varphi} = K f(\varphi) - \sum_{n=0}^{\infty} \delta(t - nT), \quad n \in \mathbb{Z}$$

tijenje

uvodimo varijable (dimenz. sustav s 3 varijable)

$$\begin{aligned} x &= \varphi \\ y &= \dot{\varphi} \\ z &= t \end{aligned}$$

\Rightarrow

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\Gamma y + K f(x) - \sum_{n=0}^{\infty} \delta(z - nT) \\ \dot{z} &= 1 \end{aligned}$$

s 2 varijable

Neautonomi sustav
Transformirati je u
autonomi sustav
3 nezavisne varijable.

Formalnim interpretiranjem sustava po vremenu (z) dolje se
dokazat će da je slobodna masa (to je slobodna δ -funkcija) fizički
opis.

$$x_n \equiv \lim_{\varepsilon \rightarrow 0} X(nT - \varepsilon)$$

$$y_n \equiv \lim_{\varepsilon \rightarrow 0} Y(nT - \varepsilon)$$

Tražimo opće rješenje za $(n+1)T - \varepsilon > t > nT - \varepsilon$:

$$y(t) = y_n e^{-\Gamma(t-nT)} + K \sum_{m=0}^{\infty} f(x_m) \int_{nT-\varepsilon}^t dt' e^{\Gamma(t'-t)} \delta(t' - mT) \rightarrow \text{integralni oblik rješenja (Greenova funkcija)}$$

$$(*) \sum_{m=0}^{\infty} f(x_m) \int_{nT-\varepsilon}^t dt' e^{\Gamma(t'-t)} \delta(t' - mT) = \sum_{m=0}^{\infty} f(x_m) e^{\Gamma(mT-t)} = f(X(t)) = f(X_n)$$

(analogno i za $\dot{x} = y$ sledi slijedeći)

za $t = (n+1)\tau$, $\epsilon \rightarrow 0$ dobivamo:

117

$$y_{n+1} = e^{-\frac{\tau}{\Gamma}} \cdot [y_n + K f(x_n)]$$

$$x_{n+1} = x_n + \frac{1 - e^{-\frac{\tau}{\Gamma}}}{\Gamma} \cdot [y_n + K f(x_n)]$$

→ redukcija 3D din. sustava
na 2D diskontinuir mapu

$$x_{n+1} = f(x_n, y_n)$$

$$y_{n+1} = g(x_n, y_n)$$

a) limes velikog trenja ($\Gamma \rightarrow \infty$) i jakeg ulazaca ($K \rightarrow \infty$):

- za $P, K \rightarrow \infty$ je $\frac{\Gamma}{K} \rightarrow 1$

- def.: $f(x_n) = (n-1)x_n - \mu x_n^2$ (definicija parametra $\underline{\mu}$) !

$$\begin{aligned} x_{n+1} &= x_n + \frac{1 - e^{-\frac{\Gamma}{\Gamma}}}{\Gamma} \left[y_n + K (n-1)x_n - \mu x_n^2 \right] = x_n + \mu x_n - x_n - \mu x_n^2 = \\ &= \underline{\mu} x_n (1 - x_n) \xrightarrow{\text{logistička mapa}} \underline{\text{logistička mapa}} (\mu = 4r) ! \end{aligned}$$

Ključ rotator se u linijskom velikog trenju i jakog ulazaca svede na logističku mapu.

b) Hénonova mopa

Zapitimo jednostavne ključ rotatora na mrežu drugačiji način

$$\begin{aligned} y_{n+1} &= e^{-\frac{\tau}{\Gamma}} [y_n + K f(x_n)] \\ x_{n+1} &= x_n + \frac{e^{\frac{\Gamma}{\Gamma}-1}}{\Gamma} y_{n+1} \end{aligned} \quad \begin{matrix} (*) \\ (1) \\ (2) \end{matrix}$$

vjeravamo po y_{n+1} :

$$y_{n+1} = (x_{n+1} - x_n) \frac{\Gamma}{e^{\frac{\Gamma}{\Gamma}-1}} \xrightarrow{n \rightarrow n-1} y_n = (x_n - x_{n-1}) \frac{\Gamma}{e^{\frac{\Gamma}{\Gamma}-1}}$$

Stavimo radi jednostavnosti $\Gamma \geq 1$, izrazim y_n (tako da u y_{n+1} promenjuju $n \rightarrow n-1$ i uvrstimo u $(*)-1$):

$$y_{n+1} = e^{-\Gamma} \left((x_n - x_{n-1}) \frac{\Gamma}{e^{\Gamma}-1} + K f(x_n) \right)$$

$$(x_{n+1} - x_n) \frac{r}{e^{r-1}} = \frac{r e^{-r}}{e^{r-1}} (x_n - x_{n-1}) + e^{-r} K f(x_n) \quad | : r \quad | \cdot (e^r - 1)$$

$$x_{n+1} + e^{-r} x_{n-1} = (1 + e^{-r}) x_n + \frac{1 - e^{-r}}{r} K \cdot f(x_n)$$

def:

$$(1) \quad \frac{1 - e^{-r}}{r} K f(x_n) \equiv - (1 + e^{-r}) x_n + 1 - a x_n^2 \rightarrow \text{definicija parametra } a$$

$$(2) \quad -e^{-r} \equiv b \rightarrow \text{definicija parametra } b; |b| \leq 1$$

