Solution of Supplee's submarine paradox through special and general relativity

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In 1989 Supplee described an apparent relativistic paradox on which a submarine seems to sink in a given frame while floating in another one. If the submarine density is adjusted to be the same as the water density (when both of them are at rest) and then it is put to move, the density of the submarine will become higher than that of the water, thanks to Lorentz contraction, and hence it sinks. However, in the submarine proper frame, is the water that becomes denser, so the submarine supposedly should float and we get a paradox situation.

In this paper we analyze the submarine paradox in both a flat and a curved spacetime. In the case of a flat spacetime, we first show that any relativistic force field in special relativity can be written in the Lorentz form, so that it can always be decomposed into a *static* (electric-like) and a *dynamic* (magnetic-like) part. Taking into account the gravitomagnetic effects between the Earth and the water, a relativistic formulation of Archimedes principle can be established, from which the apparent paradox can be explained. On the other hand, considering the curved spacetime on the vicinity of Earth, we show that the gravitational force exerted by Earth on a moving body should now increase with the speed of the body. Notwithstanding, the submarine paradox can still be explained with this speed-dependent force when the corresponding gravitomagnetic effects are taken into account.

Keywords: Supplee's submarine paradox, theory of relativity, gravitomagnetism, Archimedes principle, Lorentz force.

I. THE PARADOX

When a submarine is submerged underwater it can sink or float, depending on whether its density is higher or lower than the density of the water. Suppose that we adjust the density of the submarine when it is at rest to be the same as the density of the standing water, so that the submarine remains in equilibrium when submerged. What should happen, then, when the submarine is put to move with a high velocity in the water? Disregarding any hydrodynamic effects as drag, viscosity, turbulence, etc. (which we shall always assume hereafter), observers fixed to the ocean would claim that the submarine sinks, since its density becomes higher than the water density thanks to the Lorentz contraction. On the other hand, observers within the submarine would claim instead that the submarine should float, since is the water that now becomes denser. Of course, the submarine cannot float in a frame and sink in another, so we get a contradictory situation.

This apparent paradox was described in 1989 by Supplee in [1], although he had used a bullet instead of a submarine. Considering some assumptions about the gravitational force among moving bodies, Supplee gave two explanations for the problem, from which he concluded that the submarine should sink in both frames. In the first explanation, he avoided to use the theory of gravitation by considering a uniformly upwards accelerated lake which, according to Einstein's equivalence principle, behaves likely a uniform gravitational field; he showed that in this case the bullet acceleration is less than that of the lake, so that the bullet relatively sinks. In the submarine proper frame, however, the water indeed becomes denser but, since this frame is no longer inertial, the isobaric surfaces of the lake will not be flat anymore, which ultimately results in the submarine going far away from the lake surface, *i.e.*, in the bullet sinking again. In the second explanation, Supplee

considered a constant weak field in the framework of general relativity, which led him to the same conclusion.

Fourteen years after Supplee's publication, Matsas had analyzed the problem again, but this time using the full machinery of general relativity [2]. Considering a background spacetime with a Rindler chart and assuming reasonable conditions about the submarine rigidness, Matsas analyzed the motion of a submarine which accelerates from the rest to a given velocity *v*. He concluded that the submarine shape gets deformed as it accelerates, with its length contracting more and more, so that its density increases accordingly, which leads the submarine to sink. Moreover, in the proper frame of the submarine he showed that the observed gravitational field is somewhat different, which leads the submarine to sink again. Matsas also argued that this problem can be important to some questions regarding the thermodynamic of black-holes [2].

Finally, Supplee's paradox was studied once more by Jonsson through the analysis of the fictitious forces that appear in non-inertial frames [3].

Although the approaches above mentioned are interesting on their own, we believe that it is not necessary to employ accelerated frames neither to use the full theory of general relativity in order to explain the submarine paradox. In fact, first of all, we should remark that accelerated motions can be contemplated with special relativity without use of non-inertial frames, so the introduction of non-inertial frames to explain the submarine paradox is not necessary. Moreover, the behavior of the submarine (if it sinks or floats) depends only on the balance between the Archimedes (buoyancy) force and the gravitational (weight) force acting on it. On the surface of Earth, the gravitational field is relatively very small (in the sense that spacetime curvature can be neglected for any practical purpose), which enable us to interpret the gravitational interaction as an ordinary force field in a flat spacetime. This, of course, is only an approximation, since general relativity show us that in an exact flat spacetime there is no gravity. Nevertheless, the spacetime in the vicinity of Earth can be regarded, with a very high accuracy, as consisting

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of a *flat space* plus a *curved time*. We shall show that even in this case the gravitational field can still be interpreted as a force field, although it must become dependent on the speed of the falling bodies. Special theory of relativity can also be employed with some care and, hence, the submarine paradox can be explained in both a flat as well as in a curved spacetime.

