The physics of stone skipping

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The motion of a stone skimming over a water surface is considered. A simplified description of the collisional process of the stone with water is proposed. The maximum number of bounces is estimated by considering both the slowing down of the stone and its angular stability. The conditions for a successful throw are discussed.

I. INTRODUCTION

Nearly everyone has tried to throw a stone on a lake and count the number of bounces the stone was able to make. Of course the more, the better.[1] Our intuition gives us some empirical rules for the best throw: the best stones are flat and rather circular; one has to throw them rather fast and with a small angle with the water surface; a small kick is given with a finger to give the stone a spin. Of course these rules can be understood using the laws of physics: the crucial part of the motion is the collisional process of the stone with the water surface. The water surface exerts a reaction (lift) force on the stone, allowing it to rebound. This process is quite complex because it involves the description of the flow around the immersed stone.[2,3] Some energy is also dissipated during a collision, so that after a few rebounds, the initial kinetic energy of the stone is fully dissipated and the stone sinks.

The purpose of this paper is to propose a simplified description of the bouncing process of a stone on water, in order to estimate the maximum number of bounces performed by the stone. This problem provides an entertaining exercise for undergraduate students, with simple explanations for empirical laws that almost everyone has experienced.

II. BASIC ASSUMPTIONS

Consider a flat stone, with a small thickness and a mass M. The stone is thrown over a flat water surface. The angle between the stone surface and the water plane is θ. A schematic view of the collisional process is shown in Fig. 1. The velocity V is assumed to lie in a symmetry plane of the stone (the plane of the paper). The difficult part of the problem is, of course, to model the reaction force due to the water, which results from the flow around the stone during the stone-water contact. It is not the aim of this paper to give a detailed description of the fluid flow around the colliding stone. Rather I shall use a simplified description of the force keeping only the main ingredients of the problem. First, the velocity V of the stone is expected to be (at least initially) the order of a few meters per second. For a stone with a characteristic size a of the order of a few centimeters, the Reynolds number, defined as $\mathcal{R}e = Va/\nu$, with ν the kinematic viscosity ($\nu \sim 10^{-6}\,\mathrm{m^2\,s^{-1}}$ for water), is of or-

der $\mathcal{R}e \sim 10^5$, that is, much larger than unity.[4] In this (inertial) regime, the force due to the water on the stone is expected on dimensional grounds to be quadratic in the velocity and proportional to the apparent surface of the moving object and the mass density of the fluid.[5] Because the stone is only partially immersed in water during the collisional process, we expect the force to be proportional to the immersed surface (see Fig. 1). The force can be adequately decomposed into a component along the direction of the stone (that is, along $\mathbf{t}$, see Fig. 1) and a component perpendicular to it (that is, along $\mathbf{n}$). The latter corresponds to the lift component of the force, and the former corresponds to a friction component (of water along the object). I write the reaction force due to water, $\mathbf{F}$, as:

$$\mathbf{F} = \frac{1}{2}C_l\rho_w V^2 S_{\mathrm{im}}\mathbf{n} + \frac{1}{2}C_f\rho_w V^2 S_{\mathrm{im}}\mathbf{t}\,, \qquad (1)$$

where C_l and C_f are the lift and friction coefficients, ρ_w is the mass density of water, S_{im} is the area of the immersed surface, and $\mathbf{n}$ is the unit vector normal to the stone (see Fig. 1). Note that in general, both C_l and C_f are func-
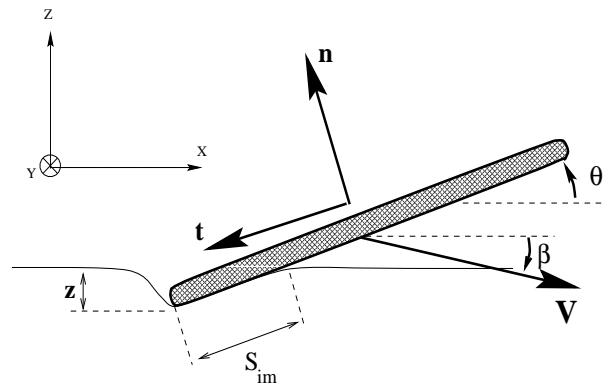

FIG. 1: Schematic view of the collisional process of a flat stone encountering a water surface. The stone has a velocity V, with an incidence angle β, while θ is the tilt angle of the stone. The immersed area S_{im} represents the area of the stone in contact with the water surface. The depth of the immersed edge is z.

tions of the tilt angle θ and incidence angle β, defined as the angle between velocity V and the horizontal. In the simplified analysis I will assume that both C_l and C_f are constant and independent of tilt and incidence angles.[6]

This assumption is not a strong one because ricochets are generally performed with a small tilt angle, θ, and a small incidence angle, β. If one denotes the initial components of the incident velocity by V_{x0} and V_{z0} (parallel and perpendicular to the water surface, respectively), the latter assumption amounts to $V_{z0} \ll V_{x0}$.

