Pion condensation in the $O(4)$ constituent quark model

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- Motivation
- The constituent quark model and its renormalization
- Propagators
- OPT and parametrization
- Equations at finite temperature
- Condition for pion condensation

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Where can pion condensation occur in nature?

Quark matter can exist in neutron stars $\longrightarrow$ at very large bariochemical potential ($\mu_{\text{B}} \approx 1$ GeV)

If the isospin chemical potential is also different from zero $\longrightarrow$ possibility of pion condensation

In 2 flavoured NJL model (L. He *et al* Phys. Rev. **D74**, 036005 (2004)):

- if 140 MeV$< \mu_I < 230$ MeV $\rightarrow$ BEC phase
- if $\mu_I > 230$ MeV $\rightarrow$ BCS phase

Neutrino emission from pion condensed quark matter $\rightarrow$ direct Urca processes:

$$d \rightarrow u + e^- + \bar{\nu}$$
$$u + e^- \rightarrow d + \nu$$

$\Longrightarrow$ It might will be investigated experimentaly

Interesting feature of pion condensation found in $SU(2)$ PNJL model:

At sufficiently high temperature the condesate evaporates
above a certain μ_I. (Z. Zhang, Y. Liu: hep-ph/0610221v3)

Up to now:

- Investigations in $SU(2)$ NJL and PNJL model (BEC, BCS and CFL phases)
- Investigation in $O(4)$ model in the large N_c limit (leading order) (BEC phase)

Possible investigations:

- In different models such as constituent quark model
- In three flavour
- Dependence on other chemical potentials
- Neutrino production

The model and its renormalization

Our starting point is the renormalized $O(4)$ symmetric Lagrangian with explicit symmetry breaking term

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_0^2\phi^2\right) - \frac{\lambda}{4}\phi^4 + h\phi_0 + i\bar{\psi}\gamma_\mu\partial^\mu\psi - \frac{g_F}{2}\bar{\psi}T_i\phi_i\psi \\ &+ \frac{1}{2}\left(\delta Z\partial_\mu\phi\partial^\mu\phi - \delta m_0^2\phi^2\right) - \frac{\delta\lambda}{4}\phi^4\end{aligned}$$

$\psi = (u,\, d)^T \quad \longrightarrow$ quark fileds

$\phi = (\phi_0, \phi_1, \phi_2, \phi_3) \equiv (\sigma, \pi_1, \pi_2, \pi_3) \quad \longrightarrow$ sigma and pion scalar fields

$h \quad \longrightarrow$ symmetry breaking external field

$T_i = (\tau_0, i\tau_i\gamma_5) \quad \longrightarrow$ quark–boson coupling matrix

The unknown renormalized parameters of the Lagrangian: $m_0, \quad \lambda, \quad g_F$

$\delta z, \delta m_0^2, \delta\lambda$ are the usual (infinite) counterterms
(Fermions are treated at tree level $\rightarrow$ no wavefunction renormalization)

The genarating functional:

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\Pi \mathcal{D}\psi^\dagger \mathcal{D}\psi \, \mathrm{e}^{\int_0^\beta d^4x (i\Pi_i\dot{\phi}_i + i\psi^\dagger\dot{\psi} - \mathcal{H} + \mu_B Q_B + \mu_I Q_I)},$$

where the the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\left(\Pi_i^2 + (\nabla\phi_i)^2 + m_0^2\phi_i^2\right) + \frac{\lambda}{4}(\phi_i^2)^2 - h\phi_0 - i\bar{\psi}\gamma_i\partial_i\psi + \frac{g}{2}\bar{\psi}T_i\phi_i\psi \\ &- \frac{1}{2}\delta Z\Pi_i^2 + \frac{1}{2}\delta Z(\nabla\phi_i)^2 + \frac{\delta\lambda}{4}(\phi_i^2)^2 + \frac{1}{2}\delta m_0^2\phi_i^2\end{aligned}$$

and the renormalized canonical momenta of the scalar fields

$$\Pi_i = \frac{\delta\mathcal{L}}{\delta\dot{\phi}_i} = (1+\delta Z)\dot{\phi}_i$$

Q_B, Q_I are the conserved barion and isospin charges

$$\begin{aligned}Q_B &= \int \mathrm{d}^3x \frac{1}{3}(u^\dagger u + d^\dagger d) \\ Q_I &= \int \mathrm{d}^3x \left[(1+\delta Z)(\pi_2\dot{\pi}_1 - \pi_1\dot{\pi}_2) + \frac{1}{2}\left(u^\dagger u - d^\dagger d\right)\right].\end{aligned}$$

Symmetry breaking:

At small T when either $h \neq 0$ or $h = 0$ and $m_0^2 < 0 \Rightarrow \langle \phi_0 \rangle \equiv \langle \sigma \rangle \equiv v \neq 0$