To correctly explain the submarine paradox, however, it is necessary to impose covariance of the gravitational force regarding the Lorentz transformations. This lead us to a *covariant theory of gravitation* in a flat spacetime [4]. This theory also holds in a flat space but a curved time when the speed dependence of the gravitational force is taken into account. This covariance requirement implies that gravitomagnetic effects, which play key hole in our explanation of the Supplee's submarine paradox, must be present whenever there is a relative movement between the interacting bodies.

This paper is organized as follows. In section II we proof a theorem stating that every force field in special relativity can be written in the Lorentz form, so that any force can be split into a *static* (electric-like) and a *dynamic* (magnetic-like) part. In the section III we use these results to explain the submarine paradox in a flat spacetime; the gravitomagnetic effects involved enable us to formulate a relativistic version of the Archimedes principle which gives the correct Archimedes force acting on the submarine, providing in this way the solution of the paradox. Next, in the section IV, we discuss the curvature of spacetime in the vicinity of Earth and we show that in this case the gravitational force between the Earth and a very fast moving body must increase with the speed of the body. Finally, we revisit the submarine paradox considering this speed-dependent force law in the section V and we compare the results of both approaches.

II. ANY COVARIANT FORCE FIELD OF SPECIAL RELATIVITY CAN BE WRITTEN IN LORENTZ FORM

Special theory of relativity tells us that the force is not a four-vector. In fact, if $F = dp/dt$ is the same force acting in a given body, as measured from an inertial frame R, and $F' = dp'/dt'$ is the force measured by another inertial frame R' (with R' moving w.r.t. R with the velocity $v = v\hat{x}$, all the axes being coincident at $t = t' = 0$), then we get that [5, 6]

$$F'_x = F_x - \frac{v}{c^2}\frac{u_y F_y + u_z F_z}{1 - \frac{u_x v}{c^2}}, \tag{1a}$$

$$F'_y = F_y \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u_x v}{c^2}}, \tag{1b}$$

$$F'_z = F_z \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{u_x v}{c^2}}, \tag{1c}$$

where u is the body velocity as measured by R.

In this section, we shall show that although the force is not a four-vector, it can be always written in a Lorentz form,

$$F = G + u \times H, \tag{2}$$

w.r.t. any inertial frame. In (2), G is defined as the part of the force F which does not depend on the body velocity u − we may call it the *static* (electric-like) part of the force. Similarly, $M = u \times H$ is defined by the part of the force which does depend on u − we may call it the *dynamic* (magnetic-like) part of the force.

To prove the statement above, let us assume that F, the force acting on the body in the frame R, does not depend on u[1]. Hence, in the frame R the force is already written in Lorentz form with $G = F$ and $H = 0$. Now we have to proof that the same is true in the frame R'. To find the force in the frame R', we can use (1). Notice however that force F' depends on velocity u that the particle has in the frame R. However, the observers in R' do not measure u, instead, it is u' that is actually measured. This fact suggests us to eliminate u through the velocity transformation formulæ,

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}}, \quad u_y = u'_y \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}}, \quad u_z = u'_z \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{u'_x v}{c^2}}, \tag{3}$$

in order to rewrite (1) in terms of u'. Inserting (3) into (1) and simplifying, we get the formulæ

$$F'_x = F_x - \frac{v}{c^2}\frac{u'_y F_y + u'_z F_z}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{4a}$$

$$F'_y = F_y \frac{1 + \frac{u'_x v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{4b}$$

$$F'_z = F_z \frac{1 + \frac{u'_x v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4c}$$

Thus, from the part of F' which does not depend on u' we get the static part of the force, G', namely,

$$G'_x = F_x, \quad G'_y = \frac{F_y}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad G'_z = \frac{F_z}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{5}$$

and, from that part of F' which does depend on u', we get the dynamical part of the force, M',

$$M'_x = -\frac{v}{c^2}\frac{u'_y F_y + u'_z F_z}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad M'_y = \frac{\frac{u'_x v}{c^2} F_y}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad M'_z = \frac{\frac{u'_x v}{c^2} F_z}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{6}$$

Now it is just a matter of fact that (6) can be written as the vector product $M' = u' \times H'$, with

$$H' = -\frac{v}{c^2} \times G', \tag{7}$$

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[1] For speed-dependent force fields, in general there is no inertial frame where the force becomes independent of the body velocity. Nonetheless, this fact does not invalidate the use of (1) and, hence, the results which follow will still hold. The words *static* and *dynamic*, however, becomes inappropriate in this case, since now both G as H may depend on the body velocity (perhaps the terms *electric-like* and *magnetic-like* are more suitable here).

that is,

$$H'_x = 0, \quad H'_y = \frac{v}{c^2} \frac{F_z}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad H'_z = -\frac{v}{c^2} \frac{F_y}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (8)$$

