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## Abstract

A well studied problem in elementary mechanics is the determination of the height which, for a particle released from rest, is sufficient to traverse the total length of a frictionless vertical loop of a circular shape. A generalization of this problem including the effects of sliding friction is studied and the diagram exposing diverse behavior of the particle subjected to varying friction parameter is presented. A comparison with the case of different elliptical shapes of the track is discussed.

## I. INTRODUCTION

A classic handbook problem in introductory mechanics is to find the height of a slope, from which a particle must be dropped in order to go freely through a total length of a circular loop of radius $R$, placed at the bottom of the slope. In case of a frictionless track the minimal height is equal to $\frac{5}{2}R$. The issue can be also reformulated to search for the value of the initial speed of a particle that is sufficient to traverse the whole loop. One finds easily that the kinetic energy at the bottom of the loop must be equal to potential energy of the particle at its initial position on the slope and that leads immediately to the conclusion that the initial speed must be at least $\sqrt{5gR}$, where $g$ is the acceleration due to gravity.

Although the idea of the particle moving through the vertical loop might seem quite trivial at first glance, the problem becomes much more complex after considering the influence of frictional effects resulting from the existence of sliding friction between the particle and the track. It would be challenging to analyze not only how the frictional effects influence the motion on the circular loop and what the conditions for the particle to go though it are, but one might also ask, what will happen if we change the shape of the loop (e.g. elliptical loop)? What will be required speed to hold the track? How will the friction affect the motion in this case? Will the time needed to traverse the track be greater or lesser?

To try to answer those questions there is no need to implement any sophisticated theoretical apparatus. Nevertheless, some basic knowledge of the dynamics on curved tracks and little calculus are advisable. The analyses of the problem give good insight into interplay between forces of different kinds in reference to the law of conservation of energy. The paper is organized in the following manner: In Sec. II the basic geometry of the problem in case of a circle is presented, followed by an analytical solution for this case. Then the universal conditions for traversing the loop are formulated and in case of the circular loop these criteria are resolved. In Sec. III the loop of elliptical shape of different orientations is concerned and its results are given. In Sec. IV some comparisons between different loops are made and in Sec. V symmetries with similar problems are shown.

## II. DYNAMICS ON A CIRCULAR LOOP

Consider a circular loop of radius $R$ which is described in a parametric form:

$$\begin{aligned} x &= R\sin\phi, \\ y &= R(1-\cos\phi). \end{aligned} \tag{1}$$

An additional frictional force opposing the motion in a tangent plane of the track (see Fig. 1) can be modeled as $F_f = \mu F_N$, where $F_N$ is the normal force, i.e. the pressure that the particle exerts on the track, and $\mu$ is the coefficient of kinetic friction. Applying Newton's second law for the radial forces:

$$F_N - mg\cos\phi = mv^2(\phi)/R, \tag{2}$$

where $v$ is the speed of the particle of mass $m$, one finds the frictional force to be of the form:

$$F_f = \mu\left(mg\cos\phi + \frac{mv^2(\phi)}{R}\right). \tag{3}$$

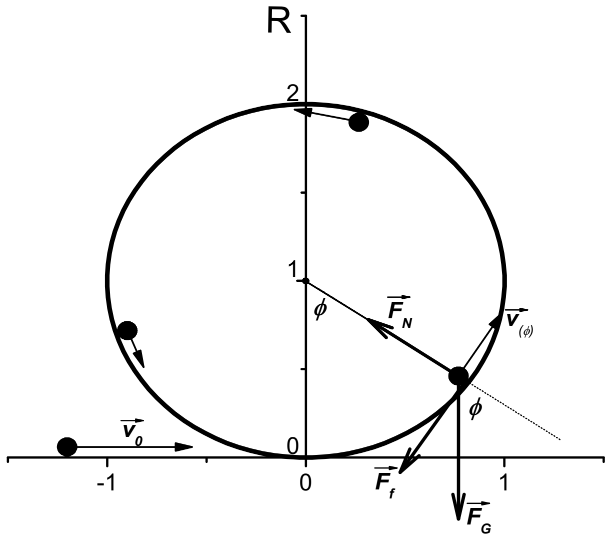


FIG. 1: General setup of a problem concerning a circular loop. A particle with initial speed $v_0$ moves through the loop.

