Radiation rate of metastable states in scalar theories

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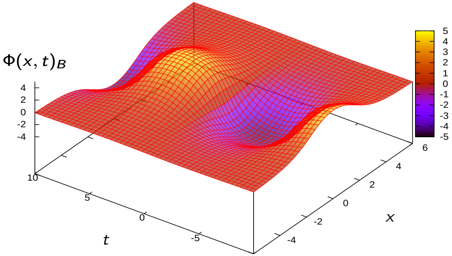
The sG breather

In the 1+1 dimensional sine-Gordon theory a soliton-antisoliton dublet solution is present:

$$\Phi(x,t)_B = 4\,\mathrm{arctg}\left[\frac{\sin\left(\frac{ut}{\sqrt{1+u^2}}\right)}{u\cdot \mathrm{ch}\,\frac{x}{\sqrt{1+u^2}}}\right] \quad \text{with} \quad \omega = \frac{u}{\sqrt{1+u^2}}\,.$$

Figure: The sG breather with $u = 0.5$



The idea of quasi-breathers (QBs)

- Long living oscillating lumps have been observed numerically.
- Such configurations emerge e.g. in soliton-antisoliton collisions.
- QBs are the stationary counterparts of oscillons. QBs are infinite energy configurations. Oscillons are localized, finite energy, long-living oscillating lumps. They emerge if we omit the incoming radiation from a QB.
- Small amplitude QBs can be represented by a series expansion just like the sG breather.

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- We introduce $\epsilon = \sqrt{1-\omega^2}$, which is small close to the mass threshold. This generates the series:

$$X := \epsilon x \qquad S(X) = \frac{\sqrt{2}}{\mathrm{ch}\, X}$$

$$\Phi(x,t)_B = \epsilon \left[2\sqrt{2}\, S \cdot \sin(\omega t) \right] +$$
$$+ \epsilon^3 \left[\frac{\sqrt{2}}{4} (4S - S^3) \cdot \sin(\omega t) + \frac{\sqrt{2}}{12} S^3 \cdot \sin(3\omega t) \right] + \mathcal{O}(\epsilon^5)$$

- An important difference between oscillons and breathers is whether they decay or not. This property is reflected in the precedent series expansion; for a breather the series converge, while for a QB the series proves to be an asymptotic series.
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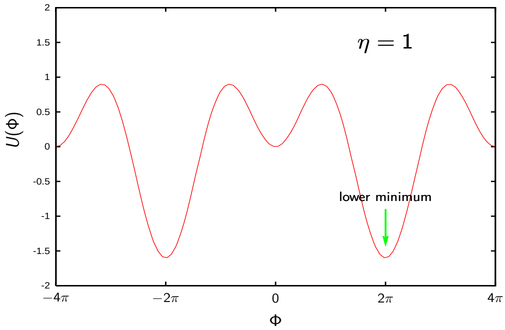
The dsG quasi-breather

The double sine-Gordon potential is:

$$U(\Phi) = -\frac{4}{1+|4\eta|}\left[-\cos\left(\frac{\Phi}{2}\right)+\eta\cos\Phi\right] .$$

We examine the QB in the $\eta > -\frac{1}{4}$ case in the lower minimum of the potential (2π), about which the potential is symmetric.

Figure: The dsG potential with $\eta = 1$



We employ rescalings in order to have $m^2 = 1$ and $g_3 = -1$:

$$\tilde{x} = mx \quad \tilde{t} = mt \quad \tilde{\Phi} = \sqrt{\frac{m^2}{|g_3|}}(\Phi - 2\pi)$$

$$U(\tilde{\Phi}) = -\frac{4}{1+|4\eta|}\left[\cos\left(\sqrt{\frac{|g_3|}{m^2}}\frac{\tilde{\Phi}}{2}\right) + \eta\cos\left(\sqrt{\frac{|g_3|}{m^2}}\tilde{\Phi}\right)\right]$$

and get the field equation ($\tilde{\Phi} \to \Phi$):

$$-\partial_{tt}\Phi + \partial_{xx}\Phi = \Phi - \Phi^3 + \sum_{k=2}^{\infty} g_{2k+1}\Phi^{2k+1}$$

We get the QB in the asymptotic series representation with the aid of the following formulae:

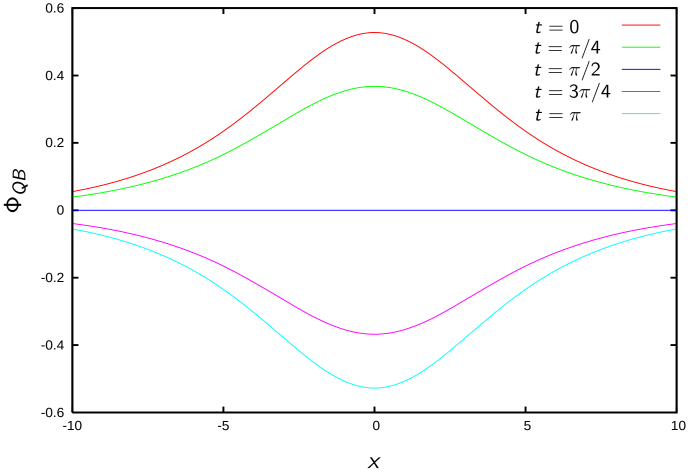
$$\Phi_{QB} = \sum_{k=1}^{\infty} \epsilon^k \Phi_k \qquad \Phi_1 = \frac{2}{\sqrt{3}}\, S \cdot \cos(\omega t)$$

$$\partial_{xx} S - S + S^3 = 0 \qquad S = \frac{\sqrt{2}}{\operatorname{ch} X}\ .$$

The QB takes the form:

$$\Phi_{QB} = \epsilon \frac{2}{\sqrt{3}}\, S \cdot \cos(\omega t) + \epsilon^3 \left[\frac{2}{3\sqrt{3}} \left(\frac{1}{24} + \frac{10 g_5}{9} \right) \right] \left(-S^3 + 4S \right) \cdot \cos(\omega t) + \\ + \epsilon^3 \frac{1}{12\sqrt{3}}\, S^3 \cdot \cos(3\omega t) + \dots\ .$$

Figure: The dsG QB with $\epsilon = 0.3$



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- Intuition: the theory has a characteristic speed (c) and the lump has a characteristic length (L), by dimensional analysis we get $T = \frac{L}{c} = \frac{1}{\epsilon}$ for the lifetime of the lump.
- Quantitatively we could calculate the radiation by the Green function method. We use a mode expansion of the field equation and write down the first two modes:

$$\Phi_{QB} = \sum_{k=0}^{\infty} \phi_{2k+1} \cos((2k+1)\omega t)$$

$$\left[\partial_{xx} + \underbrace{(\omega^2 - 1)}_{-\epsilon^2}\right] \phi_1 = \frac{3}{4}\phi_1^3 + \frac{3}{4}\phi_1^2\phi_3 + \cdots = \sum_{k=0}^{\infty} f(1)_{2k+1} S^{2k+1}$$

$$\left[\partial_{xx} + (9\omega^2 - 1)\right] \phi_3 = \frac{1}{4}\phi_1^3 + \frac{3}{2}\phi_1^2\phi_3 + \cdots = \sum_{k=0}^{\infty} f(3)_{2k+1} S^{2k+1}$$

$$f(l)_m = \epsilon^m (a_1 + \epsilon^2 a_2 + \dots)$$

The first radiating mode turns out to be ϕ_3, as we remain under the mass threshold. The leading order calculation gives:

$$\Phi_{osc} = \Phi_{osc}^{(1)} + \mathcal{O}(\epsilon^3) = \epsilon \frac{2}{\sqrt{3}} S \cos(\omega t) + \mathcal{O}(\epsilon^3)$$

$$\left[\partial_{xx} + \underbrace{(9\omega^2 - 1)}_{8}\right] \underbrace{\phi_3}_{\phi_{rad}} = J^{(1)} = \frac{1}{4}\left(\phi_1^{(1)}\right)^3 = \epsilon^3 \frac{2}{3\sqrt{3}} S^3 + \mathcal{O}(\epsilon^5)$$

$$\phi_3(x) = \int_{-\infty}^{\infty} \mathrm{d}\xi \, G(x,\xi) J^{(1)}(\xi)$$

In order to satisfy the outgoing radiation asymptotics we choose $\gamma_\pm = e^{\pm i\sqrt{8}x}$ and

$$G(x,\xi) = \begin{cases} \dfrac{\gamma_-(x)\cdot\gamma_+(\xi)}{W(\xi)} & \text{if} \quad \xi \le x, \\[2ex] \dfrac{\gamma_+(x)\cdot\gamma_-(\xi)}{W(\xi)} & \text{if} \quad x \le \xi. \end{cases}$$