Thus, the force $\boldsymbol{F}'$ can be written in the Lorentz form (2) w.r.t. the frame R' as well. Moreover, since the frame R' is quite arbitrary, we had proved that in any inertial frame every physically acceptable force field can be written in the Lorentz form[2]. This can be also proved considering another inertial frame R'' moving w.r.t. R' with a velocity $\boldsymbol{w} = w\hat{\boldsymbol{x}}$. Supposing that in R' the force acting on the body is $\boldsymbol{F}' = \boldsymbol{G}' + \boldsymbol{u}' \times \boldsymbol{H}'$, then, repeating the above procedure, we can show that in R'' the force is still given by $\boldsymbol{F}'' = \boldsymbol{G}'' + \boldsymbol{u}'' \times \boldsymbol{H}''$, with the static and dynamic forces in each frame related with themselves by the formulæ (see [4] for details),

$$G''_x = G'_x, \quad G''_y = \frac{G'_y - wH'_z}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad G''_z = \frac{G'_z + wH'_y}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad (9)$$

and

$$H''_x = H'_x, \quad H''_y = \frac{H'_y + \frac{w}{c^2}G'_z}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad H''_z = \frac{H'_z - \frac{w}{c^2}G'_y}{\sqrt{1 - \frac{w^2}{c^2}}}. \quad (10)$$

The most known example of such a force is the electromagnetic one. In this case we have $\boldsymbol{G} = q\boldsymbol{E}$ and $\boldsymbol{M} = q\boldsymbol{u} \times \boldsymbol{B}$, with q denoting the electric charge, $\boldsymbol{E}$ the electric field and $\boldsymbol{B}$ the magnetic field, respectively. Another example is the gravitational force in the approximation where Newton's law is valid (*i.e.*, in a flat spacetime background). In this case, we have $\boldsymbol{G} = m\boldsymbol{g}$ and $\boldsymbol{M} = m\boldsymbol{u} \times \boldsymbol{h}$, where $\boldsymbol{g}$ is the static gravitational field and $\boldsymbol{h}$ the dynamic gravitational field − the gravitational analogue of the magnetic field. The consequences of this covariant theory of gravitation were recently discussed in [4].

III. SOLUTION OF THE SUBMARINE PARADOX IN A FLAT SPACETIME

Let us now analyze the submarine paradox considering by now only the special theory of relativity. This means that in this section the gravitational interaction will be regarded as an ordinary force field in a flat spacetime (in the same way as, for instance, the electromagnetic interactions are usually treated). We shall assume therefore that the gravitational force that Earth exerts in a given particle is given by Newton's law,

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}, \quad (11)$$

when the Earth is at rest, no matter what is the particle motion. In (11), M is the mass of Earth, m the mass of the particle, $\boldsymbol{r}$ is the distance vector from the Earth to the particle position,

G the Newton constant. In the present case, we shall consider actually only the constant gravitational force $\boldsymbol{F} = -mg\hat{\boldsymbol{z}}$ on the surface of Earth, where g is the acceleration of gravity. Finally, we shall also consider the (inertial and gravitational) mass as an invariant quantity, which is of course the most logical way to proceed and to avoid misunderstandings[3].

Before analyze the original formulation of Supplee's paradox, let us consider first a slight modified version on which no acceleration is involved. This is obtained supposing a submarine moving with velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ in the standing water of the ocean and letting its density be adjusted by the observers at rest within the ocean (frame R) in such a way that the submarine remains in equilibrium in this frame. From Archimedes principle this means that the submarine density must be adjusted to be the same as the water density when both are measured by the frame R. The paradox situation arises because it seems, at first sight, that the submariners (frame R') would conclude that submarine should float, since in this frame the submarine density happens to be lesser than that of the moving water, thanks to the Lorentz length effects. We shall see, however, that this apparent paradox is due to an incorrect use of usual Archimedes principle: if special relativity is correctly employed, we get that the submarine actually does not sink neither float in both frames.

A straightforward confirmation of this result could be given directly from the transformation formulæ (1) or (4). In fact, since in the frame R the total force acting on the submarine is null, the same will be true in the frame R', so that the submarine cannot accelerate in neither frame. However, in order to provide a physical explanation of the problem, a more detailed exposition is necessary. We shall explain in the sequel what happens with the weight and Archimedes forces separately, explaining why the Archimedes principle cannot be directly applied in the frame R'. Nevertheless, a relativistic formulation of Archimedes principle can be formulated if we take into account the gravitomagnetic effects that will be present in R'.