The law of conservation of energy is represented by the equation:

$$\frac{mv_0^2}{2} = \frac{mv^2(\phi)}{2} + mgy(\phi) + \int_0^{l(\phi)} F_f(s)\,ds, \tag{4}$$

where $v_0$ is the initial speed of the particle at the bottom of the loop, whereas the integral represents work done against friction at a distance $l(\phi) = R\phi$. One observes that since $F_f$ at a given $\phi$ depends on the value of speed, the speed itself depends on the history of the particle's motion up to its position given by $\phi$. It becomes much more apparent if one expresses the integrand explicitly as a function of $\phi$ and the element of arc length as $ds = \sqrt{dx^2 + dy^2} = Rd\phi$. Then, by integrating the part with $\cos\phi$, rearranging terms and redefining units of speed by $V = v/\sqrt{gR}$, one reaches the equation for $V(\phi)$:

$$V^2(\phi) = V_0^2 + 2(\cos\phi - \mu\sin\phi - 1) - 2\mu\int_0^{\phi} V^2(\phi')\, d\phi'. \tag{5}$$

Equation (5) is a simple Volterra integral equation of the second kind, for which the solution may be given straightforwardly (see App. A). Hence, $V^2(\phi)$ is found to be:

$$V^2(\phi) = \mathrm{e}^{-2\mu\phi}\left(V_0^2 + \frac{2\xi}{\zeta}\right) - \frac{2}{\zeta}\left(\xi\cos\phi + 3\mu\sin\phi\right), \tag{6}$$

with $\zeta = 1 + 4\mu^2$ and $\xi = 2\mu^2 - 1$.

Now comes the right moment to establish the conditions which have to be met in order to go through the total length of the loop:

1. $V(\phi) > 0$, for $0 < \phi < \frac{\pi}{2}$;

2. $F_N(\phi) > 0$, for $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$;

3. $V(\phi) > 0$, for $\frac{3\pi}{2} < \phi < 2\pi$.

Condition 1 imposes the requirement on the particle to remain its motion through the 1st quarter of the loop. If that condition is violated (i.e. $V(\phi)$ reaches 0 for some $0 < \phi < \frac{\pi}{2}$) the particle will stop. The same simple idea stays behind formulating condition 3. On the other hand, fulfilling condition 2 ensures that the particle will stick to the upper half of the track. Otherwise, i.e. when $F_N(\phi)$ reaches 0 for some $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$, the pressure that the particle exerts on the track vanishes and consequently it will drop due to gravity. Recalling Eq. (3) (for the circular loop) this condition is equivalent to: $V^2(\phi) > -\cos\phi$.

Having defined the criteria above, one is now able to search for the values of $V_0$ and $\mu$, for which each condition is fulfilled for every $\phi$ in the given range. Although in general case it may be done numerically, at this point it is possible to tackle with the problem analytically. In order to do it, little reformulation of the conditions is needed. Putting $V(\frac{\pi}{2}) = 0$ one obtains the equation for the boundary of a region on a $\mu - V_0$ plane for which the particle is unable to go beyond the 1st quarter. The same can be done with respect to the 4th quarter by putting $V(2\pi) = 0$ which then gives the critical values of $V_0$ and $\mu$ upon which the particle stops at the end of the track.

In case of condition 2 however, it is not clear what would be the angle $\phi_0$ for which $F_N(\phi_0) = 0$, since $F_N$ is a quite complex function of $V_0$, $\mu$ and $\phi$. Nevertheless, from condition 2 one finds that $V_0^2 > f(\phi,\mu)/\zeta$, where $f(\phi,\mu) = 3\mathrm{e}^{2\mu\phi}(2\mu\sin\phi - \cos\phi) - 2\xi$. In other words $V_0^2\zeta$ must be greater than $\max f(\phi,\mu)$, which is easily found to be $f(\pi,\mu)$, since $\partial f(\phi,\mu)/\partial\phi = 0$ only when $\phi = \pi$, for which $(\partial^2 f(\phi,\mu)/\partial\phi^2)|_{\phi=\pi} < 0$.