We expect the lift force to be maximum when the object is only partially immersed due to the lack of symmetry between the two sides of the stone. Therefore, if the object reaches a depth such that it becomes completely immersed, the lift force would be greatly diminished and would probably not be able to sustain the weight of the stone anymore. For simplicity, I will assume that the lift force vanishes for completely immersed objects. The model for the force in Eq. (1) is crude, but it is expected to capture the main physical ingredients of the stone-water interaction. It might fail for lower stone velocities or larger incidence angles, where a bulge of water could be created and affect the lift and friction forces on the stone.[2] However, in this case it is expected that the stone will be strongly destabilized during the collision process and perform only a very small number of bounces. We will restrict ourselves to large initial velocities and small incidence angles, such that the number of bounces is sufficiently large.

III. EQUATIONS OF MOTION

Consider the collisional process, that is, the time during which the stone is partially immersed in water. I will assume in this section that the incidence angle θ between the stone and the water surface is constant during the collisional process. The validity of this assumption is considered in detail in Sec. V. The origin of time, $t = 0$, corresponds to the instant when the edge of the stone reaches the water surface. During the collisional process, the equations of motion for the center of mass velocity are

$$M\frac{dV_x}{dt} = -\frac{1}{2}\rho_w V^2 S_{\text{im}}(C_l \sin\theta + C_f \cos\theta) \quad (2a)$$

$$M\frac{dV_z}{dt} = -Mg + \frac{1}{2}\rho_w V^2 S_{\text{im}}(C_l \cos\theta - C_f \sin\theta) \quad (2b)$$

with $V^2 = V_x^2 + V_z^2$ and g is the acceleration due to gravity. Note that in Eq. (2) the area S_{im} depends on the immersed depth and thus varies during the collisional process.

Equation (2) is non-linear due to the V^2 terms on the right-hand side, but also due to the dependence of the immersed area, S_{im}, on the height z. However, we can propose a simple approximation scheme: the magnitude of the velocity, V, is not expected to be strongly affected by the collision process (as I shall show in Sec. VI). I thus make the approximation that $V^2 \simeq V_{x0}^2 + V_{z0}^2 \simeq V_{x0}^2$ on the right-hand side of Eq. (2). The validity of this assumption requires a sufficiently high initial velocity, V_{x0},

and it might fail in the last few rebounds of a stone skip sequence.

With this approximation, Eq. (2b) decouples from Eq. (2a). I thus first focus the discussion on the equation for the height z, which is the height of the immersed edge (see Fig. 1). Note that the equation for z is equivalent to the equation of the center of mass position, Eq. (2b) because θ is assumed to be constant (see Sec. V for a detailed discussion of this point). Hence, we may identify V_z with dz/dt and Eq. (2b) yields a closed equation for the height z.

IV. COLLISIONAL PROCESS

To solve Eq. (2b) we need to prescribe the z dependence of the immersed area S_{im}. This quantity depends on the precise shape of the stone. A natural choice is circular, which I will treat in Sec. IV B. However, it is enlighting to first consider a square shape; this shape greatly simplifies the mathematics and already contains the basic mechanisms involved.

A. A Square Stone

In this case, the immersed area is simply $S_{\text{im}} = a|z|/\sin\theta$ (see Fig. 1), with a the length of one edge of the stone. The equation for z thus becomes

$$M\frac{d^2z}{dt^2} = -Mg - \frac{1}{2}\rho_w V_{x0}^2 C\frac{az}{\sin\theta}, \quad (3)$$

where $C = C_l \cos\theta - C_f \sin\theta \simeq C_l$, and I have used $|z| = -z$ ($z < 0$). We define the characteristic frequency ω_0 as

$$\omega_0^2 = \frac{C\rho_w V_{x0}^2 a}{2M\sin\theta}, \quad (4)$$

and rewrite Eq. (3) as

$$\frac{d^2z}{dt^2} + \omega_0^2 z = -g. \quad (5)$$

With the initial conditions at $t = 0$ (first contact with water), $z = 0$ and $\dot{z} = V_{z0} < 0$, the solution of Eq. (5) is

$$z(t) = -\frac{g}{\omega_0^2} + \frac{g}{\omega_0^2}\cos\omega_0 t + \frac{V_{z0}}{\omega_0}\sin\omega_0 t. \quad (6)$$

Equation (6) characterizes the collisional process of the stone with water. After a collision time t_{coll} defined by the condition $z(t_{\text{coll}}) = 0$ ($t_{\text{coll}} \simeq 2\pi/\omega_0$), the stone emerges totally from the water surface. It is easy to show that the maximal depth attained by the stone during the collision is

$$|z_{\text{max}}| = \frac{g}{\omega_0^2}\left[1 + \sqrt{1 + \left(\frac{\omega_0 V_{z0}}{g}\right)^2}\right]. \quad (7)$$