\hookrightarrow dissipacija!

$$\Rightarrow x_{n+1} = 1 - a x_n^2 + b x_{n-1}$$

a to se daće napisati u obliku 2D mape:

$x_{n+1} = 1 + y_n - a x_n^2$
 $y_{n+1} = b x_n$

dissipacija

\rightarrow HÉNONOVA MAPA

(2D extrema logisticke mape)

neilinearnost

Hénonova mapa sadrži STRANI ATTRAKTOR - fraktalne strukture (ranositnosti) na manim skalaama prostora).

(*znaci Hénonove mape:

0)



1) $x' = x$

$$y' = y + 1 - a x^2$$



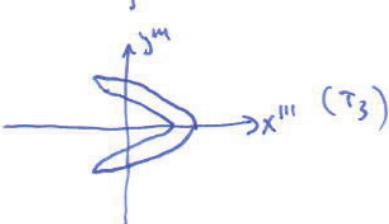
2) $x'' = b x'$

$$y'' = y'$$



3) $x''' = y''$

$$y''' = x''$$



i.e.

$$x''' = y'' = y' = y + 1 - a x^2$$

$$y''' = x'' = b x' = b x$$

$$x_{n+1} = y_n + 1 - a x_n^2$$

$$y_{n+1} = b x_n \quad \checkmark$$

djelovanje Hénonove mape:

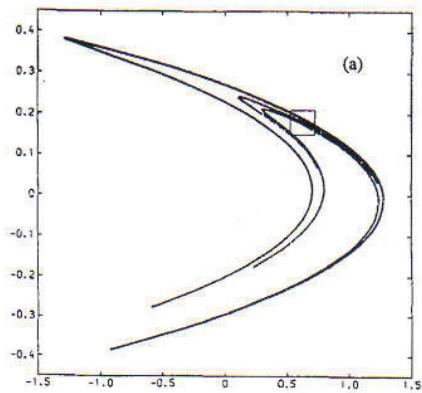
$$T = T_3 \cdot T_2 \cdot T_1$$

$$\begin{pmatrix} x \\ y \end{pmatrix}_{\text{uht}} = T \begin{pmatrix} x \\ y \end{pmatrix}_u$$

Hénonov atraktor

$$\begin{aligned}x_{k+1} &= y_k + 1 - \alpha x_k^2 \\y_{k+1} &= \beta x_k\end{aligned}$$

↑ linearnost
↑ dissipacija



$$\alpha = 1.4 \\ \beta = 0.3 \rightarrow$$

numerički: $D \approx 1.26$

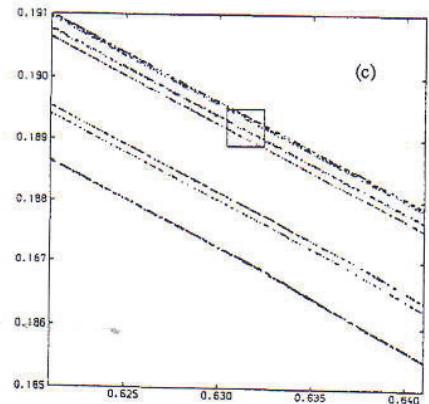
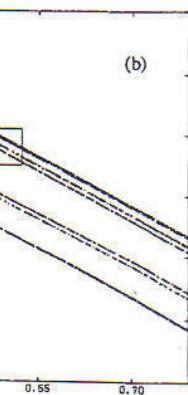
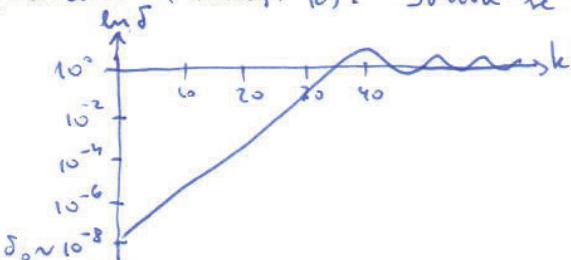
Javljaju:

$$J = \begin{vmatrix} \frac{\partial x_{k+1}}{\partial x_k} & \frac{\partial x_{k+1}}{\partial y_k} \\ \frac{\partial y_{k+1}}{\partial x_k} & \frac{\partial y_{k+1}}{\partial y_k} \end{vmatrix} = \begin{vmatrix} -2\alpha x_k & 1 \\ \beta & 0 \end{vmatrix} = -\beta$$

kontrolnačka F.P. za $|D| < 1$

Lj-exponent:

Gledamo težitko je se uveljati da je početno bliske točke nelson dve trosko "svemir" (iteracija 1). Doliva se da su varijacije za dimenziju atraktora ~ 1 .



najlježe re:

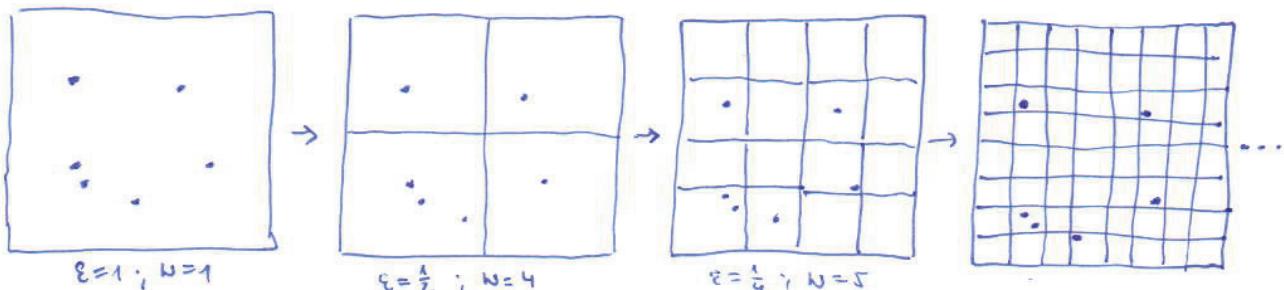
$$\lim_{N \rightarrow \infty; \varepsilon \rightarrow 0} \varepsilon e^{N \bar{\lambda}} = |f^N(x_0) - f^N(x_0)|$$

$$\delta_k = \delta_0 e^{\bar{\lambda} k} \rightarrow \bar{\lambda} = \frac{1}{k} \ln \frac{\delta}{\delta_0}$$

(pojavljaje)

Kako numerički odrediti H-B dimenziju? Računalom generirati sljep P.P. obnovi je konzidan i ne ide do sre skale.

↳ BOX-COUNTING DIMENSION (aproximacija H-B dimenzije za konzidan sljep)



$$D = \lim_{\varepsilon \rightarrow 0} \frac{\ln N(\varepsilon)}{\ln \frac{1}{\varepsilon}}$$

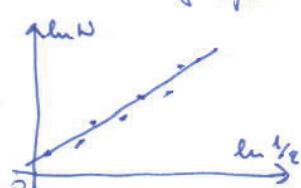
$$\ln N = D \ln \frac{1}{\varepsilon}$$

$$y = ax + b$$

$$a = D$$

(*) Smjerajući mrezu u nelson horizont na polje: brojimo "kutile" u kojoj su razvite točkom iz skupa,

linearna regresija

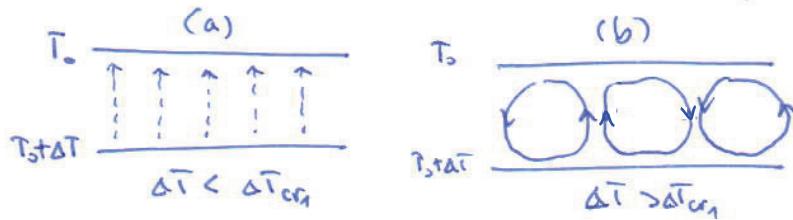


Rayleigh-Bénardov experiment:

Promatra se tok fluida u gravitacijskom polju. Fluid se velazi između 2 beskonačne ravne ploče na različitim temperaturama. Donja ploča je toplija što uzrokuje jače strujanje fluida uz nju te se uspostavlja konvekcija toplijeg (njedeg) fluida prema gore, a hladniji pada prema dolje.

forne probleme:

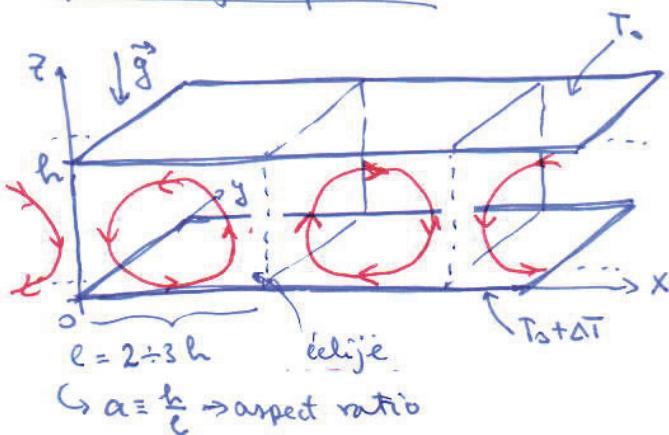
Inje plohe su ne različitim temperaturama. Da bi se uspostavilo strujanje fluida, treba sasladati viskoznost. Zato će među temperaturnim razlikom ΔT neće strujanja fluida već samo uniformni toplinski transport (a).