At large $\mu_I \Rightarrow \langle \phi_1 \rangle \equiv \langle \pi_1 \rangle \equiv \rho \neq 0$ and $\langle \phi_i \rangle \equiv \langle \pi_i \rangle = 0$ for $i = 2, 3$

Shifting the corresponding fields

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\psi^\dagger \mathcal{D}\psi \left[\exp\left(-\int_0^\beta d^4 x_E (\tilde{\mathcal{L}}_F + \tilde{\mathcal{L}}_I) \right) \int \mathcal{D}\Pi \exp\left(\int_0^\beta d^4 x_E (i\Pi_i \dot{\phi}_i - \tilde{\mathcal{H}}_B) \right) \right],$$

where the Π dependent part of $i\Pi_i\dot{\phi}_i - \tilde{\mathcal{H}}_B$:

$$\begin{aligned} i\Pi_i\dot{\phi}_i - \tilde{\mathcal{H}}_B = & - \frac{1}{2}(1-\delta Z)\Big(\Pi_0\Pi_0 - 2(1+\delta Z) i\Pi_0\dot{\phi}_0\Big) \\ & - \frac{1}{2}(1-\delta Z)\Big(\Pi_3\Pi_3 - 2(1+\delta Z) i\Pi_3\dot{\phi}_3\Big) \\ & - \frac{1}{2}(1-\delta Z)\Big(\Pi_1\Pi_1 - 2(1+\delta Z)(i\Pi_1\dot{\phi}_1 + (1-\delta Z)\mu_I\phi_2)\Big) \\ & - \frac{1}{2}(1-\delta Z)\Big(\Pi_2\Pi_2 - 2(1+\delta Z)(i\Pi_2\dot{\phi}_2 - (1-\delta Z)\mu_I(\phi_1+\rho))\Big). \end{aligned}$$

Making whole squares in the above brakets

For instance the first one becomes

$$-\Pi_0'^2 - \frac{1}{2}(1-\delta Z)(1+2\delta Z)\dot{\phi}_0^2 = -\Pi_0'^2 - \frac{1}{2}(1+\delta Z)\dot{\phi}_0^2 + \text{2-loop}$$

Than performing the Π integration $\rightarrow$ produce the tree-level EoS (linear terms) and the inverse propagators

$$\begin{aligned} E_v &= v(m_0^2 + \lambda(v^2+\rho^2)) - h \\ E_\rho &= v(m_0^2 + \lambda(v^2+\rho^2) - \mu_I^2(1-\delta Z)) \end{aligned}$$

Renormalized propagator of the π_3 particle in euclidian space:

$$G_{\pi_3} = \frac{1+\delta Z}{p^2 + m^2_{\pi,\text{tree}} + \delta\lambda v^2 + \delta m_0^2 + \Sigma(p^2=0) + p^2\Sigma'(p^2=0) + \tilde{\Sigma}(p^2)}$$

Conditions:

- momentum independent part of the inverse propagator must be finite:
  $(1-\delta Z)m_{\pi,\text{tree}}^2 + \delta\lambda v^2 + \delta m_0^2 + \Sigma(p^2=0) = \text{finite}$

- coefficient of p^2 in the inverse propagator must be one:
  $(1+\Sigma'(p^2=0))(1-\delta Z) = 1$

Note: there is some arbitrariness in choosing the finite parts

With cut-off regularization:

$$\begin{aligned} \delta m_0^2 &= -6\lambda(\Lambda^2 - m_0^2 \ln \frac{\Lambda^2}{l_b^2}) \\ \delta\lambda &= 12\lambda^2 \ln \frac{\Lambda^2}{l_b^2} \\ \delta Z &= -N_c \frac{g_F^2}{16\pi^2} \ln \frac{\Lambda^2}{l_f^2 e^2} \end{aligned}$$

$l_b, l_f \rightarrow$ bosonic and fermionic renormalization scales

Bosonic and fermionic propagators

Going into Fourier space and introducing Matsubara frequencies the bosonic inverse propagators are:

$$-iD^{-1}_{\pi_3} = \omega_n^2 + \mathbf{k}^2 + m_\pi^2 + \lambda\rho^2$$

$$-iD^{-1}_{\pi_1,\pi_2,\sigma} =$$

$$\begin{pmatrix} \omega_n^2 + \mathbf{k}^2 + m_\pi^2 - \mu_I^2 + 3\lambda\rho^2 & -2\mu_I\omega_n & 2\lambda v\rho \\ 2\mu_I\omega_n & \omega_n^2 + \mathbf{k}^2 + m_\pi^2 - \mu_I^2 + \lambda\rho^2 & 0 \\ 2\lambda v\rho & 0 & \omega_n^2 + \mathbf{k}^2 + m_\sigma^2 - \mu_I^2 + \lambda\rho^2 \end{pmatrix}$$

where the tree-level bosonic masses

$$m_\pi = m_0^2 + \lambda v^2, \quad m_\sigma = m_0^2 + 3\lambda v^2$$

$\Longrightarrow$ Characteristic equation is of third degree $\Longrightarrow$ Propagators would be very complicated $\Longrightarrow$ Hard to perform the Matsubara sums