As discussed in Sec. I, the stone will rebound if it stays only partially immersed during the collision. The rebound condition can be written as $|z_{max}| < a\sin\theta$. If we use Eqs. (7) and (4), this condition can be written after some straightforward calculations as

$$V_{x0} > V_c = \frac{\sqrt{\frac{4Mg}{C\rho_w a^2}}}{\sqrt{1 - \frac{2\tan^2\beta M}{a^3 C\rho_w \sin\theta}}}, \qquad (8)$$

where the incidence angle β is defined as $V_{z0}/V_{x0} = \tan\beta$. Therefore, we obtain a *minimum critical velocity* for skimming. Using the typical values, $M = 0.1\,\mathrm{kg}$, $L = 0.1\,\mathrm{m}$, $C_l \approx C_f \approx 1$, $\rho_w = 1000\,\mathrm{kg\,m^{-3}}$, and $\beta \sim \theta \sim 10°$, we obtain $V_c \simeq 0.71\,\mathrm{m\,s^{-1}} \sim 1\,\mathrm{m\,s^{-1}}$.

The physical meaning of this condition is clear: it simply expresses the fact that the lift force $\frac{1}{2}C\rho_w V^2 a^2$ has to balance the weight of the stone Mg in order for it to bounce.

B.   A circular stone

For a circular stone, the immersed area is a more complex function of the height z, and is given in terms of the area of a truncated circle. A simple integral calculation yields

$$S_{im}(s) = R^2[\arccos(1-s/R) - (1-s/R)\sqrt{1 - (1 - s/R)^2}], \qquad (9)$$

with $s = |z|/\sin\theta$ (the maximum immersed length) and $R = a/2$ as the radius of the stone.

The equation of motion for z, Eq. (2b), thus becomes non-linear. However, it is possible to describe (at least qualitatively) the collisional process and obtain the condition for the stone to bounce.

I first introduce dimensionless variables to simplify the calculations. The dimensionless height, $\tilde{z}$, time, τ, and immersed area, $\mathcal{A}$, are defined as $\tilde{z} = -z/R\sin\theta$, $\tau = \omega_0 t$, and $\mathcal{A}(\tilde{z}) = S_{im}/R^2$. (The minus sign in $\tilde{z}$ is introduced for convenience.) If we use these variables, Eq. (2b), and $V_z = dz/dt$, we obtain

$$\frac{d^2\tilde{z}}{d\tau^2} = \alpha - \frac{1}{2}\mathcal{A}(\tilde{z}), \qquad (10)$$

with $\alpha = g/(R\omega_0^2 \sin\theta)$. Equation (10) is the equation of a particle (with unit mass) in the potential $\mathcal{V}(\tilde{z}) = \int(\frac{1}{2}\mathcal{A}(\tilde{z}) - \alpha)d\tilde{z}$. We can use standard techniques for mechanical systems to solve Eq. (10). In particular, Eq. (10) can be integrated once to give the "constant energy" condition

$$\frac{1}{2}\left(\frac{d\tilde{z}}{d\tau}\right)^2 + \mathcal{V}(\tilde{z}) = E, \qquad (11)$$

where E is the energy of the system and is given in terms of the initial conditions

$$E = \frac{1}{2}\left(\frac{d\tilde{z}}{d\tau}\right)^2\Big|_{\tau=0} + \mathcal{V}(\tilde{z} = 0) = \frac{1}{2}(V_{z0}/(R\omega_0\sin\theta))^2. \qquad (12)$$

The potential $\mathcal{V}(\tilde{z})$ can be calculated analytically using the expression for the immersed area S_{im} given in Eq. (9). An integral calculation gives

$$\mathcal{V}(\tilde{z}) = \frac{1}{2}\left(\sqrt{1 - (1 - \tilde{z})^2}\left[\frac{2}{3} + \frac{1}{3}(1 - \tilde{z})^2\right] \right.$$
$$\left. -(1 - \tilde{z})\arccos(1 - \tilde{z})\right) - \alpha\tilde{z}. \qquad (13)$$

This potential is plotted in Fig. 2 as a function of $\tilde{z}$. As a consequence of the constant energy condition, Eq. (11), $\tilde{z}$ exhibits a turning point at a maximum depth defined by $\mathcal{V}(\tilde{z}_{max}) = E$.
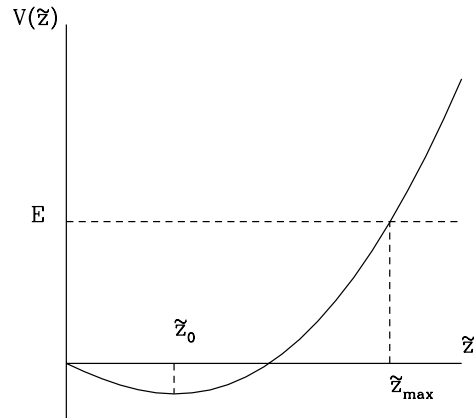

FIG. 2: Plot of the potential $\mathcal{V}(\tilde{z})$. The horizontal line is the constant energy E of the system.