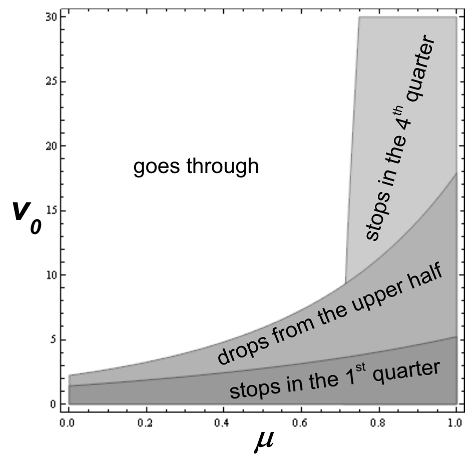


FIG. 2: Phase diagram delineating four regions in the $\mu - V_0$ plane representing different behavior of the particle on the circular loop.

The corresponding diagram summarizing the above analyses is depicted in Fig. 2. As expected, the initial speed $V_0$ required to pass each part of the track raises with increasing value of $\mu$. Although the region for which the particle stops in the 4th quarter might seem quite strange at first glance in comparison with the others, it becomes understandable if one recalls Ref.[1], in which it was shown that when $V(\frac{3\pi}{2}) = 0$ the particle is able to traverse the last section of the loop for any $\mu < 0.603$. In the present case however, after crossing the upper half for which a larger amount of speed was needed, the particle at this point possesses some remaining speed which shifts the boundary towards greater values of $\mu$. In addition, while for $\mu < 0.713$ the friction is insufficient to stop the particle after passing the upper half, with larger values of $\mu$ it becomes the critical factor determining the particle's motion in the last quarter, when the value of initial speed needed to go through the loop raises rapidly and in the limit of $\mu = 1$ it reaches $V_0 = 338.674$.

The nonfriction limit $\mu = 0$ however, is easy to tackle, since Eq. (5) reduces to the form which, after using conditions 1 and 2, determines the critical values of $V_0$ to be $\sqrt{2}$ and $\sqrt{5}$, respectively, as expected.

## III. DYNAMICS ON AN ELLIPTICAL LOOP

The purpose of this section is the comparison of how different shapes of the loop influence the dynamics of the particle, which might seem to be much more complex to analyze than the previous case, but in fact it still remains manageable. Consider an ellipse, of which two orientations are of special interest in this analysis: one, for which semi-major axis is oriented horizontally, will be henceforth referenced as horizontal ($H$), whereas another, for which semi-major axis is oriented vertically – vertical ($V$).

A reliable comparison is possible when the particle is subjected to the frictional force which act over the same distance in each case (namely $2\pi R$), hence it is required to introduce such parametrization of the ellipse that meets that requirement. Yet, since the circumference $C$ of an ellipse cannot be given by any explicit analytical form, one is forced to use some approximation, e.g. one due to Ramanujan[2]:

$$C \approx \pi \left( 3(a+b) - \sqrt{(3a+b)(a+3b)} \right), \tag{7}$$

where $a$ ($b$) is semi-major (semi-minor) axis. (For ratios $\frac{a}{b}$ considered in this paper, this approximation differs less than 0.007% from the actual value of $C$). With the approximation the ellipses are then described by (horizontal):

$$\begin{aligned} x &= Rp\sin\phi, \\ y &= Rp\varepsilon(1-\cos\phi), \end{aligned} \tag{8}$$

and (vertical):

$$\begin{aligned} x &= Rp\varepsilon\sin\phi, \\ y &= Rp(1-\cos\phi), \end{aligned} \tag{9}$$

where $p = 2/(3(1+\varepsilon) - \sqrt{(3+\varepsilon)(1+3\varepsilon)})$ and $\varepsilon = \sqrt{1-e^2}$, where $e = \sqrt{1-\frac{b^2}{a^2}}$ is the eccentricity of the ellipse. Obviously, for $e = 0$ one finds $\varepsilon = 1$ and $p = 1$, for which Eqs. (8) and (9) reduce to (1).