For $x \gg \frac{1}{\epsilon}$ we get:

$$\phi_{rad} = e^{-i\sqrt{8}x} \frac{i}{3\sqrt{3}} \int_{-\infty}^{\infty} \mathrm{d}\xi \; e^{i\sqrt{8}\xi} \; \frac{\epsilon^3}{\mathrm{ch}^3(\epsilon\xi)} \approx$$
$$\approx e^{-i\sqrt{8}x} \frac{i}{3\sqrt{3}} \; 2\pi \exp\left[-\frac{\sqrt{8}\pi}{2\epsilon}\right] \; .$$

However this approximation cannot be ameliorated, as from every ϵ order in the source a $\mathcal{O}(1)$ correction comes:

$$\int_{-\infty}^{\infty} \mathrm{d}\xi \; e^{i\sqrt{8}\xi} \; \frac{\epsilon^n}{\mathrm{ch}^n(\epsilon\xi)} \approx \frac{\sqrt{8}^{n-1}}{(n-1)!} \; \frac{\pi}{2} \exp\left[-\frac{\sqrt{8}\pi}{2\epsilon}\right] \; .$$

With the coefficients of S^n increasing like $\frac{(n-1)!}{8^{n/2}}$ these corrections sum up to give a divergent result for the outgoing radiation.

Correction beyond all orders

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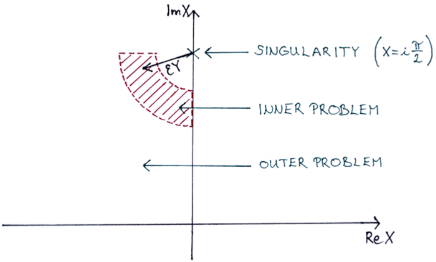
- Kruskal and Segur: The correction can be found by solving the complexified mode equations in the neighborhood of the singularity ($X = iR = i\frac{\pi}{2}$) in the complex plane closest to the real axis.

$$X = \epsilon x = i\frac{\pi}{2} + \epsilon y \qquad S(X) = -\frac{i\sqrt{2}}{\epsilon y} + \frac{i\sqrt{2}\epsilon y}{6} + \mathcal{O}\left(\left(\epsilon y\right)^3\right)$$

- The geometry of the matching region:
$\{|\epsilon y| \ll 1 \;\; (\epsilon y \to 0), \;\; |y| \gg 1 \;\; (|y| \to \infty), \;\; -\pi \leq \arg(y) \leq -\frac{\pi}{2}\}$

Figure: The geometry of the matching region



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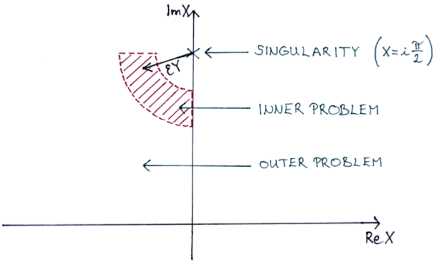

- The mode equations on the complex plane:

$$\left[\partial_{xx} + (n^2\omega^2 - 1)\right]\phi_n = \frac{1}{4}\sum_{k,l,m=\mathrm{odd}} \phi_k\phi_l\phi_m\delta_{n,\pm k\pm l\pm m} + \dots$$

- Equations of the inner problem (leading order in ϵ):

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- Matching in the overlap region yields:

$$\phi_1 = -\frac{i2\sqrt{2}}{\sqrt{3}} \frac{1}{y} + \frac{i4\sqrt{2}}{3\sqrt{3}} \left(\frac{1}{24} + \frac{10 g_5}{9} \right) \frac{1}{y^3} + \dots$$

$$\phi_3 = -\frac{i\sqrt{2}}{6\sqrt{3}} \frac{1}{y^3} + \dots$$

$$\vdots$$

- Let $\phi_n =: A_n + iB_n$. For $\mathrm{Im}\, y \to -\infty$ along $\mathrm{Re}\, y = 0$ the asymptotic series for each B_n converges, because every term vanishes. We will find the corrections beyond all orders there, thus our method makes sense.