Let us begin our analysis in the frame R. Here, the water of the ocean is at rest, while the submarine has velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$. The submarine is subject to two forces: a small constant gravitational (weight) force, $\boldsymbol{W} = -mg\hat{\boldsymbol{z}} = -\rho_{\text{sub}}V_{\text{sub}}g\hat{\boldsymbol{z}}$, and the Archimedes (buoyancy) force, $\boldsymbol{A}$. Archimedes force is a response from the water to the action of gravity: a gradient of pressure arises in order to kept its static equilibrium. The gradient of pressure present in a fluid suited in a gravitational field equals the density of that gravitational force, $\boldsymbol{\nabla}p = \boldsymbol{f} = -\rho_{\text{wat}}g\hat{\boldsymbol{z}}$. Hence, Archimedes force acting on the submarine can be found integrating $-\boldsymbol{\nabla}p$ over the submarine volume:

$$\boldsymbol{A} = \int_{V_{\text{sub}}} (-\boldsymbol{\nabla}p)\,dV = \int_{V_{\text{sub}}} \rho_{\text{wat}}g\hat{\boldsymbol{z}}dV = \rho_{\text{wat}}V_{\text{sub}}g\hat{\boldsymbol{z}}. \quad (12)$$

Thus we can see that the total force acting on the submarine will be null if we set $\rho_{\text{sub}} = \rho_{\text{wat}} = \rho$.

In the proper submarine frame, R', on the other hand, the water is moving with the velocity $\boldsymbol{u}'_{\text{wat}} = -v\hat{\boldsymbol{x}}$. Lorentz

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[2] For a deduction of this result using the concept of four-force, see [4].

[3] See Refs. [7–11] for discussions about the concept of mass in relativity.
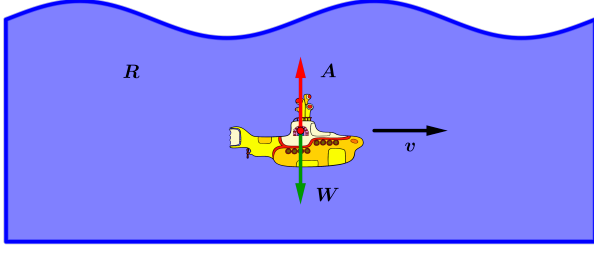
Figure 1. *Submarine paradox in the frame R*. A submarine is moving with velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$ in a standing ocean. The submarine density is adjusted to be the same as the water density, so that the intensity of Archimedes force $\boldsymbol{A}$ equals the intensity of gravitational (weight) force $\boldsymbol{W}$ and the submarine does not sink neither float.
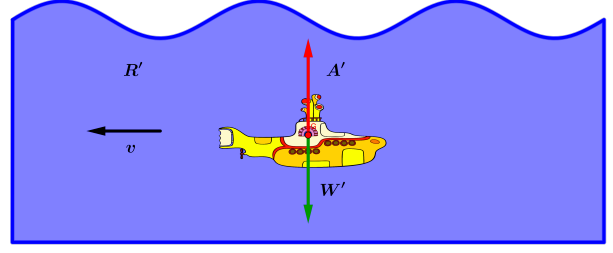

Figure 2. *Submarine paradox in the frame R′*. In this case the submarine is at rest and the water is moving with the velocity $\boldsymbol{u}'_{\text{wat}} = -v\hat{\boldsymbol{x}}$. Thus the submarine density becomes lower than the water density. Notwithstanding the difference of densities, a relativistic Archimedes principle shows that Archimedes force $\boldsymbol{A}'$ still equals the weight force $\boldsymbol{W}'$ in intensity, so that the submarine still remains in equilibrium. The dynamic (magnetic-like) gravitational force that the moving Earth exerts on the moving water contributes significantly to this result.

length effects implies that water's density increases, while the submarine density decreases by the same factor:

$$\rho'_{\text{wat}} = \frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad \rho'_{\text{sub}} = \rho\sqrt{1 - \frac{v^2}{c^2}}. \qquad (13)$$

In this frame, the submarine is also subjected to the gravitational and Archimedes forces. The gravitational force, however, is not given by Newton's law anymore, neither can the Archimedes force be deduced directly from the usual Archimedes principle. The gravitational force acting on the submarine should be found through (4), remembering that in R' the submarine velocity $\boldsymbol{u}'_{\text{sub}}$ is zero:

$$\boldsymbol{W}' = \frac{\boldsymbol{W}}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{mg\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{\rho V_{\text{sub}}g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}}. \qquad (14)$$

On the other hand, to find the Archimedes force in the frame R', we need to know first what happens with the gradient of pressure on the water. The gradient of pressure can also be regarded as a response from the water to the gravitational force of the Earth, of course, but now we should realize that both the water as the Earth move w.r.t. R' with the velocity $\boldsymbol{u}'_{\text{wat}} = -v\hat{\boldsymbol{x}}$. Hence, the water will be subject to both a static (electric-like) gravitational force as well as a dynamic (magnetic-like) one. A unit volume of water in the frame R is subject to the gravitational force $\boldsymbol{F} = -\rho g\hat{\boldsymbol{z}}$ and, from (5) and (6), follow that the static and dynamic gravitational forces acting on this element of volume, as measured in the frame R', will be, respectively,

$$\boldsymbol{G}' = -\frac{\rho g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \boldsymbol{M}' = \frac{v^2}{c^2}\frac{\rho g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}}. \qquad (15)$$