Here again, the condition for the stone to bounce is that this maximum depth be reached before the stone is fully immersed, that is, $|z_{max}| < 2R\sin\theta$. In terms of dimensionless variables, we obtain the condition: $\tilde{z}_{max} < 2$, with $\tilde{z}_{max}$ defined by $\mathcal{V}(\tilde{z}_{max}) = \frac{1}{2}(V_{z0}/(R\omega_0\sin\theta))^2$. This condition can be explicitly solved. Let me introduce $\tilde{z}_0$ such that $d\mathcal{V}/d\tilde{z} = 0$ at $\tilde{z} = \tilde{z}_0$: $\mathcal{V}(\tilde{z})$ is a monotonically increasing function of $\tilde{z}$ for $\tilde{z} > \tilde{z}_0$. Now it is easy to show that $\tilde{z}_{max} > z_0$ (because $\mathcal{V}(\tilde{z}_{max}) > 0$ and $\mathcal{V}(\tilde{z}_0) < 0$), and the condition $\tilde{z}_{max} < 2$ is therefore equivalent to $\mathcal{V}(\tilde{z}_{max}) < \mathcal{V}(2) = \frac{\pi}{2} - 2\alpha$, that is, $\frac{1}{2}(V_{z0}/(R\omega_0\sin\theta))^2 < \frac{\pi}{2} - 2g/(R\omega_0^2\sin\theta)$. Then the condition for skimming can be rewritten (recalling that $V_{z0}/V_{x0} = \tan\beta$)

$$V_{x0} > V_c = \frac{\sqrt{\frac{16Mg}{\pi C\rho_w a^2}}}{\sqrt{1 - \frac{8M\tan^2\beta}{\pi a^3 C\rho_w \sin\theta}}}. \qquad (14)$$

Up to (slightly different) numerical factors this condition is the same as in Eq. (8) for a square stone. Note moreover, that the reasoning used for the potential $\mathcal{V}$ is quite general and can be applied to the square shape as well.

This reasoning yields the same condition as Eq. (8) in this case.

Note also that for the circular stone, a simplified analysis of the motion could have been performed. First if $\tilde{z}$ remains small during the bounce of the stone, a small $\tilde{z}$ expansion of $\mathcal{V}(\tilde{z})$ is possible, yielding $\mathcal{V}(\tilde{z}) = 4\sqrt{2}/15\tilde{z}^{5/2} - \alpha\tilde{z}$ (corresponding to a parabolic approximation for the shape of the stone near its edge). Moreover, we remark that for small V_{z0}, the energy E goes to zero, so that $\tilde{z}_{\max}$ is defined in this case by $\mathcal{V}(\tilde{z}_{\max}) = 0$. If also use the previous approximation, we obtain $\tilde{z}_{\max} = (15\alpha/4\sqrt{2})^{2/3}$. The condition for the stone to bounce, $\tilde{z}_{\max} < 2$, therefore yields $\alpha < 16/15$. In terms of V_{x0}, this condition gives again a minimum critical velocity for skimming, defined as $V_c = \sqrt{\zeta M g/C\rho_w a^2}$ with $\zeta = 15/4 \simeq 3.75$. This result is thus close to the "exact" condition found in Eq. (14) for the $V_{z0} = 0$ case.

C. Energy Dissipation

I have so far described the rebound of the stone by analyzing its vertical motion. This analysis gave a minimum velocity for skimming which results from the balance between the weight of the stone and the lift of the force due to water. However, some energy is dissipated during the collision due to the "friction" contribution of the force (the component along x). This mechanism of dissipation leads to another minimum velocity condition, in terms of the balance between dissipation and initial kinetic energy. Only a qualitative description of the dissipation is given here.

As shown by Eq. (2), the component F_x of the reaction force in the x direction (parallel to the water surface) will decrease the velocity of the stone. Then after a few bounces, the condition for the stone to bounce, Eq. (8) or Eq. (14), will no longer be satisfied and the stone will stop. It is possible to estimate the decrease in the x-component of the velocity using the equation for the center of mass position, Eq. (2a). If we multiply both sides of Eq. (2a) by V_x and integrate over a collision time, we obtain the decrease in the kinetic energy in the x direction in terms of the work of the reaction force:

$$\mathcal{W} \equiv \frac{1}{2}MV_{xf}^2 - \frac{1}{2}MV_{x0}^2 = -\int_0^{t_{\text{coll}}} F_x(t)V_x(t)dt, \quad (15)$$

where V_{x0} and V_{xf} are the x-components of the velocity before and after the collision, t_{coll} is the collision time, and $F_x = \frac{1}{2}\tilde{C}\rho_w V_x^2 S_{\text{im}}$ is the x-component of the reaction force, with $\tilde{C} = C_l \sin\theta + C_f \cos\theta$.