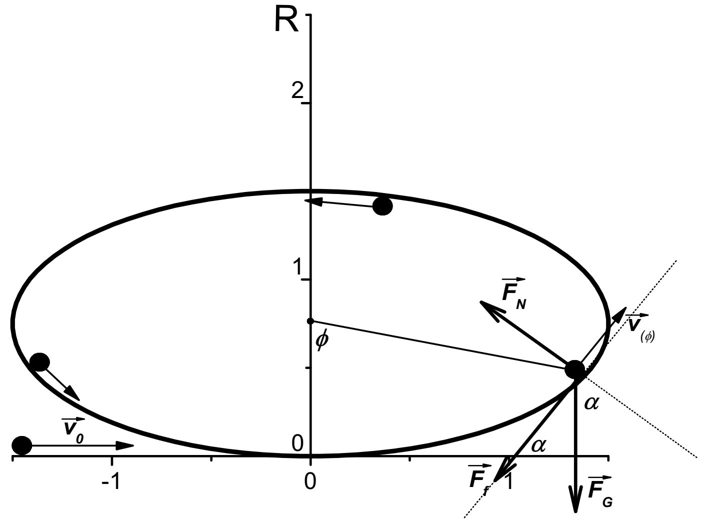


FIG. 3: Geometry of a problem concerning elliptical loop in horizontal orientation.

The dynamics on the elliptical loop differs considerably from that on circular one (see Fig. 3). In this case the centripetal force is no longer directed into the center of the loop, due to the fact that curvature of the ellipse is no longer constant for an arbitrary $\phi$. Yet, the normal force (obviously!) holds its perpendicular orientation relative to the tangent plane of the track. This time however, Newton's second law for normal forces yields:

$$F_N - mg\cos\alpha_{(\phi)} = mv^2(\phi)/\rho(\phi), \tag{10}$$

where $\rho(\phi)$ stands for radius of curvature at a given $\phi$ (certainly, $\rho(\phi) = R$ for a circle). The frictional force then takes the form:

$$F_f(\phi) = \mu\left(mg\cos\alpha_{(\phi)} + \frac{mv^2(\phi)}{\rho(\phi)}\right). \tag{11}$$

For the sake of clarity further analyses will be performed only for the horizontal ellipse (for analogous derivations in case of vertical ellipse, see App. B).

Firstly, the $F_f(\phi)$ has to be given as an explicit function of $\phi$ and natural question emerges, how $\cos\alpha$ depends on $\phi$. With only a little glance at Fig. 3 one might convince oneself, that:

$$f'(x)|_\phi = \tan\alpha, \tag{12}$$

i.e. the slope of the tangent line is equal to the derivative of the function at the given point.

Since

$$f'(x) = \frac{dy}{dx} = \frac{dy/d\phi}{dx/d\phi}, \tag{13}$$

and after using simple trigonometric identities one can easily reach the solution:

$$\cos \alpha_H = \frac{\gamma}{\varepsilon Q_H}, \tag{14}$$

where $Q_H = \sqrt{\tan^2 \phi + \varepsilon^{-2}}$ and $\gamma = \text{sig}(\cos \phi)$.

$\rho(\phi)_H$ can be calculated with assistance of a concept of curvature $\kappa$ of a curve given in the parametric form[3] (Eq. (8)):

$$\frac{1}{\rho(\phi)_H} = \kappa(\phi)_H = \frac{y''_\phi x'_\phi - x''_\phi y'_\phi}{(x'^2_\phi + y'^2_\phi)^{\frac{3}{2}}} = \frac{1}{Rp\varepsilon^2 |\cos^3 \phi| Q_H^3}. \tag{15}$$

Eq. (4) also requires the knowledge of the ellipse's arc length which is given by:

$$ds_H = \sqrt{dx^2 + dy^2} = Rp\varepsilon |\cos \phi| Q_H d\phi. \tag{16}$$

Substituting Eqs. (14) and (15) into Eq. (11) one obtains the explicit form of $F_f(\phi)$. Then by inserting it along with Eq. (16) into Eq. (4), integrating the part of the integrand with $\cos \phi$, rearranging terms and redefining units of speed one reaches the equation for $V(\phi)$ on the elliptical track:

$$\begin{aligned} V^2(\phi) &= V_0^2 + 2p\varepsilon(\cos \phi - \frac{\mu \sin \phi}{\varepsilon} - 1) + \\ &- 2\mu \int_0^\phi V^2(\phi') \text{sic}_H(\phi', \varepsilon)\, d\phi'. \end{aligned} \tag{17}$$

where:

$$\text{sic}_H(\phi, \varepsilon) = (\varepsilon \sin^2 \phi + \varepsilon^{-1} \cos^2 \phi)^{-1}. \tag{18}$$