Notice that, according to (6) and (7) we can write the dynamical force as $\boldsymbol{M}' = \boldsymbol{u}'_{\text{wat}} \times \boldsymbol{H}'$, where $\boldsymbol{H}' = -\boldsymbol{v}/c^2 \times \boldsymbol{G}'$. From (2) we get, therefore, the total force acting on that element of volume:

$$\boldsymbol{F}' = -\frac{\rho g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{v^2}{c^2}\frac{\rho g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}} = -\rho g\hat{\boldsymbol{z}}\sqrt{1 - \frac{v^2}{c^2}}. \qquad (16)$$

Now we should realize that this quantity of water no longer occupies a unit volume in the frame R'. In fact, it is contracted by a factor of $\sqrt{1 - v^2/c^2}$, so that, in order to get the force per unit volume in the frame R', we need further to divide (16) by this factor. Whence, we get,

$$\boldsymbol{f}' = \boldsymbol{\nabla}'p' = -\rho g\hat{\boldsymbol{z}}. \qquad (17)$$

The conclusion is that the gradient of pressure is not proportional to the higher water's density − rather, it is an invariant quantity, which could be anticipated already from the fact that pressure is a scalar and from $\partial'_z = \partial_z$. This is why Archimedes principle cannot be directly applied in R'.

Integrating $\boldsymbol{f}' = -\boldsymbol{\nabla}'p'$ over the submarine volume, we get the Archimedes force acting on it:

$$\boldsymbol{A}' = \int_{V'_{\text{sub}}} (-\boldsymbol{\nabla}'p')\,dV' = \int_{V'_{\text{sub}}} \rho g\hat{\boldsymbol{z}}\,dV' = \rho V'_{\text{sub}}g\hat{\boldsymbol{z}}. \qquad (18)$$

Finally, since $V'_{\text{sub}} = V_{\text{sub}}/\sqrt{1 - v^2/c^2}$, we get,

$$\boldsymbol{A}' = \frac{\rho V_{\text{sub}}g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}} = -\boldsymbol{W}'. \qquad (19)$$

Therefore, the Archimedes force intensity equals the weight of the submarine and, thus, it does not float neither sink in the frame R' as well − the submarine remains in equilibrium in both frames.

We would like to highlight that Archimedes force can also be obtained from a *relativistic Archimedes principle*. Remember that the original formulation of Archimedes principle states that the intensity of the Archimedes force equals the weight of water displaced by the submersed body. This principle is still valid in the frame R', we must distinguish between the *static weight* of the mass displaced by the body, defined as the mass displaced times the static (electric-like) gravitational field, $\boldsymbol{W}' = \rho'_{\text{wat}}V'_{\text{bod}}\boldsymbol{g}'$, and its *dynamic weight*, which is given by the current of mass times the dynamic (magnetic-like) gravitational

field, $\boldsymbol{M}' = V'_{\mathrm{bod}}\boldsymbol{j}'_{\mathrm{wat}} \times \boldsymbol{h}'$, where $\boldsymbol{j}'_{\mathrm{wat}} = \rho'_{\mathrm{wat}}\boldsymbol{u}'_{\mathrm{wat}}$ and $\boldsymbol{h}' = -\boldsymbol{v}/c^2 \times \boldsymbol{g}'$. Therefore, we get for the static and dynamic weight of the submarine, respectively,

$$\boldsymbol{W}' = -\frac{\rho'_{\mathrm{wat}}V'_{\mathrm{sub}}g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \boldsymbol{M}' = \frac{v^2}{c^2}\frac{\rho'_{\mathrm{wat}}V'_{\mathrm{sub}}g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (20)$$

Notice that the dynamical weight is contrary to the static weight and hence it can be thought as a "negative weight" due to the repulsive dynamic gravitational force between the Earth and the water. The sum of these two terms (with the opposite signs) provides, of course, the Archimedes force (19).

Finally, let us consider the original formulation of Supplee's paradox. In this case the density of the submarine is adjusted to the water density when both of them are at rest (let m be the submarine mass and V_0 its proper volume, so that its proper density is $\rho_0 = m/V_0$). If the submarine is put to move with a velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$, the gravitational force acting on it will still be $\boldsymbol{W} = -mg\hat{\boldsymbol{z}} = -\rho_0 V_0 g\hat{\boldsymbol{z}}$, since the gravitational field is just static in the frame R. To evaluate the Archimedes force we should realize that now the submarine volume is contracted to $V_{\mathrm{sub}} = V_0\sqrt{1 - v^2/c^2}$, and then, from the Archimedes principle, we get that $\boldsymbol{A} = \rho_0 V_0 g\hat{\boldsymbol{z}}\sqrt{1 - v^2/c^2}$. Thus, the total force acting on the submarine is

$$\boldsymbol{F} = -\rho_0 V_0 g\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)\hat{\boldsymbol{z}}. \quad (21)$$