A rough estimate of the right-hand side of Eq. (15) is

$$\int_0^{t_{\text{coll}}} F_x(t)V_x(t)dt \simeq V_{x0}\int_0^{t_{\text{coll}}} F_x(t)dt. \quad (16)$$

Now we have the simple relation $F_x(t) = \mu F_z(t)$, with $\mu = \tilde{C}/C$ (see Eq. (1)). Moreover, it is expected that the average vertical force during a collision, $\langle F_z(t)\rangle =$

$t_{\text{coll}}^{-1}\int_0^{t_{\text{coll}}} F_z(t)dt$, is the order of the weight of the stone, Mg. This point can be explicitly verified for the square stone case, using the expression of the force F_z in terms of the height $z(t)$ and Eq. (6). The final result is $\langle F_x(t)\rangle \simeq \mu Mg$.[8]. Moreover, as shown in the above (and in particular for the square stone, although the results remain qualitatively valid for the circular one), the collision time is given approximatively by $t_{\text{coll}} \sim 2\pi/\omega_0$. We eventually find that the loss in kinetic energy in Eq. (15) is approximatively given by

$$\mathcal{W} \simeq -\mu MgV_{x0}\frac{2\pi}{\omega_0} = -\mu Mg\ell, \quad (17)$$

where ℓ is defined as

$$\ell = V_{x0}\frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{2M\sin\theta}{C\rho_w a}}. \quad (18)$$

The quantity $\ell = V_x t_{\text{coll}}$ is the distance along x traversed by the stone during a collision. If the energy loss $\mathcal{W}$ is larger than the initial kinetic energy, the stone would be stopped during the collision. Using Eq. (15), this condition can be written explicitly as $\frac{1}{2}MV_{x0}^2 > |\mathcal{W}| = \mu Mg\ell$. We deduce that the initial velocity should be larger than the minimum velocity V_c in order to perform at least one bounce, that is,

$$V_{x0} > V_c = \sqrt{2\mu g\ell}. \quad (19)$$

If we use the same numerical values as in the previous paragraph, we obtain $\mu = 1.4$, $\ell = 13\,\text{cm}$, so that $V_c \approx 2\,\text{m s}^{-1}$. This criterion is more restrictive than the previous one, Eq. (14). I thus consider in the following that Eq. (19) is the criterion for the stone to skim over water.

V. WHY GIVE THE STONE A SPIN?

The previous calculations assumed a constant angle θ. It is obvious that the rebound of the stone is optimized when θ is small and positive (see, for example, the value of the force constant $C = C_l\cos\theta - C_f\sin\theta$ which decreases when θ increases). Now, if after a collision, the stone is put in rotation around the y-axis (see Fig. 1), that is, $\dot{\theta} \neq 0$, its orientation would change by an appreciable amount during free flight: the incidence angle θ for the next collision has little chance to still be in a favorable situation. The stone performs, say, at most one or two more collisions. There is therefore a need for a stabilizing angular motion. This is the role of the spin of the stone.

Let us denote $\dot{\phi}_0$ as the rotational velocity of the stone around the symmetry axis parallel to $\mathbf{n}$ in Fig. 1. I neglect in the following any frictional torque on the stone (associated with rotational motion). During the collision, the reaction force due to the water is applied only to the immersed part of the stone and results in a torque applied

on the stone. For simplicity, I consider only the lift part of the force. Its contribution to the torque (calculated at the center O of the stone) can be readily calculated as $\mathcal{M}_{\text{lift}} = OP \cdot F_{\text{lift}} \mathbf{e}_y$, where $\mathbf{e}_y$ is the unit vector in the y direction in Fig. 1 and P, the point of application of the lift force, is located at the center of mass of the immersed area. This torque is in the y direction and will eventually affect the angular motion along θ. However a spin motion around $\mathbf{n}$ induces a stabilizing torque: this is the well-known gyroscopic effect.[7] The derivation of the equation of motion of the rotating object (the Euler equations) is a classic problem and is treated in standard mechanics textbooks (see for example, Ref. 7). On the basis of these equations, it is possible to derive the stabilizing gyroscopic effect. This derivation is briefly summarized in the Appendix.