Da još veću vrijednostinu $\Delta T > \Delta T_{cr}$ strujući fluidi prelazi u kaotiku.

Kako ΔT raste, dolazi do nestabilnosti: započinje stacionarno strujanje fluida rotacijom (b).

geometrija problema



δ - gustoća fluida

μ - viskoznost

p - tlak

ρ - termalna vodljivost

$\vec{F} = -\delta g \hat{z}$ - omjeseča sila (gravitacija)

$\vec{v}(x, t)$ - pdje konvekcije

$T(x, t)$ - poveća temperatura

model (fizmedice) ⭐

- Navier-Stokes :

$$\rho \frac{d\vec{v}}{dt} = \vec{F} - \vec{\nabla}p + \mu \nabla^2 \vec{v}$$

- rotirajući topline :

$$\frac{dT}{dt} = \rho \nabla^2 T$$

- srednja kontinuiteta :

$$\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}(\vec{v} \cdot \vec{v}) = 0$$

- rubni uvjeti :

$$T(x, y, z=0, t) = T_0 + \Delta T$$

$$T(x, y, z=h, t) = T_0$$

veličinom :

$$\frac{d\vec{v}}{dt} = (\vec{v} \cdot \vec{\nabla}) \vec{v} + \frac{\partial \vec{v}}{\partial t} \sim \underline{\underline{V}^2}$$

Lorenzov zavetar sferadatki (Shuster, p. 233)

postpostulace:

koh. lineare
ovinosti

- Ovinost o svih koeficijentata ΔT , osim $\gamma \approx g_0 \cdot (1 - 2 \cdot \Delta T)$
je zanemariva! (Boussinesq-ova aproksimacija)

$$\Rightarrow \text{svrh. kontinuiteta postaje } \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 ; u \equiv v_x, w \equiv v_z$$

- Uvodna funkcija $\Psi(x, z, t)$ za koga je $u = -\frac{\partial \Psi}{\partial z}$ & $w = \frac{\partial \Psi}{\partial x}$
(svrh. kont. je automatski zadovljena)
- Uvodna odstupanje $\Theta(x, z, t)$ od linearne temperaturenog profila:

$$T(x, z, t) = T_0 + \Delta T - \underbrace{\frac{\Delta T}{h} \cdot z}_{\text{linearno}} + \underbrace{\Theta(x, z, t)}_{\text{ne-linearno}}$$

$$\Rightarrow \frac{\partial}{\partial t} \nabla^2 \Psi = - \frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(x, z)} + \nu \nabla^4 \Psi + g \alpha \frac{\partial \Theta}{\partial x}$$

$$(*) \quad \frac{\partial}{\partial t} \Theta = - \frac{\partial(\Psi, \Theta)}{\partial(x, z)} + \frac{\Delta T}{h} \frac{\partial \Psi}{\partial x} + H \nabla^2 \Theta ; \quad \frac{\partial(a, b)}{\partial(x, z)} = \frac{\partial a}{\partial x} \cdot \frac{\partial b}{\partial z} - \frac{\partial a}{\partial z} \cdot \frac{\partial b}{\partial x}$$

$$\nu = \frac{\mu}{g_0} - \text{kinematicka viskoznost}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial z^4}$$

(tako je eliminiran prvični rot (Nussin-Stokes))

- Strobojni rubni uvjeti ("strobojni levi"):

$$\Theta(0, 0, t) = \Theta(0, h, t) = \Psi(0, 0, t) = \Psi(0, h, t) = \nabla^2 \Psi(0, 0, t) = \nabla^2 \Psi(0, h, t) = \phi$$

uvijek je samo najveći koeficijenti u Fourierovom razvoju funkcija Ψ : Θ ; Lorenz je predložio ansatz rješenja:

$$\frac{a}{1+a^2} \frac{1}{H} \Psi = \sqrt{2} X(t) \sin\left(\frac{\pi a}{h} x\right) \sin\left(\frac{\pi z}{h}\right) \quad (**)$$

$$\frac{\pi R}{R_c \Delta T} \Theta = \sqrt{2} Y(t) \cos\left(\frac{\pi a}{h} x\right) \sin\left(\frac{\pi z}{h}\right) - Z(t) \sin\left(\frac{2\pi}{h} z\right)$$

$$a = \frac{h}{e} \dots \text{aspect ratio}$$

$$R = \frac{g \alpha h^3}{4 \nu} \Delta T \dots \text{Rayleighov broj}$$

$$R_c = \frac{(1+a^2)^3}{a^2} \pi^4$$

Uvjetnički (xx) u (x) i odgovarajuće harmonike; definisemo
LORENZOV MODEL:

$$\begin{cases} \dot{X} = G(Y - X) \\ \dot{Y} = -XZ - Y + rX \\ \dot{Z} = XY - bZ \end{cases}$$

$$\dot{\vec{X}} = \vec{F}(\vec{X})$$

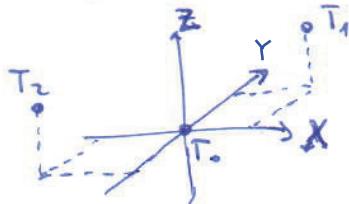
svjetlu dimenziju sustava:

F.T. $T_0 : X = Y = Z = 0$

\hookrightarrow formule vrijednosti bez gibanja fluida

$$T_{1,2} : \begin{cases} X = Y = \pm \sqrt{b(r-1)} \\ Z = r-1 \end{cases}$$

(postoje za $r > 1$)



linearizacija dws F.T.:

$$A = \begin{bmatrix} -G & G & 0 \\ -Z+r & -1 & -X \\ Y & X & -b \end{bmatrix}$$

\rightarrow Lieva derivacija:

$$\lambda = -G - 1 - b = -\frac{4}{3}$$

Velič negativni broj \Rightarrow formi volumen se lovi (ekspresivacijom) sarine u atraktor

$$T_0 : A_0 = \begin{bmatrix} -G & G & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

$$\hookrightarrow \text{w. vrijednosti: } \lambda_{1,2} = -\frac{G+1}{2} \pm \frac{1}{2}\sqrt{(G+1)^2 + 4G(r-1)}$$

$$\lambda_3 = -b$$

- Sve w. vrijednosti $\lambda_1, \lambda_2, \lambda_3 < 0$ za $0 < r < 1 \Rightarrow$ fix. točka $T_0(0,0,0)$ je STABILNA u tom intervalu kontrolnog parametra r ($T_{1,2}$ ne postoje za $r < 1$).
- Za $r=1$ je $\lambda_1=0$ (w. vrijednost potonji broj ϕ); gubi se stabilnost T_0 ne raden stranju 2 stabilne linee točke T_1 i $T_2 \Rightarrow$ PITCHFORK BIFURKACIJA! T_0 postaje nestab.

$$\underline{T_{1,2}}: A_{\lambda_{1,2}} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -C \\ C & C & -b \end{bmatrix} ; C \equiv \pm \sqrt{b(r-1)}$$

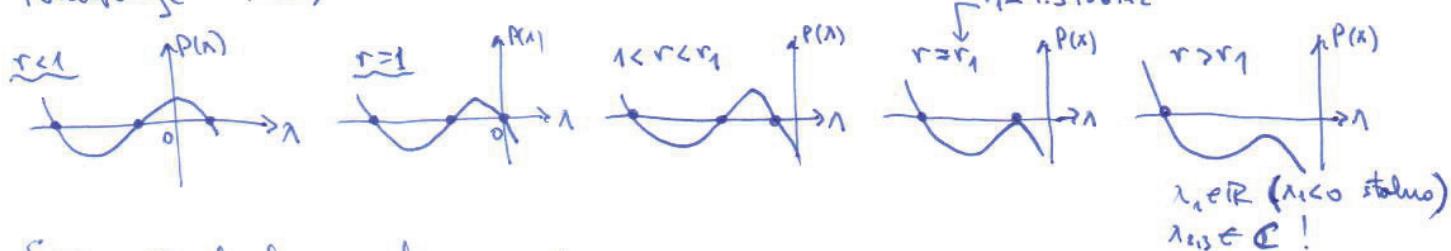
$$\begin{aligned} P(\lambda) &= \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & -C \\ C & C & -b-\lambda \end{vmatrix} = -(\lambda+\sigma) \underbrace{\left[(\lambda+1)(\lambda+b) + C^2 \right]}_{\lambda^2 + (b+1)\lambda + b} - \sigma \underbrace{\left[-(\lambda+b) + C^2 \right]}_{+b(r-2)} = \\ &= -\{(\lambda+\sigma)(\lambda^2 + (b+1)\lambda + b) + \sigma(-\lambda-b + b(r-1))\} \\ &= -\{x^3 + (b+1)x^2 + bx + \sigma x^2 + \sigma(b+1)x + \sigma br - \sigma x + b(r-2)\} \\ &= -x^3 - (\sigma+b+1)x^2 - b(\sigma+r)x - 2\sigma b(r-1) \end{aligned}$$

$P(x) = 0$ → daje 3 n. mjestočnosti $\lambda_{1,2,3}$ (juži je polinom 3. stupnja)

(za $r=1$ odmoli se reproduciru rezultati iz analize T_0 (za $r=1$) tj.