On the other hand, in the frame R' (the inertial frame that is instantaneously at rest w.r.t. the submarine at at $t = t' = 0$), the weight force acting on the submarine will be, according to (4), $\boldsymbol{W}' = -\rho_0 V_0 g\hat{\boldsymbol{z}}/\sqrt{1 - v^2/c^2}$. Notice that there is no dynamic (magnetic-like) force here again, since the submarine is at rest on R' in this instant of time. There is, however, a dynamic force between the moving Earth and the moving ocean. As we have seen, these gravitomagnetic forces combined imply that the gradient of water's pressure observed in the frame R' is the same as that measured in R. Thus, Archimedes force will be given just by $\boldsymbol{A}' = \rho_0 V_0 g\hat{\boldsymbol{z}}$. From this we find that the total force acting on the submarine w.r.t. the frame R' is

$$\boldsymbol{F}' = -\rho_0 V_0 g\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right)\hat{\boldsymbol{z}} = \frac{\boldsymbol{F}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (22)$$

The conclusion is that in both frames the submarine will sink. Notice further that is not necessary to employ accelerated frames neither general relativity in order to study the submarine behavior. We can do that, of course, but then we should take care with the geometric effects that arise in non-inertial frames, as already discussed by Supple and Matsas [1, 2].

IV. SPACETIME CURVATURE IN THE VICINITY OF EARTH AND THE IMPLICATED SPEED-DEPENDENT GRAVITATIONAL FORCE

According to Einstein's theory of gravitation, gravity is not a force but just an effect of the spacetime curvature [12].

In other words, Einstein's theory implies that there is no gravitational field in an exactly flat spacetime whatsoever. Of course, the Newtonian description of gravity as a force field in a flat spacetime can be justified as being a very good approximation in the vicinity of Earth, which is due to the very small curvature of spacetime − so small that we can only measure its effects with the most precise instruments available. The former approach presented in the last section assumes that this is indeed the case, hence, it should be treated as a first approximation for the problem.

Nevertheless, in order to get a better approximation for the gravitational phenomenon on the surface of Earth, the curvature of spacetime need to be taken into account. Since any physical measurement of distances performed on the surface of Earth does not reveal any discordance with the Euclidean geometry, it is enough to consider here a *flat space plus a curved time*. This means that the metric on the proximity of Earth can be written in the form,

$$ds^2 = -f(z)dt^2 + dx^2 + dy^2 + dz^2, \quad (23)$$

(we suppose that Earth is larger enough so that its surface can be approximated by horizontal plane), where the function $f(z)$ is to be determined. In order to do so, we shall proceed as follows: first remember that in Einstein's theory, the world-line of a particle freely falling in a gravitational field is a geodesic, which is determined by the equations [12]

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0, \quad (24)$$

where u^μ are the components of the four-velocity of the particle (in a given frame), τ is its proper time and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols, which are obtained from the metric through the formula [12]

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right). \quad (25)$$

The geodesic equation agrees with the fact that the particle four-acceleration $a^\alpha = du^\alpha/d\tau + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$ must be zero in the proper co-moving frame of the particle, since Einstein's equivalence principle states that the particle does not feel any effect of gravity as it freely fall in the gravitational field.

However, for an observer at rest on the surface of Earth, the four-acceleration of the particle will be given just by $a^\alpha = du^\alpha/d\tau$. In fact, since the spacetime can be considered locally flat in the neighborhood of the particle (which is supposed to be near from the Earth), we can use Cartesian coordinates to measure its movement, from which the Christoffel symbols will all vanish. Comparing this result with (24), we see that we can write,

$$a^\alpha = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma. \quad (26)$$

Now, consider a particle that is released from the rest in the constant gravitational field near the surface of Earth. In this very instant, the components of the particle four-velocity are

$$u^0 = c, \quad u^1 = 0, \quad u^2 = 0, \quad u^3 = 0. \quad (27)$$

Besides, we know that in this case the acceleration of the particle is just $\boldsymbol{a} = -g\hat{\boldsymbol{z}}$ and, hence, the components of its four-acceleration are

$$a^0 = 0, \quad a^1 = 0, \quad a^2 = 0, \quad a^3 = -g. \tag{28}$$

On the other hand, it follow from (23) and (25) that the only non-null Christoffel symbols are

$$\Gamma^0_{03} = \Gamma^0_{30} = \frac{1}{2}\frac{d}{dz}\log f(z), \quad \text{and} \quad \Gamma^3_{00} = \frac{1}{2}\frac{df(z)}{dz}. \tag{29}$$

Then, using (26), (27), (28) and (29) at once, we get the relation

$$a^3 = -\Gamma^3_{00}u^0u^0 = \frac{c^2}{2}\frac{df(z)}{dz} = -g. \tag{30}$$

Solving this equation, we find the function $f(z)$:

$$f(z) = \frac{2gz}{c^2} + C, \tag{31}$$

where C is the constant of integration. In order to fix C, we may realize that in the absence of the gravitational field (*i.e.*, for $g = 0$), the metric should reduce to the Minkowski metric, $ds^2 = -c^2dt + dx^2 + dy^2 + dz^2$. This lead us to the value $C = c^2$ and, hence, the spacetime metric in the vicinity of Earth's surface becomes,

$$ds^2 = -\left(1 + \frac{2gz}{c^4}\right)c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{32}$$