In our case, the equation for the angle θ can be written as

$$\ddot{\theta} + \omega^2(\theta - \theta_0) = \frac{\mathcal{M}_\theta}{J_1}, \qquad (20)$$

where $\omega = [(J_0 - J_1)/J_1]\dot{\phi}_0$, $\dot{\phi}_0$ is the initial spin angular velocity (in the $\mathbf{n}$ direction), and J_0 and J_1 are moments of inertia in the $\mathbf{n}$ and $\mathbf{t}$ directions, respectively; θ_0 is the initial tilt angle and $\mathcal{M}_\theta = OP \cdot F_{\text{lift}}$ is the projection of the torque due to the water flow in the y direction. Equation (20) shows that in the absence of spin motion, $\dot{\phi}_0 = 0$, the torque due to the lift force will initiate rotational motion of the stone in the θ direction. As discussed above, the corresponding situation is unstable. On the other hand, spin motion induces a stabilizing torque that can maintain θ around its initial value. The effect of the torque can be neglected if, after a collision with the water, the maximum amplitude of the motion of the angle θ is small: $\delta\theta = [\theta - \theta_0]_{\text{max}} \ll 1$. If we use Eq. (20), an estimate of $\delta\theta$ can be obtained by balancing the last two terms in Eq. (20), yielding $\delta\theta \sim \mathcal{M}_\theta/(J_1\omega^2)$ (note that up to numerical factors $(J_0 - J_1)/J_1 \sim 1$ and $J_1 \sim MR^2$, with R the radius of the stone). The order of magnitude of $\mathcal{M}_\theta$ can be obtained using the results of Sec. IV C. The average vertical force acting on the stone has been found to be the order of the weight of the stone (see the discussion after Eq. (16)): $\langle F_z(t) \rangle \simeq Mg$. If we take $OP \sim R$, we obtain the simple result $\mathcal{M}_\theta \sim MgR$. The estimate for $\delta\theta$ follows directly as $\delta\theta \sim g/(R\omega^2)$. Therefore, the condition for θ to remain approximately constant, $\delta\theta \ll 1$, is

$$\dot{\phi}_0 \sim \omega \gg \sqrt{\frac{g}{R}}. \qquad (21)$$

For a stone with a diameter of 10 cm, Eq. (21) gives $\dot{\phi}_0 \gg 14\,\text{s}^{-1}$, corresponding to a rotational frequency larger than a few revolutions per second ($\sim 2\,\text{Hz}$). This condition is easily fullfilled in practice and corresponds approximately to what we would expect intuitively for a successful throw. Note that the condition (21) is independent of the center of mass velocity of the stone V.

VI. AN ESTIMATE FOR THE MAXIMUM NUMBER OF BOUNCES

The estimation of the maximum number of bounces is the most difficult and tentative part of the analysis because many factors can in principle slow down or destabilize the stone, some of which are extremely difficult to model (such as irregularities of the water surface and the wind). We shall assume the idealized situation described above (perfect surface, no wind, idealized reaction force) and focus on two specific factors, which appear, at least intuitively, as natural candidates for stopping the stone.

A. Slow down of the stone

As I have discussed in Sec. IV C, energy is dissipated during a collision and the x component of the velocity of the stone will decrease during each collision: after a few collisions, all the initial kinetic energy will be dissipated. This process can be easily formulated.

I consider a succession of N collisions. Between two collisions, the motion is parabolic (wind and air friction are neglected) and the initial x component of the velocity at the next collision is equal to the final x component of the velocity at the end of the previous collision. The important point to note is that the energy loss during one collision, Eq. (17), is independent of the velocity V_{x0} before the collision. Therefore, the velocity of the stone after N collisions obeys the relation

$$\frac{1}{2}MV_x^2[N] - \frac{1}{2}MV_x^2[0] = -N\mu Mg\ell, \qquad (22)$$

so that the stone will be stopped at a collision number N_c such that the total energy loss is larger than the initial kinetic energy (similar to the argument leading to the critical velocity for skimming, V_c, in Eq. (19)). This criterion corresponds to $V_x^2[N_c] = 0$ in Eq. (22), and N_c is given accordingly by

$$N_c = \frac{V_x^2[0]}{2g\mu\ell}. \qquad (23)$$

If we use the same typical values as before ($M = 0.1\,\text{kg}$, $a = 0.1\,\text{m}$, $C_l \approx C_f \approx 1$, $\rho_w = 1000\,\text{m}^{-3}$, $\beta \sim \theta \sim 10°$, we obtain $\mu \simeq 1.4$ and $\ell \simeq 13\,\text{cm}$. We then find $N_c \approx 6$ for the initial velocity $V_{x0} = 5\,\text{m s}^{-1}$, $N_c \approx 17$ for $V_{x0} = 8$ m s^{-1}, and $N_c \approx 38$ for $V_{x0} = 12\,\text{m s}^{-1}$. The latter number of bounces corresponds to the world record.[1]