Eq. (17) is another Volterra integral equation, for which solution may be given:

$$\begin{aligned} V^2(\phi) = V_0^2 + 2p\varepsilon(\cos \phi - \frac{\mu \sin \phi}{\varepsilon} - 1) + \\ - 2\mu \int_0^\phi R_H(\phi, \phi')[V_0^2 + 2p\varepsilon(\cos \phi' - \frac{\mu \sin \phi'}{\varepsilon} - 1)]\, d\phi', \end{aligned} \tag{19}$$

where:

$$R_H(\phi, \phi') = \text{sic}_H(\phi', \varepsilon) \exp\left(-2\mu \int_{\phi'}^\phi \text{sic}_H(x, \varepsilon)\, dx\right). \tag{20}$$

Even though the integral in Eq. (20) cannot be solved, and thus the exact solution for $V(\phi)$ remains unattainable, numerical computation of $V(\phi)$ is still possible, which enables one to analyze the conditions given in Sec. II. Certainly, for $\varepsilon = 1$ (and $p = 1$) all the expressions above reduce to those obtained within the analysis of the circular motion.

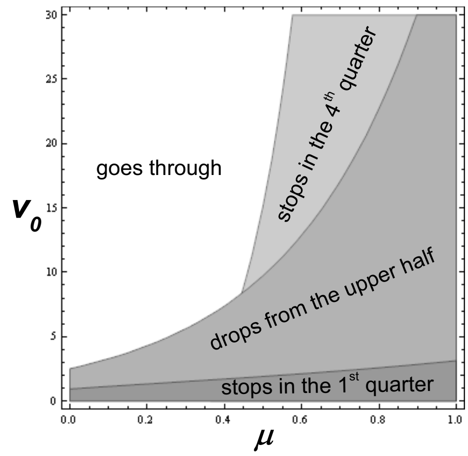


FIG. 4: Phase diagram delineating four regions in the $\mu - V_0$ plane representing different behavior of the particle on the horizontally elliptical loop for $e = 0.95$.

Figures 4 and 5 depict the behavior of the particle on the horizontal (for $e = 0.95$) and vertical ($e = 0.8$) elliptical track, respectively. In case of horizontal ellipse one observes, that flattening the originally circular loop entails decreasing the height which the particle must reach in order to pass the first quarter. While the potential energy for $\phi = \frac{\pi}{2}$ in this case is less than the corresponding one on the circular loop, the greater amount of kinetic energy is saved and the particle is more likely to climb the first quarter of the track. Hence, the first prohibited region on the $\mu - V_0$ plane is smaller than that of the circular loop. In case of vertical ellipse however, the height for which $\phi = \frac{\pi}{2}$ is greater than each of the mentioned above, namely the circular loop and the horizontally elliptical one. Thus the prohibited region becomes larger than that of the circle.

The behavior of the particle on the upper half of the track is determined not only by the relation between kinetic and potential energy – this time the curvature plays the crucial role. For the horizontal ellipse the radius of curvature in the surroundings of $\phi = \pi$ is more likely to reach a value for which the centrifugal force is no longer able to counteract the

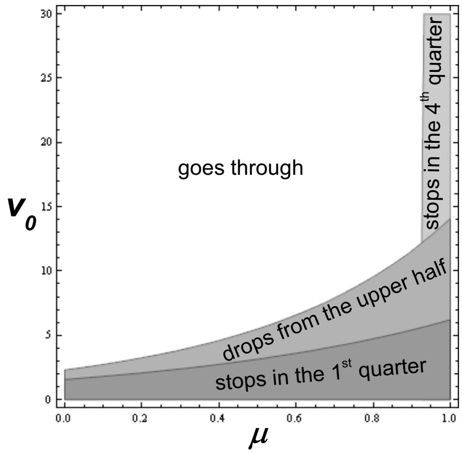


FIG. 5: Phase diagram delineating four regions in the $\mu - V_0$ plane representing different behavior of the particle on the vertically elliptical loop for $e = 0.8$.

gravitational force pulling the particle in the opposite direction. From this reason condition 2 may be satisfied by providing large initial speed, which corresponds to the increase of the second prohibited region. Yet, for the vertical ellipse the curvature at the very top is large enough to prevent the particle from leaving the track, even if the loss of kinetic energy is greater than for horizontal one, therefore the region for which $F_N(\phi) < 0$ is decreased.