The corresponding non-null Christoffel symbols reduce to

$$\Gamma^0_{03} = \Gamma^0_{30} = \frac{g}{c^4}\left(1 + \frac{2gz}{c^4}\right)^{-1}, \quad \Gamma^3_{00} = \frac{g}{c^2}. \tag{33}$$

Since the metric (32) is linear in the coordinates, Einstein's field equations are automatically satisfied, of course.

Now, let us see what should be the acceleration of the particle in the gravitational field of Earth when the particle has a given (instantaneous) velocity, $\boldsymbol{u}$. In this case its four-velocity becomes,

$$u^0 = \gamma_u c, \quad u^1 = \gamma_u u_x, \quad u^2 = \gamma_u u_y, \quad u^3 = \gamma_u u_z, \tag{34}$$

where we introduced the Lorentz factor $\gamma_u = 1/\sqrt{1 - u^2/c^2}$ for short. However, we cannot assume that the acceleration of the particle is directed along the $\hat{\boldsymbol{z}}$ direction anymore, since the theory of relativity shows us that acceleration and force are not parallel each to the other, except when the velocity is parallel or orthogonal to the force. In fact, the relationship between force and acceleration is [5, 6, 12]

$$\boldsymbol{F} = m\left[\gamma_u\boldsymbol{a} + \gamma_u^3\left(\frac{\boldsymbol{a}\cdot\boldsymbol{u}}{c^2}\right)\boldsymbol{u}\right]. \tag{35}$$

Hence, the particle four-acceleration must be written as

$$a^0 = \gamma_u^4\left(\frac{\boldsymbol{a}\cdot\boldsymbol{u}}{c^2}\right)c, \tag{36a}$$

$$a^1 = \gamma_u^2 a_x + \gamma_u^4\left(\frac{\boldsymbol{a}\cdot\boldsymbol{u}}{c^2}\right)u_x, \tag{36b}$$

$$a^2 = \gamma_u^2 a_y + \gamma_u^4\left(\frac{\boldsymbol{a}\cdot\boldsymbol{u}}{c^2}\right)u_y, \tag{36c}$$

$$a^3 = \gamma_u^2 a_z + \gamma_u^4\left(\frac{\boldsymbol{a}\cdot\boldsymbol{u}}{c^2}\right)u_z. \tag{36d}$$

On the other hand, (26), (33) and (34) furnish us,

$$a^0 = -\frac{2g}{\left(1 + \frac{2gz}{c^4}\right)}\frac{u_z\gamma_u^2}{c^3}, \quad a^1 = 0, \quad a^2 = 0, \quad a^3 = -g\gamma_u^2. \tag{37}$$

Comparing (36) with (37) we obtain the acceleration of the particle in the weak gravitational field of Earth:

$$a_x = g\left(\frac{u_x u_z}{c^2}\right), \quad a_y = g\left(\frac{u_y u_z}{c^2}\right), \quad a_z = -g\left(1 - \frac{u_z^2}{c^2}\right). \tag{38}$$

Notice that the particle acceleration will be directed along the $\hat{\boldsymbol{z}}$ direction only if $u_z = 0$ or if $u_x = u_y = 0$. In the fist case, the acceleration is just $\boldsymbol{a} = -g\hat{\boldsymbol{z}}$, while in the second case we have $\boldsymbol{a} = -g\left(1 - u^2/c^2\right)\hat{\boldsymbol{z}}$.

Finally, inserting (38) into (35) and simplifying, we get the gravitational force acting on the moving particle:

$$\boldsymbol{F} = -\frac{mg\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{39}$$

We conclude therefore that the spacetime curvature in the proximity of Earth implies a *speed-dependent* gravitational force. The gravitational force increases with the particle speed.

Notwithstanding the curved spacetime we have here, we may realize that the Lorentz transformations can still be employed, as long as the frame R' moves w.r.t. the frame R' in a direction parallel to Earth's surface (for instance in the $\hat{\boldsymbol{x}}$ direction). The only effects differing from that obtained in an exactly flat spacetime (besides the existence of the gravitational field, of course) are those on which the events should be compared in different heights coordinates, since, according to the metric (32), the higher the position, the faster the time will pass. In fact, if dt is a given interval of time measured by a clock at the surface of Earth, then an identical clock situated at the height z will mark the time $dt_z = \sqrt{1 + \frac{2gz}{c^4}}dt$, which is higher than dt. Correspondingly, if a light ray is emitted from the height z towards the surface of Earth, a blue-shift effect will take place.