It is interesting to calculate the distance between two successive collisions. As noted, the motion of the stone is parabolic out of the water: the position $\{X, Z\}$ of the particle is given by $X(t) = V_x t$, $Z(t) = -\frac{1}{2}gt^2 + |V_z|t$. The next collision will occur at a distance $\Delta X = 2V_x|V_z|/g$. The dependence of V_x on the number of collisions N is given by Eq. (22). On the other hand, V_z does not depend on the number of collisions because the stone rebounds "elastically" in the z direction, as follows from

the analysis of the collisional process in Sec. IV (see, for example, the conservation of the energy E during the collision discussed for the circular stone). If we use Eq. (22), we obtain the simple result

$$\Delta X[N] = \Delta X_0 \sqrt{1 - \frac{N}{N_c}}, \qquad (24)$$

where $\Delta X_0 = 2 V_{x0}|V_{z0}|/g$. Note that ΔX_0 is approximately equal to the distance between the two first ricochets, $\Delta X[N=1]$, when $N_c \gg 1$. For $V_{x0} = 8\,\mathrm{m\,s}^{-1}$, we obtain $\Delta X_0 \approx 2.25$ m.

Equation (24) for $\Delta X[N]$ is plotted in Fig. 3. We remark that the decrease in the distance between two successive ricochets is first rather slow ($\Delta X[N] \simeq \Delta X_0(1 - \frac{N}{2N_c})$ for $N \ll N_c$, see Eq. (24)), but strongly accelerates for the last collisions when $N \sim N_c$, due to the square root variation of $\Delta X[N]$ close to N_c. This result is in agreement with observation. Such an effect is known to specialists of stone-skipping as "pitty-pat."[1]
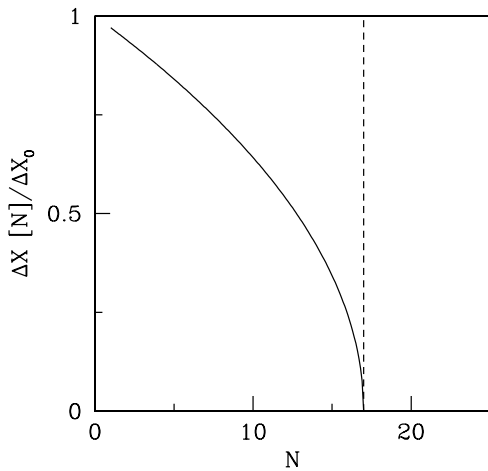

FIG. 3: Plot of the (normalized) distance between two successive collisions $\Delta X[N]/\Delta X_0$ as a function of the number of bounces N. The initial velocity is $V_{x0} = 8\,\mathrm{m\,s}^{-1}$, corresponding to $N_c = 17$ (using the same values for the parameters as those given in the text). The vertical dashed line indicates that $N_c = 17$.

B. Angular destabilization

However, there is another possible destabilizing mechanism in the collision process. As was discussed in Sec. V, the rotational stability of the stone is crucial in the collisional process. A criterion for stability has been found in the form of a minimum spin velocity of the stone. However, each collision will perturb the rotational motion and the sum of all these effects can eventually bypass the stability condition. This argument can be easily formulated. As shown above, the amplitude of the angular motion of

θ is $\delta\theta \sim g/(R\omega^2)$, with $\omega \sim \dot\phi_0$, the (constant) spin velocity of the stone. Now assume that the destabilizing effects add, a reasonable assumption. Then, after N collisions we expect that $\Delta_N \theta \sim N\delta\theta$. The stone is completely destabilized for a collision number N_c such that $\Delta_{N_c}\theta \sim 1$, yielding

$$N_c \sim \frac{R\dot\phi_0^2}{g}. \qquad (25)$$

If we use the same numerical values as before, we obtain, for example, $N_c \simeq 5$ for a initial spin velocity $\phi_0 = 5\,\mathrm{rev/s}$ and $N_c = 38$ (the world record[1]) for $\phi_0 = 14\,\mathrm{rev/s}$. Note, however, that there is a quite large uncertainty of the numerical prefactors in the above estimate of N_c, and this estimate is merely qualitative and should not be taken literally.

VII. DISCUSSION

At the level of our description, the maximum number of bounces results from the combination of the two previous mechanisms: slow down and angular destabilization. The maximum number of bounces is therefore given by the *minimum* of the two previous estimates, in Eqs. (23) and (25).

The estimate N_c^{sd} obtained in Eq. (23) from the slow down of the stone depends only (quadratically) on the initial velocity of the stone: in principle, a very large number of bounces could be reached by increasing the initial velocity of the stone. But on the other hand, the angular destabilization process results in a maximum value of N_c^{spin} which is independent of the initial velocity of the stone, as indicated by Eq. (25). This shows that even if the initial velocity of the stone is very large, that is, $N_c^{\mathrm{sd}} \gg 1$, the stone will be stopped by angular destabilization after N_c^{spin} bounces. In other words, the initial "kick" that puts the stone in rotational motion is a key factor for a good throw.