The analyses of the third prohibited region bring about an interesting feature. Since a larger part of the last quarter of the horizontal ellipse is less steep than the quarter of a circle, a lower value of $\mu$ is required to stop the particle at some point before it reaches the end of the track. On the other hand, for the vertical ellipse the trend is reversed: as the track becomes steeper with increasing $e$, the normal force for a large part of the quarter decreases almost to 0 and the particle experiences nearly free fall motion. Thus, flattening the vertical ellipse entails shifting the forbidden region towards larger $\mu$, and for some $e$ it vanishes completely.

In the absence of frictional force $\mu = 0$ conditions 1 and 2 can be easily solved using Eq. (19) (for horizontal ellipse) and Eq. (B6) (for vertical ellipse), yielding simple formulas (condition 1):

$$V_0 > \begin{cases} \sqrt{2p\varepsilon} & \text{for horizontal ellipse,} \\ \sqrt{2p} & \text{for vertical ellipse,} \end{cases} \tag{21}$$

and (condition 2):

$$V_0 > \begin{cases} \sqrt{p(4\varepsilon + \frac{1}{\varepsilon})} & \text{for horizontal ellipse,} \\ \sqrt{p(4+\varepsilon^2)} & \text{for vertical ellipse,} \end{cases} \tag{22}$$

which for $\varepsilon = 1$ coincide with corresponding values $\sqrt{2}$ and $\sqrt{5}$ for the circle.

## IV. COMPARISON BETWEEN DYNAMICS ON DIFFERENT LOOPS

While previous sections were devoted to determining the conditions which the particle must satisfy in order to go through the loops of different kind, the purpose of this section is the comparison of the particle's motion on those tracks.

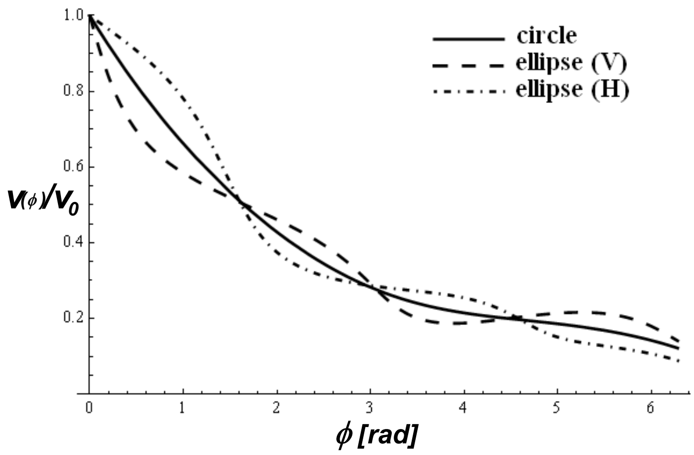


FIG. 6: The ratio $\frac{V(\phi)}{V_0}$ plotted as a function of $\phi$ for different loops, for $\mu = 0.4$, $V_0 = 10$ and $e = 0.9$.

Figure 6 shows the angular dependence of the particle's speed for the circular, horizontally and vertically elliptical loops for given values of $V_0$, $\mu$ and $e$. Since all the dynamical quantities were calculated as a function of $\phi$ one should bear in mind that for different loops the same $\Delta\phi$ does not cover the same distance $\Delta s$. Hence, if one urges to link these results to the analogous ones expressed in terms of covered distance, it is reliable only for $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

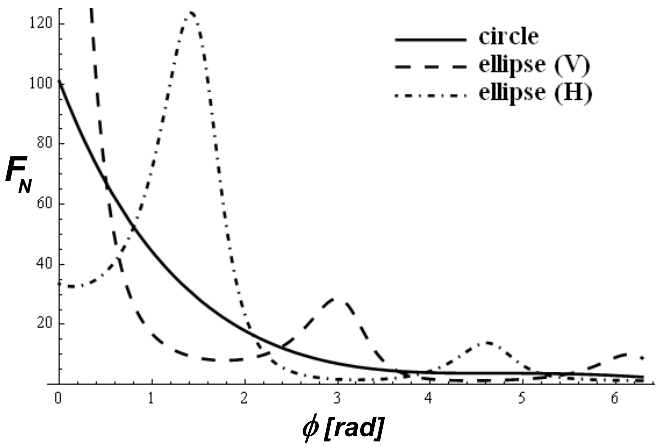


FIG. 7: $F_N(\phi)$ in units of $mg$ plotted as a function of $\phi$ for different loops, for $\mu = 0.4$, $V_0 = 10$ and $e = 0.9$.