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- Taking the imaginary part of the mode equations on this line gives us decoupled linear equations, as the B_ns are transcendentally small. The solution of the equations take the form:

$$B_3(y) = \nu_3 \exp\left[-i\sqrt{8}\, y\right] \cdot \left\{1 + \mathcal{O}\left(\frac{1}{y}\right)\right\} + \mathcal{O}\left[\frac{1}{y}\, \exp\left[-i\sqrt{24}\, y\right]\right]$$

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  We have similar formulae for other B_ns; ν_3 can be determined numerically.
- We continue the solution back to the real axis and get the radiation field configuration in ϕ_3 (a similar term comes from the neighborhood of the lower half plane singularity: $X = -i\frac{\pi}{2}$):

$$\Phi_{rad} = \underbrace{2\nu_3}_{\pi K} \cdot \exp\left[-\frac{\sqrt{8}\pi}{2\epsilon}\right] \cdot \sin\left[\sqrt{8}\, x - 3t\right] \ .$$

The background of this unfamiliar idea

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$$u_{sol}(x,t) = \frac{C}{2\,\mathrm{ch}^2\left[\frac{\sqrt{C}}{2}(x - Ct)\right]}$$

$$\frac{\mathrm{d}C}{\mathrm{d}t} \propto \exp\left[-\frac{2\pi}{\sqrt{C}\,\epsilon}\right] .$$

Determination of the amplitude K via Borel summation

- Hakim and Pomeau: The radiation amplitude K may be determined from the leading behaviour of the $a(3)_{2k+1}$ coefficients ($\phi_3 = \sum_{k=1}^{\infty} \frac{a(3)_{2k+1}}{y^{2k+1}}$).

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- It can be shown that the coupled mode equations in the vicinity of the singularity are consistent with the following asymptotics:

$$a(3)_{2m+1} = K\,(-1)^m \frac{(2m)!}{8^m} \left[1 + \mathcal{O}\left(\frac{1}{m}\right)\right] ,$$

similar formulae hold for $a(n)_{2m+1}$s, as ϕ_3 proves to be dominant among the modes and drives the leading behaviour of the other modes through source terms.

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  similar formulae hold for $a(n)_{2m+1}$s, as ϕ_3 proves to be dominant among the modes and drives the leading behaviour of the other modes through source terms.
- K can be obtained numerically by solving the mode equations up to some large order of m and matching these numerical values to the asymptotics.

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$$\phi_3(y) = \int_0^{\infty} \mathrm{d}t \; e^{-t} V(t/y) \tag{1}$$

$$V(z) = \sum_{m=1}^{\infty} \frac{a(3)_{2m+1}}{(2m+1)!} \, z^{-(2m+1)} \sim \sum_{m=1}^{\infty} K \frac{(-1)^m}{(2m+1)} \, z^{-(2m+1)} \, .$$

This new converging series has logarithmic singularities at $z = \pm i$; the leading order behaviour of coefficients $a(3)_{2m+1}$ only determine the series in the neighborhood of these singularities.

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- To complete the Borel summation procedure we have to compute the (1) integral: the logarithmic singularity does not contribute, while integrating on the branch cut we obtain the radiation field configuration:

$$\Phi_{rad} = \pi K \cdot \exp\left[-\frac{\sqrt{8}\pi}{2\epsilon}\right] \cdot \sin\left[\sqrt{8}\, x - 3t\right] \, .$$

K in the dsG theory

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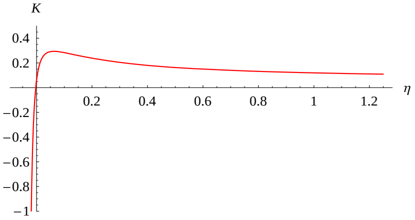
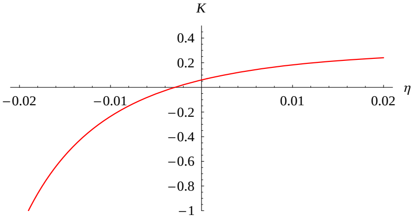
  Truncation of the infinite set of mode equations at 2 modes gives the minimal solvable system. However we need at least 3 mode equations to approach the real value of K.
- The dependence of K on η is weak on a wide interval. K tends to zero close to the η (g_5) value of the sG theory.