The results presented here are in agreement with our intuition for the conditions of a good throw. Some of the results are also in agreement with observations, for example, the acceleration of the number of collisions at the end of the throw (a phenomenon known as "pitty-pat" in stone skipping competitions[1]). Some easy checks of the assumptions underlying our calculations could be performed, even without any sophisticated apparatus. For example, taking pictures of the water surface after the ricochets would locate the positions of the collisions (because small waves are produced at the surface of water). A simple test of the variation of the distance between two collisions as a function of collision number, Eq. (24), would then be possible. A more ambitious project would be to design a "catapult," allowing one to throw stones with a controlled translational and spin velocity (together with the incidence angle of the stone on water). A measurement of the maximum number of bounces performed for various throw parameters would allow us to check the

assumptions underlying the present simple analysis and to determine some of the parameters involved in the description (such as μ and ℓ). It would be also interesting to repeat the experiments reported in Ref. 2 using modern techniques (such as fast cameras), in order to image and analyze in particular the rebound process as a function of the throw parameters. Hopefully a better understanding of the mechanisms of stone skipping will allow someone to break the actual world record!

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APPENDIX A: APPENDIX

I briefly recall the derivation of Eq. (20), from the Euler equations described in Ref. 7 The latter are written as:[7]

$$I_1\frac{d\omega_1}{dt} - \omega_2\omega_3(I_2 - I_3) = N_1 \quad \text{(A1a)}$$
$$I_2\frac{d\omega_2}{dt} - \omega_1\omega_3(I_3 - I_1) = N_2 \quad \text{(A1b)}$$
$$I_3\frac{d\omega_3}{dt} - \omega_1\omega_2(I_1 - I_2) = N_3 . \quad \text{(A1c)}$$

In Eq. (A1), I_α, ω_α, and N_α ($\alpha = 1, 2, 3$) are respectively the moment of inertia, angular velocity, and torque along the direction of a particular principal axis, denoted as α. In our case, the direction 1 is taken along the axis perpendicular to the vectors $\mathbf{n}$ and $\mathbf{t}$ (the direction 1 is along the y axis in Fig. 1), the direction 2 along $\mathbf{n}$ and the direction 3 along $\mathbf{t}$. We therefore have $\omega_1 = \dot{\theta}$, and due to the symmetry of the circular stone, $I_1 = I_3 \equiv J_1$ and $I_2 \equiv J_0$. Moreover, because only the lift component of the reaction force (along $\mathbf{n}$) is considered in the present analysis, we have $N_1 \equiv \mathcal{M}_\theta$ and $N_2 = N_3 = 0$.

Equation (A1b) yields immediately that $\dot{\omega}_2 = 0$. We therefore have $\omega_2 = \dot{\phi}_0$, with $\dot{\phi}_0$ the initial spin velocity. Equation (A1c) can be therefore written as:

$$\frac{d\omega_3}{dt} = \frac{J_1 - J_0}{J_1}\dot{\phi}_0\omega_1 . \quad \text{(A2)}$$

If we use $\omega_1 = \dot{\theta}$, Eq. (A2) can be integrated once to give

$$\omega_3 = \frac{J_1 - J_0}{J_1}\dot{\phi}_0(\theta - \theta_0) , \quad \text{(A3)}$$

with $\theta_0 = \theta(t = 0)$, the initial tilt angle. The substitution of Eq. (A3) into Eq. (A1a) leads to Eq. (20).

[1] The actual world record appears to be 38 rebounds (by J. Coleman-McGhee). See, for example, <http://www.stoneskipping.com> for more information on stone skipping competitions.
[2] Some pictures of the bouncing process of a circular stone on water and sand can be found in C. L. Stong, The Amateur Scientist, Sci. Amer. **219** (2), 112–118 (1968).
[3] H. R. Crane, "How things work: What can a dimple do for skipping stones?," Phys. Teach. **26** (5) 300–301 (1988).
[4] D. J. Tritton, *Physical Fluid Dynamics* (Oxford University Press, 1988), 2nd ed., pp. 97–105.
[5] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, 1959), pp. 168–175.
[6] Note that the nontrivial point is to assume that C_l does not vanish and reaches a finite value in the small θ and β limit. We may invoke the finite aspect ratio (thickness over lateral size) of the object. For example, if the stone is an ellipsoid of revolution with thickness h and radius a, with $h \ll a$, we expect $C_l \sim h/a$.[5] However the proportionality constant is expected to be sufficiently large so that the lift effect is non-negligible. This property is exemplified by water skiing. In this case, the lift force is sufficiently large to sustain the weight of a skier on small boards, while both tilt and incidence angles are close to zero.
[7] H. Goldstein, *Classical Mechanics* (Addison-Wesley, 1980), 2nd ed., pp. 203–213.
[8] It is amusing to note that the laws of friction for the stone are similar to those of solid friction. We have indeed $F_x = \mu Mg$, with $\mu = \tilde{C}/C$, independent of the velocity and surface of the stone. Of course, the same result holds for water skiing, which is not obvious.