Nevertheless, this graph gives some insight into how the speed changes in relation to a particular track. The intersection of those three graphs might seem quite interesting at first, but it appears to be an accidental misleading feature when one comes into details. It shows clearly that in proximity of $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ the speeds for different loops are merely comparable.

Figure 7 represents the angular dependence of the normal force for different loops. Since $F_N$ is given in units of $mg$, one might feel puzzled by the large numbers on vertical axis (in case of vertical ellipse the value $F_N(0) = 393.59$ is beyond the scale!). In fact, when one refers to the initial definitions, it becomes clear that for e.g. $R = 1\,m$, the initial speed $V_0 = 10$ is an equivalent of $70\,mph$ and coming across the loop at such a speed inevitably must involve huge G-forces!

The peaks for the ellipses correspond to the high curvature in the proximity of the ellipses' foci, while different heights of those peaks are related to the fact that, since the particle is subjected to frictional force, it loses its kinetic energy (in case of the vertical ellipse the height of the peak at $\phi = \pi$ is also affected by the increase of potential energy) which in turn entails decreasing value of $F_N$.

Finally, it is worthwhile to compare the time needed to traverse the total length of the track for different loops. Since the speed $v = \frac{ds}{dt}$, the formula for the time $t_{tr}$ can be easily found to be:

$$t_{tr} = \oint \frac{ds}{v}. \tag{23}$$

Numerical evaluation of this quantity for $V_0 = 10$ and $e = 0.9$ (note that for each shape of the loop, the arc length $ds$ must take its appropriate form) in frictionless case yields:

$$t_{tr}(\mu = 0) = \begin{cases} 0.2027\, s & \text{for the circle,} \\ 0.2018\, s & \text{for the horizontal ellipse,} \\ 0.2034\, s & \text{for the vertical ellipse.} \end{cases} \tag{24}$$

As it was expected, the shortest time for traversing the loop occurs in case of the horizontal ellipse, where the loss of speed due to an increase of potential energy is the smallest. In the presence of the frictional force however (e.g. $\mu = 0.4$):

$$t_{tr}(\mu = 0.4) = \begin{cases} 0.7462\, s & \text{for the circle,} \\ 0.8310\, s & \text{for the horizontal ellipse,} \\ 0.7047\, s & \text{for the vertical ellipse,} \end{cases} \tag{25}$$

and in this situation the shortest time is that of the vertical ellipse, even though the speed losses resulting from the increasing potential energy are most apparent.

## V. RELATED ISSUES

The main purpose of this paper was to study the motion of the particle on different elliptical loops and investigate how the existence of frictional force affects the dynamics of the particle. This topic is closely related to the problem of finding the velocity of a block as it traverses a quarter of a circular surface with friction, as presented in Ref.[1] In fact, once the velocity as a function $V(\phi)$ is calculated (see Eq. (6)) there is an easy way to skip the derivation and resolve the equation of motion in that particular case. By making simple transformation in Eq. (6):

$$\phi \longrightarrow \phi' = \phi + \frac{3\pi}{2}, \tag{26}$$

and then choosing $V_0^2$ (being at this point some artificial constant) such that it satisfies the initial condition $V(\phi' = 0) = 0$, one immediately obtains the desired function $V(\phi')$ describing the motion of the particle in this case. Moreover, from Eq. (19), the analogous analyses can now be made for the elliptical tracks as well.

Another interesting problem is the location of the release point of a particle which, given some initial speed $V_0'$ at the top, slides on the exterior surface of a sphere.[4] Since the centrifugal force inputs negative contribution to the normal force, this time Eq. (6) can be reformulated by imposing on it a transformation:

$$\begin{aligned} \phi &\longrightarrow \phi' = \phi + \pi, \\ R &\longrightarrow R' = -R, \end{aligned} \tag{27}$$

and demanding that $V(\phi' = 0) = V_0'$ for initial speed. Finally, the release point can be found by resolving the equation $F_N = 0$, which is an analogy to finding the boundary of a region in the $\mu - V_0$ plane for which the particle drops from the track.