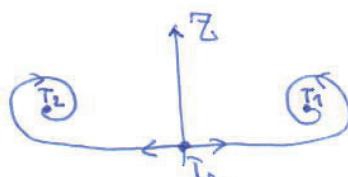
$\lambda_1=0$, $\lambda_2=-b$, $\lambda_3=-(b+1)$ tj. F.T.- T_0 je marginalno stabilna (stabilna neustabilna)).

- Ponašanje $P(\lambda)$:



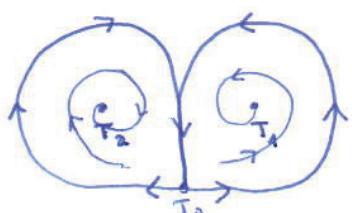
Scenarij ulaska u chaos: (ovisno o parametru r)

- Za $r \in [0, 1]$: stabilna F.T. je $T_0(0, 0)$.
- Za $r = 1$: T_0 postaje nestabilna (sedlo), a stranju se dva stabilna čvora T_1 i T_2 . Sve tri $\lambda_{1,2,3} < 0 \in \mathbb{R}$.
- Za $r \geq r_1 \approx 1.3456172$ samsu $\lambda_1 \in \mathbb{R}$, $\lambda_{2,3} \in \mathbb{C}$ (i $\operatorname{Re}(\lambda_{2,3}) < 0$!) $\Rightarrow T_1 : T_2$ postaju stabilni fokus' (od eksp.)



Kada r raste, T_1 i T_2 se razmjenjuju, a spirale oba ujedno se povećavaju.

- Za $r = r_H \approx 13.96$ spirale se "dotiču" (T_0):

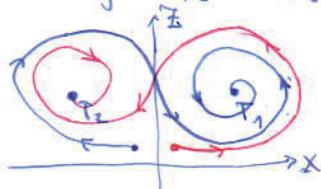


Nastaju 2 HOMOKLINIČKE TRAJEKTORIJE (ova dvoja povezuje u F.T. u $t \rightarrow -\infty$ i zavrsi u istoj F.T. u $t \rightarrow +\infty$; F.T. je sedlo!) - to je nova, tzv. HOMOKLINIČKA BIFURKACIJA: točke T_1, T_2 ostaju stabilni fokus' oba brojih se formiraju granicne kružne.

(Konoklinička bifurkacija: postaje spajajuće redlo i granidrog holo; npr.:



U ovom slučaju konokliničkom bifurkacijom oboje T_1 i T_2 formiraju se 2 nestabilne granidroe hole. Barem atraktori zauzimajući ciljni atraktor tj. trajektorije iz okoline T_1 asymptotski padeju u T_2 i obrnuto.



T_1 i T_2 su i dalje stabilni fokus u levoj i desnoj ($r > r_c$), tako se realni dio $\operatorname{Re}(\lambda_{2,3})$ ($\lambda_1 < 0 \in \mathbb{R}, \lambda_{2,3} \in \mathbb{C}$) ne više menjuje i u harmonici, za $r = r_c$, teži nuli.

$$\text{Za } r = r_c = \underbrace{\sigma \frac{6+b+3}{6-b-1}}_{= 24.7368}, \quad \operatorname{Re}(\lambda_{2,3}) = 0 \quad \text{tj. } \lambda_{2,3} = \pm i\omega.$$

Fix. točke T_1 i T_2 postaju nestabilne - redlo (jer su $\lambda_{1,2} \neq 0 \in \mathbb{R}$).

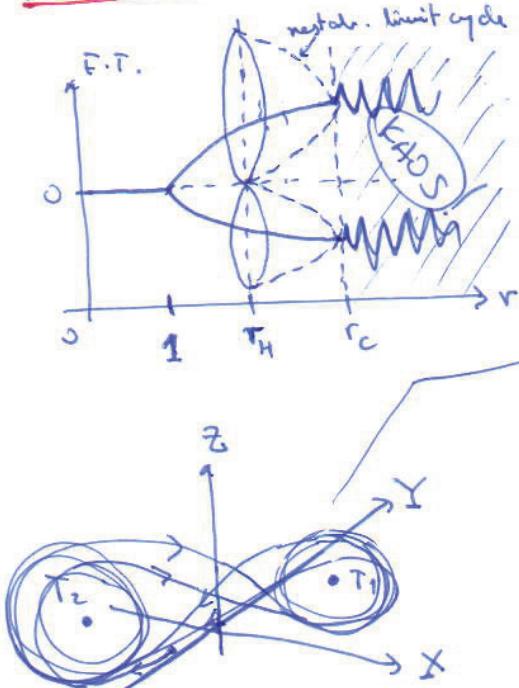
Svaka f.točka T_1 i T_2 bila je za $r < r_c$ obouzdujuće nestabilne SEOLEVIM KOLOM (to su specijalni slučaj granidre hole, moguć samo u niskim redovima 3 i više, kada ih su dva dimenzionalna nestabilna step u dva dimenzionalna stabilna step oboje sebe). Kako $r \rightarrow r_c$, redlo hole kolabira u (stabilnu) sivu troku (T_1, T_2); dogadja se HOPFOVA BIFURKACIJA kojom ova od stabilne postaje nestabilna (nestabilna) točka. Da je ta hopfova bif. nadkritična, gr. hole bila bi stabilna, no H. bifurkacija je pravitečna: one su nestabilne! Lorentz je pružao da u za $r \gg r_c$ ne mogu postojati stabilna gr. hole!