Figure: Dependence of K on η



(a) Illustration of the weak dependence



(b) The neighbourhood of the sG theory

Radiation law in non-symmetric potentials

In 1+1 dimensions with non-symmetric potential (e.g. Φ^4 theory) all Fourier modes are involved:

$$\Phi_{QB} = \sum_{k=0}^{\infty} \phi_k \cos(k\omega t) \ .$$

Dominantly the radiation is in the second mode. The radiation field configuration is:

$$\Phi_{rad} = \underbrace{2\nu_2}_{\pi K} \cdot \exp\left[-\frac{\sqrt{3}\pi}{2\epsilon}\right] \cdot \sin\left[\sqrt{3}\,x - 2t\right] \ .$$

We not yet understand how K could be calculated via Borel summation because of ϕ_0.

Radiation law in arbitrary dimensions

- For radially symmetric QBs in arbitrary dimensions we have to solve the following field equation ($\rho = \epsilon r$):

$$-\partial_{tt}\Phi + \partial_{\rho\rho}\Phi + \frac{D-1}{\rho}\partial_\rho\Phi = \Phi - \Phi^3 + \sum_{k=2}^{\infty} g_{2k+1}\Phi^{2k+1} \ .$$

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- On the real axis we have the master equation:

$$\partial_{\rho\rho}S + \frac{D-1}{\rho}\partial_\rho S - S + S^3 = 0 \, . \qquad (2)$$

QBs can be represented with series containing the powers of S and $\partial_\rho S$, except in one dimension, where the series only contains S.

Radiation law in arbitrary dimensions

- For radially symmetric QBs in arbitrary dimensions we have to solve the following field equation ($\rho = \epsilon r$):

$$-\partial_{tt}\Phi + \partial_{\rho\rho}\Phi + \frac{D-1}{\rho}\partial_\rho\Phi = \Phi - \Phi^3 + \sum_{k=2}^{\infty} g_{2k+1}\Phi^{2k+1} \, .$$

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  QBs can be represented with series containing the powers of S and $\partial_\rho S$, except in one dimension, where the series only contains S.

- The singularity of (2) may be determined numerically and we can work near the singularity just as we did in the one dimensional case.

- The complexified mode equations leading order in ϵ are the same as in the one dimensional case (if we have the same potential):

$$\rho = iR(D) + \epsilon y$$

$$\left[\partial_{yy} + \epsilon\frac{D-1}{iR(D)+\epsilon y}\partial_y + (n^2\omega^2 - 1)\right]\phi_n =$$

$$= \left[\partial_{yy} + (n^2 - 1) + \epsilon\frac{D-1}{iR(D)}\partial_y + \right.$$

$$\left. + \epsilon^2\left(-\frac{(D-1)y}{(iR(D))^2}\partial_y - n^2\right) + \mathcal{O}(\epsilon^3)\right]\phi_n \xrightarrow[\text{order in } \epsilon]{\text{leading}}$$

$$= \left[\partial_{yy} + (n^2 - 1)\right]\phi_n = \frac{1}{4}\sum_{k,l,m=\mathrm{odd}}\phi_k\phi_l\phi_m\delta_{n,\pm k\pm l\pm m} + \ldots$$

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- Thus the value of K and the transcendental correction is the same in arbitrary dimensions as in one dimension. The only difference is in the continuation back to the real axis.

- From the correction beyond all orders plane, cylindrical and spherical waves emerge on the real axis. The radiation field configuration reads:

$$\Phi_{rad} = \underbrace{2\nu_3}_{\pi K} \cdot \exp\left[-\frac{\sqrt{8}R(D)}{\epsilon}\right] \left[\frac{R(D)}{\epsilon r}\right]^{\frac{D-1}{2}} \cdot \sin\left[\sqrt{8}\, r - 3t\right] \quad r \gg \frac{1}{\epsilon}$$

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where S_D is the surface of the unit sphere.

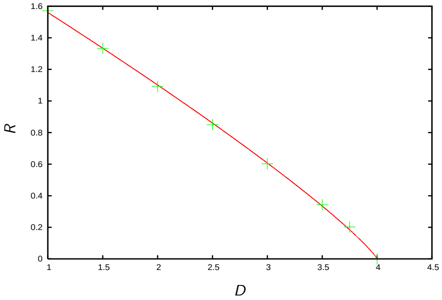
- For the radiation rate of small amplitude oscillons we get:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -3\sqrt{2}\,\pi^2 \cdot \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}\, K^2 \left[\frac{R(D)}{\epsilon}\right]^{D-1} \exp\left[-\frac{2\sqrt{8}R(D)}{\epsilon}\right] \ .$$

Discussion of the radiation law

- The position of the pole can be determined by Padé's approximation:

Figure: The position of the singularity as a function of D



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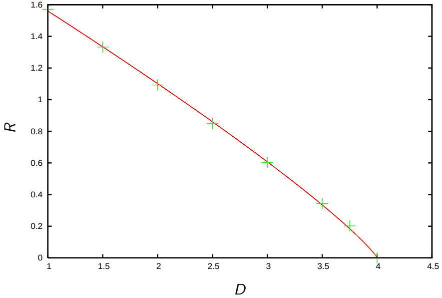

- As D increases the pole approaches the origin and the radiation rate of the oscillon increases. This result predicts $D = 4$ to be the critical dimension, but gives no information about the stability of the oscillon (i.e. for $D = 3$ the small amplitude oscillons are unstable).

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- We compute the energy density to leading order in ϵ and after integration we get:

$$E = \epsilon^{2-D} E_0 + \mathcal{O}(\epsilon^{4-D})$$

$$E = \begin{cases} \frac{4}{3}\epsilon + \mathcal{O}(\epsilon^3) & \text{if} \quad D = 1 \\ E_0 + E_1\epsilon^2 + \mathcal{O}(\epsilon^4) & \text{if} \quad D = 2 \\ \frac{E_0}{\epsilon} + \mathcal{O}(\epsilon) & \text{if} \quad D = 3 \end{cases}$$

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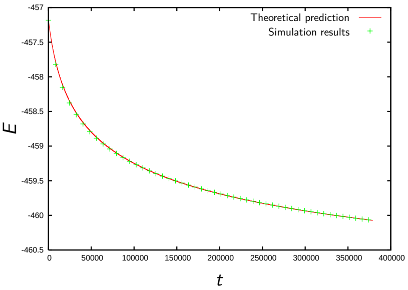
- In three dimensions the ϵ parameter increases and the frequency moves further from the mass threshold and approaches ω_m, which characterizes the oscillon with minimal energy.

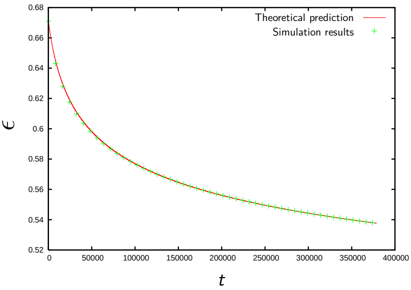
Confrontation of theoretical formulae with numerical data

The adiabatical hypothesis is confirmed by numerical simulations. We have to start from big $\epsilon = 0.65$ value in order to see ϵ change. The semi-empirical radiation law in the dsG theory reads:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -2320.97 \exp\left[-\frac{11.3646}{\epsilon}\right]$$

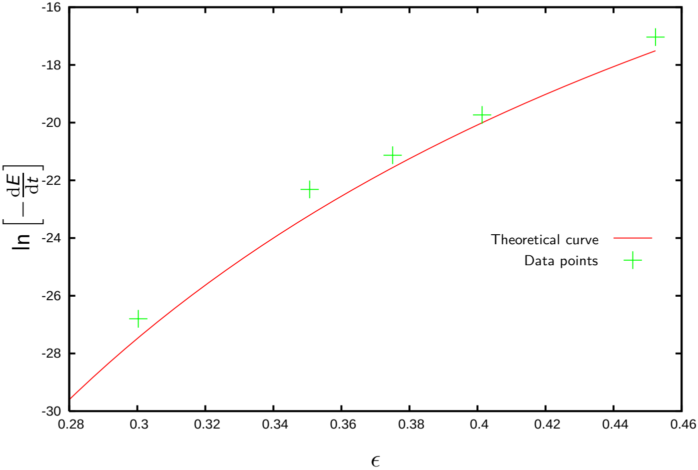
Figure: $E(t)$ and $\epsilon(t)$ according to simulation and theory





In the dsG theory we ran simulations starting with initial data with smaller $\epsilon = 0.25 - 0.4$ values. From the numerical results both the position of the pole and the value of K agree with the theoretical prediction.

Figure: R and K from the simulation



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- From the outgoing radiation we determined the radiation law for small amplitude oscillons. We explained oscillon evolution with the aid of the adiabatical hypothesis.
- We compared numerical simulations with theoretical formulae and found satisfactory agreement: the adiabatical hypothesis was confirmed by simulations starting from oscillons with 'big' ϵ values, the position of the pole and the approximate value of K was determined from oscillons with smaller ϵ values.