Questions that arise from the analysis presented here include the following:

Problem 1. Eq. (6) under the transformation $\phi \longrightarrow \phi' = -\phi$ followed by $R \longrightarrow R' = -R$ or $g \longrightarrow g' = -g$ gives the exact solution of the problem obtained in Ref.[4] What is the physical meaning of such transformation (in fact $\phi \longrightarrow \phi' = -\phi$ is mathematically equivalent to $\mu \longrightarrow \mu' = -\mu$)?

Problem 2. What is the value of the eccentricity $e$ for which the 3$^{\text{rd}}$ prohibited region in case of the vertical ellipse (see Fig. 5) vanishes completely?

Problem 3. What is the behavior of $t_{tr}$ with respect to $\mu$ and (or) $e$? Are there any such values of these parameters for which the time $t_{tr}$ in case of the circular loop is the shortest?

Problem 4. How will the friction affect the motion on a real roller coaster loop (i.e. one for which the normal force is constant along the whole length of a track[6])?

**Acknowledgments**

The author would like to thank Maciej Kasperski for his interesting discussions, which inspired this paper. Helpful comments of Andrzej Grudka are also acknowledged.

## APPENDIX A: THE SOLUTION OF THE VOLTERRA INTEGRAL EQUATION

A generalized form of the Volterra integral equation of the second kind considered in the analyses in this paper may be written as:

$$y(x) = f(x) + \int_a^x h(t)y(t)\,dt. \tag{A1}$$

The solution of Eq. (A1) is given by:

$$y(x) = f(x) + \int_a^x R(x,t) f(t)\, dt, \tag{A2}$$

where:

$$R(x,t) = h(t) \exp\left( \int_t^x h(s)\, ds \right). \tag{A3}$$

Clearly, in case of the circular loop, the function $h(t)$ is constant, and since $f(t)$ takes a simple trigonometric form, the solution is to be found straightforwardly, albeit quite laboriously.

## APPENDIX B: EXPRESSIONS CONCERNING AN ELLIPSE IN THE VERTICAL ORIENTATION

In case of the vertical ellipse $(V)$ the interplay between angles $\alpha$ and $\phi$ (see Eq. (14)) is described by:

$$\cos\alpha_V = \frac{\gamma}{Q_V}, \tag{B1}$$

where $Q_V = \sqrt{\tan^2\phi\,\varepsilon^{-2} + 1}$.

The curvature $\kappa$ (see Eq. (15)) is then calculated to be:

$$\kappa(\phi)_V = \frac{1}{\rho(\phi)_V} = \frac{1}{Rp\varepsilon^2 |\cos^3\phi| Q_V^3}, \tag{B2}$$

and the vertical ellipse's arc length (see Eq. (16)):

$$ds_V = \sqrt{dx^2 + dy^2} = Rp\varepsilon |\cos\phi| Q_V d\phi. \tag{B3}$$

The equation for $V(\phi)$ is (see Eq. (17)):

$$\begin{aligned} V^2(\phi) &= V_0^2 + 2p(\cos\phi - \varepsilon\mu\sin\phi - 1) + \\ &- 2\mu\int_0^\phi V^2(\phi')\mathrm{sic}_V(\phi', \varepsilon)\, d\phi'. \end{aligned} \tag{B4}$$

where:

$$\mathrm{sic}_V(\phi, \varepsilon) = \mathrm{sic}_H(\phi, \varepsilon^{-1}) = (\varepsilon^{-1}\sin^2\phi + \varepsilon\cos^2\phi)^{-1}. \tag{B5}$$

The solution of Eq. (B4) is:

$$\begin{aligned} V^2(\phi) = {} & V_0^2 + 2p(\cos\phi - \varepsilon\mu\sin\phi - 1) + \\ & - 2\mu\int_0^\phi R_V(\phi, \phi')[V_0^2 + 2p(\cos\phi - \varepsilon\mu\sin\phi - 1)]\, d\phi', \end{aligned} \tag{B6}$$

where:

$$R_V(\phi, \phi') = \mathrm{sic}_V(\phi', \varepsilon) \exp\left(-2\mu \int_{\phi'}^{\phi} \mathrm{sic}_V(x, \varepsilon)\, dx\right). \tag{B7}$$

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