$$\left. \begin{aligned} (*) |r_c| & (\lambda_{2,3} = \pm i\omega) \\ P(\lambda) &= -(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \\ &= -(\lambda - \lambda_1)(\lambda^2 + \omega^2) \\ &= -\lambda^3 + \lambda_1\lambda^2 - \omega^2\lambda + \omega^2\lambda_1 \\ P(\lambda) &= -\lambda^3 - (\xi + b + 1)\lambda^2 - b(\xi + r_c)\lambda - 2b\xi(r_c - 1) \end{aligned} \right\} =$$

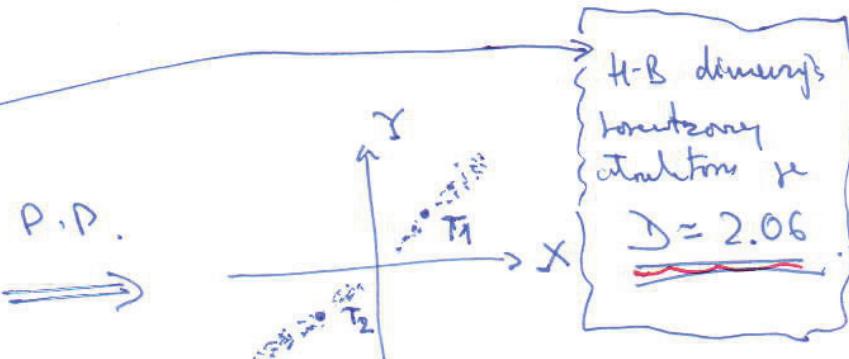
- (1) $-\lambda_1 = \xi + b + 1$
 $\Rightarrow \lambda_1 = -(\xi + b + 1)$
- (2) $\omega^2 = b(\xi + r_c)$
- (3) $\omega^2 \lambda_1 = -2b\xi(r_c - 1)$

$$\left. \begin{aligned} \sqrt{\xi(\xi + b + 1)(\xi + b + 3)} &= \sqrt{2b\xi(r_c - 1)} \\ \xi(\xi + b + 1) + r_c(\xi + b + 1) &= 2b\xi - 2b \\ r_c(-\xi - b - 1) &= -\xi(\xi + b - 3) \\ r_c = \sigma \frac{\xi + b - 3}{\xi - b - 1} & \text{ku} \end{aligned} \right\}$$

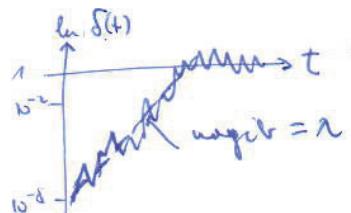
- 125
- za $r > r_c$; $\operatorname{Re}(\lambda_{1,2}) > 0 \Rightarrow$ F.T. $T_{1,2}$ postaju NOTA-
BIUNE: i odvijaju fazi tok u sebi. Kako ga održaju kod tihu dimenzija ukoristi ne sasvimanje (jako) savnog prostora? Odgovor je: STRANI ATRAKTOR t.
LORENZOV ATRAKTOR!



Trajektorije se evolucioniraju održavajući T_1 i u T_2 prelazeći kroz razne i jednog dobro mogu ih obnoviti na drugi.

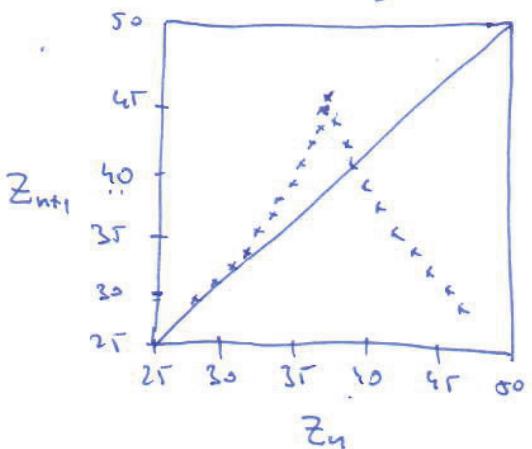
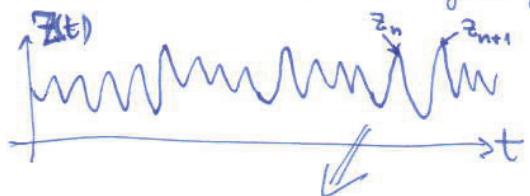