Since we would only like to determine T, μ_I, μ_B where ρ becomes nonzero $\Longrightarrow$ Perturbative diagonalization in ρ to $\mathcal{O}(\rho^2)$

After diagonalization and perturbative inversion

$$i\tilde{D}_1 = \frac{1}{(\omega_n + i\mu_I)^2 + \mathbf{k}^2 + m_\pi^2} - \rho^2 \frac{\lambda(2\mu_I^2 + 2\lambda v^2 - 4i\mu_I\omega_n)}{((\omega_n + i\mu_I)^2 + \mathbf{k}^2 + m_\pi^2)^2(\mu_I^2 + 2\lambda v^2 - 2i\mu_I\omega_n)}$$

$$i\tilde{D}_2 = \frac{1}{(\omega_n - i\mu_I)^2 + \mathbf{k}^2 + m_\pi^2} - \rho^2 \frac{\lambda(2\mu_I^2 + 2\lambda v^2 + 4i\mu_I\omega_n)}{((\omega_n - i\mu_I)^2 + \mathbf{k}^2 + m_\pi^2)^2(\mu_I^2 + 2\lambda v^2 + 2i\mu_I\omega_n)}$$

$$i\tilde{D}_\sigma = \frac{1}{\omega_n^2 + \mathbf{k}^2 + m_\sigma^2} - \rho^2 \frac{\lambda(\mu_I^2 + 2\lambda v^2)(\mu_I^2 + 6\lambda v^2 + 4\mu_I^2\omega_n^2)}{(\omega_n^2 + \mathbf{k}^2 + m_\sigma^2)^2((\mu_I^2 + 2\lambda v^2)^2 + 4\mu_I^2\omega_n^2)}$$

Tilde denotes that these propagators belong to the transformed particles

Important to note $\longrightarrow$ the transformation matrix depends on ρ, ω_n, μ

$\Rightarrow$ To perform the Matsubara sums first one should transform the coefficient matrices and do the sums for coefficients and loops together

The fermion inverse propagator

$$iD_F^{-1} = \begin{pmatrix} (-i\omega_n + \mu_u)\gamma_0 - \gamma_i k_i - m_F & -i\frac{g_F}{2}\gamma_5\rho \\ -i\frac{g_F}{2}\gamma_5\rho & (-i\omega_n + \mu_d)\gamma_0 - \gamma_i k_i - m_F \end{pmatrix}$$

where

$$\begin{aligned} m_F &= \frac{g_F}{2} v \\ \mu_u &= \frac{1}{3}\mu_B + \frac{1}{2}\mu_I \\ \mu_d &= \frac{1}{3}\mu_B - \frac{1}{2}\mu_I \end{aligned}$$

Diagonalization must be performed with extreme care $\rightarrow$ non-commutative matrix elements

hermitian diagonalizator $\quad O_F = \begin{pmatrix} 1 + \frac{g_F^2}{32k_0^2}\rho^2 & -i\frac{g_F}{4k_0}\gamma_0\gamma_5\rho \\ -i\frac{g_F}{4k_0}\gamma_0\gamma_5\rho & 1 + \frac{g_F^2}{32k_0^2}\rho^2 \end{pmatrix}$

$$k_0 = (-i\omega_n + \frac{1}{3}\mu_B)\gamma_0$$

Fermionic propagators to $\mathcal{O}(\rho^2)$:

$$
\begin{aligned}
-i\tilde{D}_u &= \frac{1}{\not{k}_u - m_f} + \rho^2 \frac{g_F^2}{8k_0} \frac{1}{\not{k}_u - m_f} \gamma_0 \frac{1}{\not{k}_u - m_f} \\
-i\tilde{D}_d &= \frac{1}{\not{k}_d - m_f} + \rho^2 \frac{g_F^2}{8k_0} \frac{1}{\not{k}_d - m_f} \gamma_0 \frac{1}{\not{k}_d - m_f}
\end{aligned}
$$

where

$$
\not{k}_{u/d} = (-i\omega_n + \mu_{u/d})\gamma_0 - \gamma_i k_i
$$

Optimized perturbation theory and parameterization

8 unknown parameters:

couplings	m_0, λ, g_F
condensates	x, ρ
external fields	h
renormalization scales	l_f, l_b