V. SOLUTION OF THE SUBMARINE PARADOX IN THE CURVED SPACETIME OF EARTH

Finally, let us analyze what changes we get in the description of the submarine paradox, when the effects of the tiny spacetime curvature on the surface of the Earth are taken into account. It is sufficient for this end to consider a gravitational force law given by (39) instead of the Newton's law (11).

Let us first consider that version of the paradox on which the submarine does not accelerate. In this case, where the spacetime is weakly curved, the gravitational force $\boldsymbol{W}$ acting on the submarine w.r.t. the frame R, will be

$$\boldsymbol{W} = -\frac{mg\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{\rho_{\text{sub}}V_{\text{sub}}g\hat{\boldsymbol{z}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{40}$$

since the submarine moves with the velocity $\boldsymbol{v} = v\hat{x}$ and we are using the force law (39). In order to the submarine stay in equilibrium underwater, its density should now be adjusted (by the observers at rest within the water) to $\rho_{\text{sub}} = \rho_{\text{wat}}\sqrt{1 - v^2/c^2}$, so that the intensity of Archimedes force $\boldsymbol{A}$ equals the intensity of the weight force $\boldsymbol{W}$. In the frame R', of course, the same will be true. In fact, the gravitational force acting on the submarine can be found through (4), and it is given by

$$\boldsymbol{W'} = -\frac{mg\hat{z}}{\left(1 - \frac{v^2}{c^2}\right)} = -\frac{\rho_{\text{sub}}V_{\text{sub}}g\hat{z}}{\left(1 - \frac{v^2}{c^2}\right)}. \qquad (41)$$

The density of the water in the frame R' is still given by (13) and, hence, the gradient of pressure in the frame R' remains the same as that measured in the frame R. Thus, Archimedes force becomes,

$$\boldsymbol{A'} = \frac{\rho_{\text{wat}}V_{\text{sub}}g\hat{z}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\rho_{\text{sub}}V_{\text{sub}}g\hat{z}}{\left(1 - \frac{v^2}{c^2}\right)}, \qquad (42)$$

from which we can see that in frame R' the submarine will remain in equilibrium as well.

For the original formulation of Supplee's paradox, we get a similar explanation. Here the submarine density ρ_0 is matched with the water density when both are at rest. In the frame R, the submarine moves with the velocity $\boldsymbol{v} = v\hat{x}$ and gravitational force acting on it is

$$\boldsymbol{W} = -\frac{mg\hat{z}}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{\rho_0 V_0 g\hat{z}}{\sqrt{1 - \frac{v^2}{c^2}}}. \qquad (43)$$

The Archimedes force is $\boldsymbol{A} = \rho_0 V_{\text{sub}}g\hat{z} = \rho_0 V_0 g\hat{z}\sqrt{1 - v^2/c^2}$ and, thus, the total force acting on the submarine is

$$\boldsymbol{F} = -\rho_0 V_0 g\hat{z}\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} + \sqrt{1 - \frac{v^2}{c^2}}\right) = -\frac{v^2}{c^2}\frac{\rho_0 V_0 g\hat{z}}{\sqrt{1 - \frac{v^2}{c^2}}}. \qquad (44)$$

In the frame R' the total force acting on the submarine can be found through (4), and then we get

$$\boldsymbol{F'} = -\frac{\rho_0 V_0 g\hat{z}}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} + \sqrt{1 - \frac{v^2}{c^2}}\right) = -\frac{v^2}{c^2}\frac{\rho_0 V_0 g\hat{z}}{\left(1 - \frac{v^2}{c^2}\right)}. \qquad (45)$$

The first and second terms in this equation refer respectively to the weight and Archimedes force. Taking into account the gravitomagnetic effects, we can still get the Archimedes force from a relativistic Archimedes principle, as before.

Equations (44) and (45) agree with those obtained by Supplee and Matsas [1, 2]. It should be mentioned, however, that Supplee's argument is not quite correct, since he assumed (as many others assume, see Refs. [7–11]) that the gravitational mass of a body is determined by Einstein's formula $E = mc^2$, which leads to a speed-dependent mass: $m_u = m_0/\sqrt{1 - u^2/c^2}$. However, the relation between mass and energy expressed by Einstein's formula holds only when the momentum of the body is null − in fact the correct expression is $E = \sqrt{m^2c^4 + p^2c^2}$. Besides, we must remember that in Einstein's theory of gravitation the source of gravitational interaction is energy-momentum tensor, not the energy alone. Therefore, it is only a coincidence that the speed-dependent mass considered by Supplee gives the same speed-dependent force law (39) deduced from general relativity. Matsas, on the other hand, bypassed this issue of what should be gravitational force acting on a moving body by replacing the special relativity analysis with a full general relativistic approach [2].

In this paper we showed that the submarine paradox can be explained through both the special and general relativity. The only difference arising from the two approaches is that the force acting on the submarine (and its acceleration) is higher in the curved spacetime of Earth than in an exact flat spacetime.

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