U eksponentijalne raspodjeljivost na poč. uvjetu:
 $\delta(t) \sim S_0 e^{\lambda t}; \lambda \approx 0.9 > 0$ (Ly-exponent)



Lorenzova mappa

Promatram u P.P. na hiperbolici održavajućoj mafotonu $\dot{z} = 0 \Rightarrow z = \frac{1}{b}xy$; to su maksimumi vrijednosti $Z(t)$:



Lorenzova mappa približno je jednaka trokutnoj mapi (tak mep) s nagibom $> 1 \Rightarrow$ KAOŠČNE SEKVENCE!

(*) Deterministički kaos: kaos potječe isključivo od nelinearnosti sustava, a ne od eratične sile tijevanja, ţima ili parametara koji su konstantni.

LORENZON MODEL

125-a

$$\sigma = 10$$
$$b = 8/3$$

```
In[5]:= fx[x_, y_, z_] := 10 * (y - x)
fy[x_, y_, z_] := -x * z - y + r * x
fz[x_, y_, z_] := x * y - (8 / 3) * z

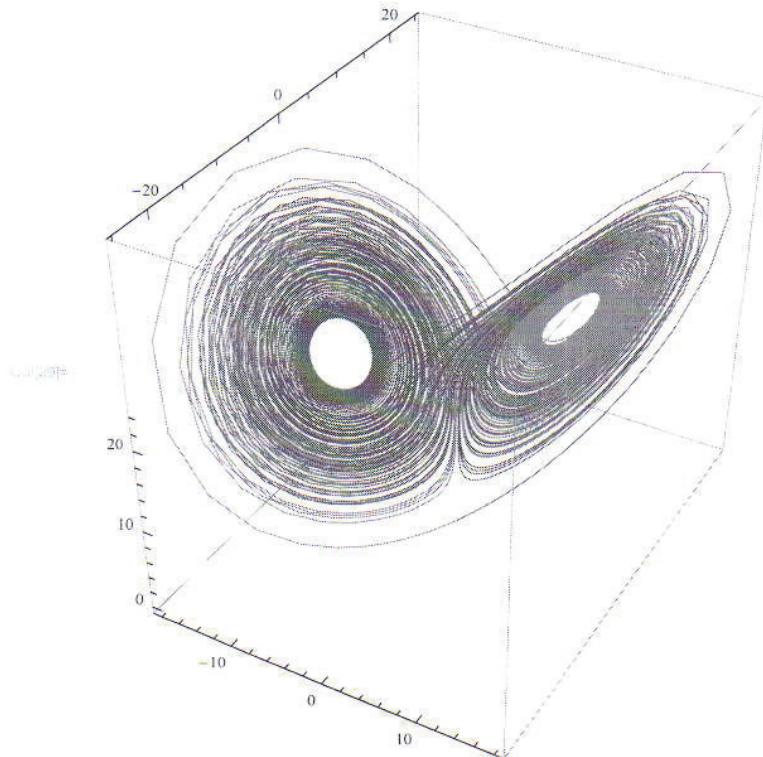
In[6]:= r = 25;
s1 = NDSolve[{x'[t] == fx[x[t], y[t], z[t]],
y'[t] == fy[x[t], y[t], z[t]], z'[t] == fz[x[t], y[t], z[t]],
x[0] == 0.1, y[0] == 0.1, z[0] == 0.}, {x, y, z}, {t, 0, 100}]
s2 = NDSolve[{x'[t] == fx[x[t], y[t], z[t]], y'[t] == fy[x[t], y[t], z[t]], z'[t] ==
fz[x[t], y[t], z[t]], x[0] == -0.1, y[0] == -0.1, z[0] == 0}, {x, y, z}, {t, 0, 100}]
ParametricPlot3D[Evaluate[{{x[t], y[t], z[t]} /. s1, {x[t], y[t], z[t]} /. s2}],
{t, 0, 85}, PlotRange -> All]

NDSolve::mxst : Maximum number of 10000 steps reached at the point t == 85.99876951294765`.
```

```
In[7]:= {{x -> InterpolatingFunction[{{0., 85.9988}}, <>],
y -> InterpolatingFunction[{{0., 85.9988}}, <>],
z -> InterpolatingFunction[{{0., 85.9988}}, <>]}}
```

```
In[8]:= NDSolve::mxst : Maximum number of 10000 steps reached at the point t == 85.99876951294765`.
```

```
In[9]:= {{x -> InterpolatingFunction[{{0., 85.9988}}, <>],
y -> InterpolatingFunction[{{0., 85.9988}}, <>],
z -> InterpolatingFunction[{{0., 85.9988}}, <>]}}}
```



LORENZOV MODEL - EVOLUCIJA

125-b