Renormalization scales are not fixed $\rightarrow$ instead physical results as a function of scales

Problem: m_0^2 can be negative at finite T
Solution: resummation using optimized perturbation theory
Chiku & Hatsuda, PRD58:076001

change: $m_0^2 \rightarrow m^2 \quad \Rightarrow \quad \mathcal{L}_{mass} = \frac{1}{2}m^2\phi^2 - \frac{1}{2}\underbrace{(m^2 - m_0^2)\phi^2}_{\Delta m^2\text{: one-loop counterterm}}$

Effect: One has to change the m_0^2 mass in the propagators to $m^2 \rightarrow$ gap-equation for m^2

Note: Zero temperature cunterterms of the original theory still renormalize the OPT

Parameterization is carried out at $T = \mu_I = \mu_B = 0$, where also $\rho = 0$

Equations (5 is needed):

- 1-loop mass equation for σ: $m^2_{\sigma,\text{tree}} + \Sigma_\sigma(p^2=0) = m^2_{\sigma,\text{phys}}$,

  $m_\sigma = 500 - 700$ MeV $\rightarrow$ complicated particle its mass is still uncertain

- 1-loop mass equation for π_3 (gap-equation):

  $m^2_{\pi_3,\text{tree}} + \Sigma_{\pi_3}(p^2=0) = m^2_{\pi_3,\text{phys}}$, $m_{\pi_3} = 138$ MeV

- Tree-level PCAC relation for the pion: $f_\pi = v$, $f_\pi = 93$ MeV

- Tree-level constituent quark mass: $m_f \equiv m_u = m_d = \frac{g_F}{2} v$,

  $m_f = 330$ MeV (one third of the proton mass)

- 1-loop EoS for v:

  $E_v = v(m_0^2 + \lambda(v^2+\rho^2)) - h + v\sum_i \text{Tad}(m_i) = 0$,

Equations at finite temperature

3 equations $\rightarrow$ gap-equation and two equations of state (for m, v and ρ)

- gap-equation for π_3: $$m^2_{\pi_3,\text{tree}} + \Sigma^\beta_{\pi_3}(m,v,\rho)\Big|_{p^2=0} = m^2_{\pi_3,\text{phys}}$$
- 1-loop EoS for v: $$E_v = v(m_0^2 + \lambda(v^2+\rho^2) + \sum_i \text{Tad}_i^\beta(m,v,\rho)) = h$$
- 1-loop EoS for ρ:
$$E_\rho = \rho(m_0^2 + \lambda(v^2+\rho^2) - \mu_I^2 + \sum_i \text{Tad}_i^\beta(m,v,\rho)) = 0,$$

It can be shown that at one-loop level the following Ward-identity is satisfied:

$$v \cdot G^{-1}_{\pi_3}(p^2=0) = h$$

This also guaranties the Goldstone-theorem

Condition for pion condensation

From the second EoS it can be seen that non-zero pion condensation can occur only if:

$$m_0^2 + \lambda(v^2 + \rho^2) - \mu_I^2 + \sum_i \mathsf{Tad}_i^\beta(m, v, \rho) = 0$$

This can be reformulated in the following way:

$$m^2_{\pi_3,1-\mathsf{loop}} - \mu_I^2 + f(m,v) + \lambda\rho^2 + \rho^2 g(m,v) = 0$$

$$\Longrightarrow \rho = \sqrt{\frac{\mu_I^2 - m^2_{\pi_3,1-\mathsf{loop}} - f(m,v)}{g(m,v) + \lambda}}$$

With increasing μ_I pion condensation occurs at:

$$\mu_I^2 - m^2_{\pi_3,1-\mathsf{loop}} - f(m,v) = 0$$

Because in this limiting case $\rho = 0 \Longrightarrow$ we can solve the remaining two equation at zero ρ.

Finally with the help of the Ward identitiy the following two equations will give the μ_I, μ_B, T dependence of v and m:

- $v \cdot m^2_{\pi_3,1-\text{loop}} = h$
- $E_v = v(m^2 + \lambda v^2 + \sum_i \text{Tad}_i^\beta(m, v)) = h$

Solving this system of equations one can determine the regions of pion condensation as a function of T, μ_I, μ_B by demanding:

$$\mu_I^2 - m^2_{\pi_3,1-\text{loop}}(T, \mu_I, \mu_B) - f(T, \mu_I, \mu_B) = 0$$

$\longrightarrow$ This gives a surface in 3 dimension

Instead of conclusion

To be done:

- Solving the above described system of equations

Can be done:

- Astrophysical applications of pion condensation (neutrino emission)
- Investigation of pion condensation in $SU(3)$ constituent quark model
- Polyakov-loop coupling